$$= \sum_i f_i(x,t) \left[\sum_j \sigma_j(x;t)\ \phi_j(\xi)\ \delta_{ij} \right]$$

$$= \sum_i f_i(x,t)\ \sigma_i(x;t)\ \phi_i(\xi) \tag{25}$$

By combining the preceding two conditions, the total effect on (18) is a complete finitization of each equation in the system of equations, thereby rendering them more effective for numerical computations. We may summarize these constructions as follows:

Let X be an arbitrary isotropic, inhomogeneous optical medium with internal emission radiance function N_η *and general time-dependent radiance field N as governed by the equation of transfer (1). Let* $\{\phi_0, \phi_1, \phi_2, \ldots\}$ *be an orthonormal family of functions defined on the unit sphere* Ξ *such that: the family (a) possesses the completeness property (see (19) of Sec. 6.1); (b) possesses the finite recurrence property (16); (c) satisfies an addition theorem (24). Then each member of the general abstract harmonic system of partial differential equations (18) reduces to the following finite form: For some positive integer* ν*:*

$$\boxed{\frac{1}{v}\frac{\partial f_k}{\partial t} + \sum_{j=0}^{\nu} f_j D_{jk} = -\alpha f_k + f_k \sigma_k + f_{\eta,k} \qquad k = 0,1,2,\ldots}$$

(26)

6.3 Classical Spherical Harmonic Method: General Media

The general theory of the abstract harmonic method developed in the preceding section will now be illustrated for the classical case in which the orthonormal family is constructed from familes of associated Legendre functions of the first kind and circular (trigonometric) functions. The optical medium X will be generally inhomogeneous and isotropic, with time varying inherent optical properties, and given internal sources.

The Orthonormal Family

We begin by observing that the classical spherical harmonic method customarily uses the ordered pair (μ,ϕ) of numbers to specify a point ξ in Ξ, where we have written "μ" for $\cos\theta$, and where (θ,ϕ) are the two angles customarily used to specify ξ in Ξ (see Sec. 2.4 and also example 14 of Sec. 2.11 for an earlier use of μ in conjunction with Legendre polynomials). The range of the variable μ is thus the interval $[-1,1]$, and the range of ϕ, $[0,2\pi]$. Every ξ in Ξ determines a unique (θ,ϕ), that is a unique μ in $[-1,1]$ and a unique ϕ in $[0,2\pi]$.

Conversely, any pair (μ,ϕ) in $[-1,1] \times [0,2\pi]$ determines a unique ξ in Ξ.

The values of associated Legendre functions are usually denoted by "$P_n^m(\mu)$". The integer n is nonnegative, i.e., $n \geq 0$ and the integer m satisfies the inequalities: $-n \leq m \leq n$. The general relations in the theory of Legendre polynomials we shall use below may be found fully developed, e.g., in [318], [289], and [119]. In particular we shall note that:

$$P_n^{-m} = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m \tag{1}$$

and that:

$$\left.\begin{aligned} P_n^o &= P_n \\ P_n^m &= 0 \quad \text{for} \quad m > n \end{aligned}\right\} \tag{2}$$

where "P_n" denotes the Legendre function of the first kind and of degree n. For our present purposes, we note that the associated Legendre function P_n^m is a real valued function with domain $[-1,1]$ and defined for all integers m,n such that n is nonnegative and $|m| \leq n$. The associated Legendre functions include, by (2), the Legendre polynomials as special cases. Any functions P_n^m arising in the subsequent discussions for which $n < 0$, are to be zero-valued functions. In view of (1) and (2) only P_n^m with n+1 nonnegative indices m need be tabulated.

The orthogonality property of the family of associated Legendre functions takes the form:

$$\int_{-1}^{1} P_n^m(\mu)\, P_r^m(\mu)\, d\mu = \begin{cases} 0, \quad , & \text{whenever } n \neq r \\ \dfrac{2}{2n+1} \cdot \dfrac{(n+m)!}{(n-m)!}, & \text{whenever } n = r \end{cases} \tag{3}$$

The integral properties of the family of circular functions needed here are summarized by the equations:

$$\begin{gathered} \int_0^{2\pi} \sin m\phi \, d\phi = 0 \\ \int_0^{2\pi} \cos m\phi \, d\phi = \begin{cases} 0 & \text{if } m \neq 0 \\ 2\pi & \text{if } m = 0 \end{cases} \end{gathered} \tag{4}$$

where m is confined to integral values. These properties can be succinctly summarized by using complex variables. Thus, all three equations in (4) may be expressed by writing:

$$\int_0^{2\pi} e^{im\phi}\, d\phi = 2\pi\delta_{om} \tag{5}$$

where δ_{om} is an instance of the general Knonecker delta δ_{ij}. The use of complex variables will considerably facilitate our work in this section, and so they will be retained throughout. One can always return to the real number setting by finding and considering separately the real and imaginary parts of a complex term.

The details of the construction of the requisite orthonormal family on Ξ are clearly indicated by considering (3) and (5). Thus to an arbitrary ξ in Ξ, (to which corresponds a unique pair (μ,ϕ)) and to every pair of integers m, n, with $n \geq 0$, $|m| \leq n$ we assign the complex number $\Phi_n^m(\xi)$ where we have written:

$$\text{"}\Phi_n^m(\xi)\text{"} \quad \text{for} \quad A_n^m\, P_n^m(\mu)\, e^{im\phi} \tag{6}$$

where in turn we have written

$$\text{"}A_n^m\text{"} \quad \text{for} \quad \left[\frac{(2n+1)(n-m)!}{4\pi(n+m)!}\right]^{1/2} \tag{7}$$

By observing that:

$$A_n^{-m} = \frac{(n+m)!}{(n-m)!}\, A_n^m$$

we can limit tabulations of A_n^m to nonnegative indices m. Furthermore, by recalling (1), the complex conjugate of $\Phi_n^m(\xi)$ may be expressed as follows:

$$\overline{\Phi_n^m(\xi)} = A_n^m\, P_n^m(\xi)\, e^{-im\phi} = (-1)^m\, \Phi_n^{-m}(\xi) \tag{8}$$

The orthonormality property of the family of functions Φ_n^m over Ξ may now be verified. For example:

$$\begin{aligned}\int_\Xi \Phi_n^m(\xi)\, \overline{\Phi_r^m(\xi)}\, d\Omega(\xi) &= A_n^m\, A_r^m \quad 2\pi \int_{-1}^{+1} P_n^m(\mu)\, P_r^m(\mu)\, d\mu \\ &= 2\pi A_n^m\, A_r^m\, \delta_{nr}\, \frac{2}{(2n+1)} \cdot \frac{(n+m)!}{(n-m)!} \\ &= \delta_{nr}\end{aligned}$$

The remaining case where the upper indices of Φ_n^m may differ is straightforward using (5). Hence we have:

$$\int_{\Xi} \Phi_n^m(\xi)\ \Phi_a^b(\xi)\ d\Omega(\xi) = \delta_{mb}\ \delta_{na} \tag{9}$$

for every n, a $\geq$ 0 and b, m such that $|b| \leq a$, $|m| \leq n$.

An exact one-to-one correspondence can be established between the abstract family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of Sec. 6.2 and the spherical harmonic family presently under consideration. Thus to ϕ_j of the earlier discussion we pair Φ_n^m, where $j = n^2 + m + n$. This correspondence arises when one contemplates Fig. 6.1 in which each dot in the figure is paired with the integer couple (m,n), $n \geq 0$, $|m| \leq n$, corresponding to the indices of Φ_n^m. Then counting each row of dots by reading from left to right and counting rows from bottom to top, each dot is given a single index j. For example the dot in the first row, corresponding to (0,0) is given the index 0. The dot corresponding to (-1,1) is given the index 1, (0,1) the index 2, (-3,4) the index 17, etc. In general:

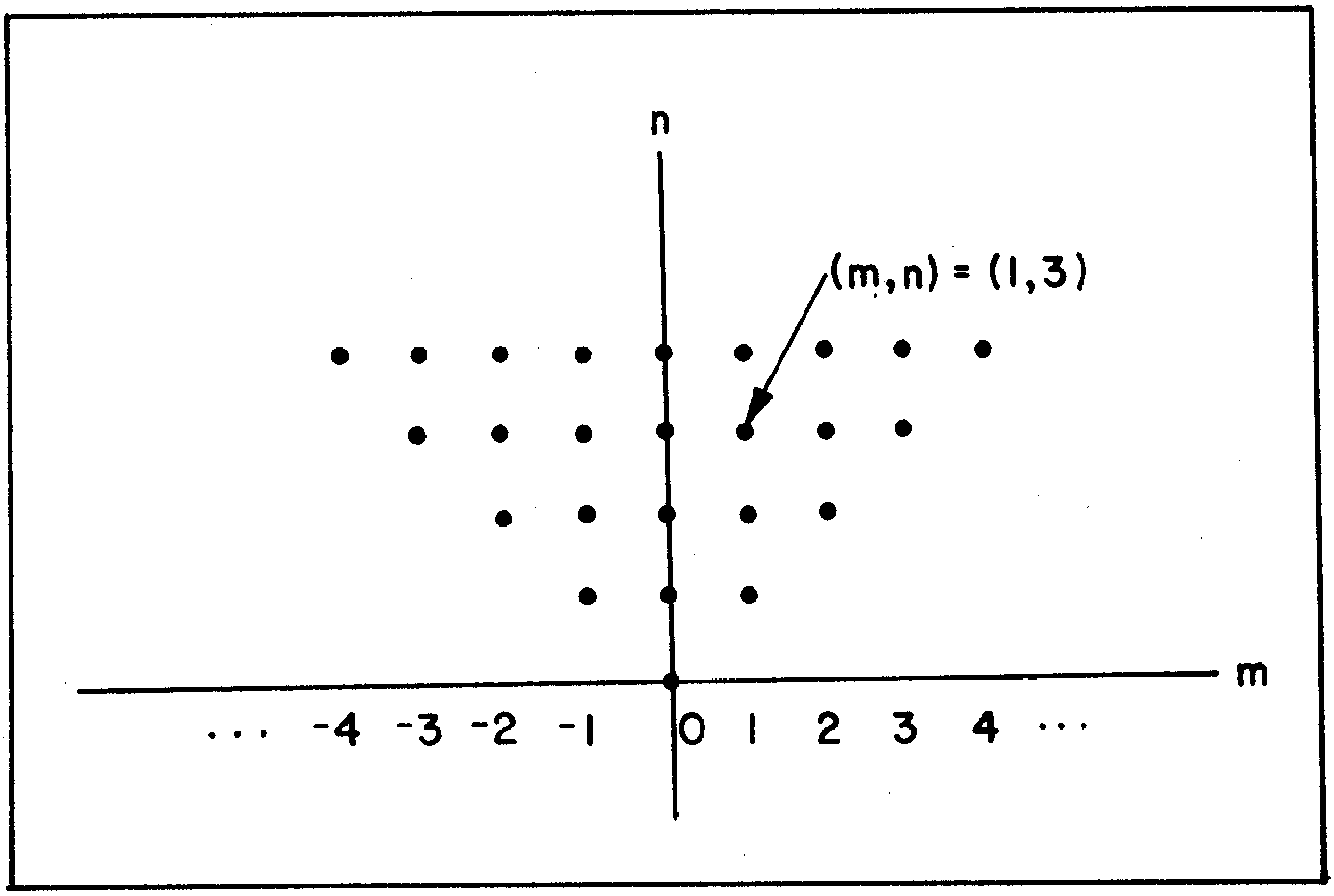

FIG. 6.1 Scheme for establishing the correspondence between the abstract and classical spherical harmonic method.

$$(m,n) \text{ is paired with the index } j = n^2 + m + n \tag{10}$$

and

$$\Phi_n^m \text{ is paired with } \phi_j \tag{11}$$

Observe that the pairings are unique: given (m,n) there is precisely one $j > 0$ corresponding to this pair; given $j > 0$, there is precisely one pair (m,n) on the array corresponding to j and is readily obtained under the conditions on m,n described above.

Properties of the Orthonormal Family

We shall now show that the family of spherical harmonics Φ_n^m on Ξ possesses the three main properties sufficient to insure a reduction of the general abstract harmonic system (18) of Sec. 6.2 to its finite version (26) of Sec. 6.2. (The proof of the orthonormality of the family of spherical harmonics was outlined in the discussion leading to (9).)

The *completeness property* of the set of spherical harmonics holds. However, the property depends on some relatively advanced arguments, and the interested reader is referred to Chapter 7 of [47] for the general theory of completeness of families of functions arising from nth order differential equations.

The *addition theorem* for Legendre functions holds and takes the form (see, e.g., [119]):

$$P_n(\xi \cdot \xi') = P_n(\mu)\, P_n(\mu') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n(\mu)\, P_n(\mu') \cos m(\phi - \phi') \tag{12}$$

where ξ and ξ' are any two directions in Ξ and (μ,ϕ), (μ',ϕ') are their corresponding angular representations. Using (1), (2), the evenness of cosine, and the oddness of sine, (12) may be compactly written as:

$$P_n(\xi \cdot \xi') = \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\mu)\, P_n^m(\mu')\, e^{im(\phi-\phi')} \tag{13}$$

The argument of P_n in (13) is the scalar product of ξ' and ξ. This scalar product is reminiscent of the isotropy condition for an optical medium. We now show how the isotropy condition leads in the present case to the representation of σ in the form of (24) of Sec. 6.2. When isotropy holds, the value of σ (for a fixed x and t) is known once $\xi \cdot \xi'$ is known, i.e., once a number $\mu = \xi \cdot \xi'$ in the interval $[-1,1]$ is specified. This value of σ under isotropy conditions will be denoted by "$\sigma(x;\xi\cdot\xi';t)$". Therefore, the family of Legendre polynomials P_n being complete (a fact also supplied by the general theory in [47] cited above), we may express $\sigma(x;\xi\cdot\xi;t)$ as follows:

$$\sigma(x;\xi\cdot\xi';t) = \frac{1}{2\pi} \sum_{j=0}^{\infty} \left[\frac{2j+1}{2}\right] \sigma_j(x;t)\ P_j(\xi\cdot\xi') \qquad (14)$$

where we have written:

$$\text{"}\sigma_j(x;t)\text{"} \quad \text{for} \quad 2\pi \int_{-1}^{1} \sigma(x;\mu;t)\ P_j(\mu)\ d\mu \qquad (15)$$

Using (13) to represent $P_j(\xi\cdot\xi')$ in (14), we have:

$$\begin{aligned} \sigma(x;\xi\cdot\xi';t) &= \frac{1}{2\pi} \sum_{j=0}^{\infty} \left[\frac{2j+1}{2}\right] \sigma_j(x;t) \sum_{m=-j}^{j} \frac{(j-m)!}{(j+m)!} P_j^m(\mu)\ P_j^m(\mu')\, e^{im(\phi-\phi')} \\ &= \sum_{j=0}^{\infty} \sigma_j(x;t) \sum_{m=-j}^{j} \overline{\Phi_j^m(\xi')}\ \Phi_j^m(\xi) \end{aligned} \qquad (16)$$

This is reducible to the form of (24) of Sec. 6.2 as may be seen by using the correspondence between ϕ_j and Φ_n^m established above. (To show the correspondence in complete detail, let $\sigma_j(x;t)$ be denoted *ad hoc* as "$\sigma_j^m(x;t)$" and require it to have value $\sigma_j(x;t)$ for m in the range $-j \leq m \leq j$.)

In this way we see how the addition theorem for the P_n^m and the isotropy condition on scattering combine to form the extremely useful representation (16). The reader may now extend this idea to still other complete orthonormal families of functions defined on $[-1,1]$ provided an addition theorem of the kind (13) is available for the family.

Next, we observe that the orthonormal family of functions Φ_n^m satisfies the *finite recurrence property* of degree 2. This observation is based on the following three well-known recurrence properties of associated Legendre functions (see, e.g., [289], [119]):

$$\mu P_n^m(\mu) = \frac{(n+m)\ P_{n-1}^m(\mu) + (n+1-m)\ P_{n+1}^m(\mu)}{2n+1} \tag{17}$$

$$\sin\theta\ P_n^m(\mu) = \frac{P_{n+1}^{m+1}(\mu) - P_{n-1}^{m+1}(\mu)}{(2n+1)} \tag{18}$$

$$\sin\theta\ P_n^m(\mu) = \frac{(n-m+2)(m-n-1)\ P_{n+1}^{m-1}(\mu) + (n+m-1)(n+m)\ P_{n-1}^{m-1}(\mu)}{(2n+1)} \tag{19}$$

As an example of how these recurrence relations give rise to instances of the general recurrence property (19) of Sec. 6.2, consider (17). Here we recall that "μ" denotes $\xi\cdot\mathbf{k}$; $\mathbf{k}$ is the unit vector along the positive z-axis. Hence ξ' in (19) of Sec. 6.2 is now $\mathbf{k}$. Next, multiply each side of (17) by $A_n^m\, e^{im\Phi}$. Applying the general definition (6) and making some algebraic rearrangements, the net result is:

$$\xi\cdot\mathbf{k}\ \Phi_n^m(\xi) = C(n,m)\ \Phi_{n-1}^m(\xi) + C(n+1,m)\ \Phi_{n+1}^m(\xi) \tag{20}$$

where we have written:

$$\text{"}C(n,m)\text{"} \quad \text{for} \quad \left[\frac{(n-m)(n+m)}{(2n-1)(2n+1)}\right]^{1/2} \tag{21}$$

Hence in (19) of Sec. 6.2, we have $\nu = 2$, and the A_{jk} are now in the form of $C(j,k)$, with $j = n^2 + m + n$, and $\alpha_1 = (n-1)^2 + m + (n-1)$, $\alpha_2 = (n+1)^2 + m + (n+1)$. The specific representation of $\xi \cdot \mathbf{k}\ \Phi_n^m(\xi)$ in (20) is now used in (20) of Sec. 6.2 to effect an evaluation of the number c_{jk}, and hence the sum:

$$\sum_{j=0}^{\nu} c_{jk} \frac{\partial f_j}{\partial x_3} \tag{22}$$

which forms part of the operation:

$$\sum_{j=0}^{\nu} f_j D_{jk} \tag{23}$$

in (26) of Sec. 6.2. To see how (22) is evaluated, let us represent $N(x,\xi,t)$ by means of the functions Φ_n^m:

$$N(x,\xi,t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_n^m(x,t)\ \Phi_n^m(\xi) \tag{24}$$

where we have written:

$$\text{"}F_n^m(x,t)\text{"} \quad \text{for} \quad \int_\Xi N(x,\xi,t)\ \overline{\Phi_n^m(\xi)}\ d\Omega(\xi) \tag{25}$$

Thus F_n^m in the present context corresponds to f_j in the abstract context of Sec. 6.2, just as Φ_n^m corresponds to ϕ_j. Furthermore, the correspondence of j in f_j with the pair of indices (m,n) of F_n^m is once again that established above. (See Fig. 6.1 and (10), (11).)

Returning to (22), we consider it in the context of (18) of Sec. 6.2, but now using the present family $\{\Phi_n^m\}$ of orthonormal functions. We threfore are to consider:

$$\begin{aligned}\sum_{j=0}^{\infty} c_{jk} \frac{\partial f_j}{\partial x_3} &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \int_\Xi \xi\cdot\mathbf{k}\,\Phi_n^m(\xi)\,\overline{\Phi_a^b(\xi)}\,d\Omega(\xi) \right\} \frac{\partial F_n^m}{\partial x_3} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \int_\Xi \left[C(n,m)\,\Phi_{n-1}^m(\xi) + C(n+1,m)\ \Phi_{n+1}^m(\xi) \right] \Phi_a^b(\xi)\,d\Omega(\xi) \right\} \frac{\partial F_n^m}{\partial x_3} \\ &= C(a+1,b)\ \frac{\partial F_{a+1}^b}{\partial x_3} + C(a,b)\ \frac{\partial F_{a-1}^b}{\partial x_3} \end{aligned} \tag{26}$$

in which $k = a^2 + b + a$.

Thus the infinite sum of z-derivatives in (18) of Sec. 6.2 is reduced to a sum of two such derivatives.

The general procedure should now be clear: by placing the recurrence relations (18) and (19) into their appropriate counterparts of (20), the numbers a_{jk} and b_{jk} in (21), (22) of Sec. 6.2 are readily evaluated. Then the sums:

$$\sum_{j=0}^{\nu} a_{jk} \frac{\partial f_j}{\partial x_1} \quad , \quad \sum_{j=0}^{\nu} b_{jk} \frac{\partial f_j}{\partial x_2}$$

are evaluated analogously to the manner displayed in (26). These details may be left to the reader.

General Equations for Spherical Harmonic Method

The net result of the reduction calculations on (26) outlined above may be written in the form:

$$\boxed{\begin{aligned}
&\frac{1}{v}\,\frac{\partial F_a^b(x,t)}{\partial t} + \left[C(a,b)\,\frac{\partial F_{a-1}^b}{\partial x_3} + C(a+1,b)\,\frac{\partial F_{a+1}^b}{\partial x_3}\right] \\
&\quad + \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)\left[B(a,b)F_{a-1}^{b-1}(x,t) - B(a+1,-b+1)\;F_{a+1}^{b-1}(x,t)\right] \\
&\quad + \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)\left[-B(a,-b)F_{a-1}^{b+1}(x,t) + B(a+1,b+1)\,F_{a+1}^{b+1}(x,t)\right] \\
&\quad = \left[-\alpha(x,t) + \sigma_a(x;t)\right] F_a^b(x,t) + F_{\eta,a}^n(a,t) \\
&\qquad a = 0,1,2,\ \ldots;\ |b| \le a.
\end{aligned}}$$

(27)

where we have written:

$$\text{"}B(a,b)\text{"} \quad \text{for} \quad \left[\frac{(a+b)(a+b-1)}{(2a-1)(2a+1)}\right]^{1/2} \tag{28}$$

and where C(a,b) is defined generally in (21). Furthermore, we have written:

$$\text{"}F_{\eta,a}^b(x,t)\text{"} \quad \text{for} \quad \int_{\Xi} N_\eta(x,\xi,t)\;\overline{\Phi_a^b(\xi)}\;d\Omega(\xi) \tag{29}$$

analogously to (25), so that N_η has the representation:

$$N_\eta(x,\xi,t) = \sum_{n=0}^{\infty}\ \sum_{m=-n}^{n} F_{\eta,n}^m(x,t)\ \Phi_n^m(\xi) \quad . \tag{30}$$

The set of equations (27) forms a coupled infinite system of equations in the unknown functions F_a^b, $a = 0,1,2,\ldots$, $|b| \le a$. The functions F_a^b are generally complex valued, according to their defined construction (25), and such that $N(x,\xi,t)$ is real valued, according to (24). The general initial conditions for the system (27) are:

$$F^b_a(x,0) = \int_{\Xi} N^o(x,\xi,0)\ \overline{\Phi^b_a(\xi)}\ d\Omega(\xi) \quad , \qquad (31)$$

for every x in X, and where N^o is the given initial radiance function on X x Ξ at t = 0. For steady state versions of (27), the time derivative term is zero. The functions F^b_a then have domain X and (31) is replaced by:

$$F^b_a(x_o) = \int_{\Xi} N^o(x_o,\xi)\ \overline{\Phi^b_a(\xi)}\ d\Omega(\xi) \qquad (32)$$

for x_o over some appropriate subset of the boundary of X (cf., e.g., (26) of Sec. 6.4).

The system (27) is of sufficient generality to solve such problems as point source, beam source, and general internal source problems in the sea; natural light field problems in lakes, harbors, and the sea. Observe that the inherent optical properties in the form of α and σ_a may be quite general, and that the term $F^b_{\eta,a}$ provides for internal sources of radiant flux, such as artificial light sources (laser beams, searchlights, submerged incandescent point sources, etc.) or natural light sources (phosphorescence, animal sources, etc.). The general methods of solution of (27) and its manifold variants are well known and may be implemented by programmed machine procedures. If the model is sufficiently simple (as, e.g., in the illustration of Sec. 6.4) the associated simplified form of system (27) may be solved by hand and evaluated numerically or even used for general theoretical reasoning.

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6.4 Classical Spherical Harmonic Method: Plane-Parallel Media

The classical spherical harmonic method developed in the preceding section for general media will now be illustrated in a setting of primary importance in hydrologic (and meteorologic) optics: the plane-parallel optical medium. Throughout this section, then, we shall assume that X is a plane-parallel medium of arbitrary (finite or infinite) depth. The incident light field and the optical properties of X are assumed to be in the steady state and independent of the x and y coordinates throughout X, thus establishing a stratified medium and stratified steady radiance field throughout X.

Under the present conditions on the medium X, the general system of equations (27) of Sec. 6.3 reduces to:

$$\boxed{\begin{array}{r} C(a,b)\,\dfrac{\partial F^b_{a-1}}{-\partial z} + C(a+1,b)\,\dfrac{\partial F^b_{a+1}}{-\partial z} = (-\alpha + \sigma_a)\ F^b_a + F^b_{\eta,a} \\ a = 0,1,2,\ \ldots;\ |b| \le a \end{array}} \qquad (1)$$