the threshold of the invariant imbedding domain of radiative transfer theory. Thus the equation (26), say, may be viewed on the one hand, as the logical culmination of the train of deductions begun in Sec. 6.2 in the development of the classical spherical harmonic method; and on the other hand (26) forms a bridge between the classical method of solution of the equation of transfer and the invariant imbedding techniques for the solution of the equation of transfer. These latter techniques will be considered in Sec. 7.10.

Summary

In the preceding four Secs. 6.1 to 6.4 the spherical harmonic method is developed and applied after an appropriate motivation of the method in Sec. 6.1. The main purpose of the discussions is to make clear the fundamental ideas on which the method rests, in particular the general role of the orthonormal family of functions used to represent the radiance function as a sum of products of purely spatial and directional terms. This was done in Secs. 6.2 and 6.3. To show the applicability of the method to the case of plane-parallel media, the setting of greatest utility in the study of hydrologic and meteorologic optics, the discussion of the present section is added to the general remarks. In particular, equation sets (14) to (17) above explicitly exhibit the truncated forms of the spherical harmonic equations, where the truncation arbitrarily sets to zero all functions F_a^b with indices $a > m$. The resultant system (24) can be used to solve for the unknown complex valued functions F_a^b, $0 \leq a, \leq m$, $|b| \leq a$. To solve (24) directly we must know N^o (in (31) or (32) of Sec. 6.3) from experiments. If N^o is to be found theoretically, we may use invariant imbedding methods which will give the aerosol's or hydrosol's reflectance to incident light (Volume IV, *et seq.*).

6.5 Three Approaches to Diffusion Theory

The term "diffusion theory" in the context of radiative transfer theory denotes a discipline based on not any single equation, but rather a collection of more or less loosely interconnected theories each springing from some analytic expression which, in turn, is based on the fundamental equation of transfer. For our present purposes we may broadly classify this collection of diffusion theories into two main groups: the *approximate* and the *exact* theories. A diffusion theory is approximate to a greater or lesser degree depending on the amount of modification undergone by the analytic structure of the equation of transfer as the equation is subject to simplifying assumptions. In the present section our purpose is to approach this complex of diffusion theories from three different directions so as to gain a useful overall perspective of the sub-discipline of diffusion theory within general radiative transfer theory. In particular we shall approach one of the more useful approximate diffusion theories (called *classical diffusion* theory, for reasons which will eventually become clear) by starting from the equation of transfer and proceeding to transform the equation by adopting the assumption of Fick's law for diffusing photons. Then we shall start again, this time proceeding via spherical harmonic theory which, depending on the order of terms retained in the basic system (27) of Sec. 6.3, opens up a multitude of paths into the domain of approximate diffusion theory. This approach serves to show the extremely large number of diffusion-type theories generally possible, and to throw light on the classical diffusion theory by appropriately placing the latter in the hierarchy of approximate diffusion theories springing from the system of spherical harmonic equations of Sec. 6.3. Finally, we start afresh once more from the equation of transfer and develop the basic equation for an important exact diffusion theory which applies rigorously to optical media whose volume scattering functions σ are independent of the directions ξ' and ξ.

The Approach via Fick's Law

We begin with the general time-dependent equation of transfer (re (4) of Sec. 3.15) with source term in a generally inhomogeneous optical medium X:

$$\frac{1}{v}\frac{\partial N(x,\xi,t)}{\partial t} + \xi \cdot \nabla N(x,\xi,t) = -\alpha(x,t)\, N(x,\xi,t) + N_*(x,\xi,t) + N_\eta(x,\xi,t) \tag{1}$$

Diffusion theory is characteristically interested in the description of the scalar irradiance h(x,t) rather than the radiance N(x,ξ,t). That is, the density of the total flow at x in all directions is of interest rather than the density of the flow in each direction ξ at x. Thus we are led to integrate each term of (1) over direction space Ξ. The reduction of the resulting integrated form of (1) is facilitated by recalling from (4) of Sec. 4.2 that:

$$\alpha(x,t) = a(x,t) + s(x,t) \tag{2}$$

and from (2) of Sec. 2.8 that we write:

$$\text{"}\mathbf{H}(x,t)\text{"} \quad \text{for} \quad \int_{\Xi} N(x,\xi,t)\,\xi\, d\Omega(\xi) \quad , \tag{3}$$

where $\mathbf{H}(x,t)$ is the *vector irradiance* at x at time t.

The reduced integrated form of (1) is:

$$\frac{1}{v}\frac{\partial h(x,t)}{\partial t} + \nabla \cdot \mathbf{H}(x,t) = -a(x,t)\, h(x,t) + h_\eta(x,t) \tag{4}$$

where we have written:

$$\text{"}h_\eta(x,t)\text{"} \quad \text{for} \quad \int_\Xi N_\eta(x,\xi,t)\, d\Omega(\xi) \quad .$$

Equation (4) lacks utility in our present efforts to describe the scalar irradiance throughout X. The presence of the divergence term for the vector irradiance blocks immediate usage of (4) in this respect: If, somehow, $\nabla \cdot \mathbf{H}$ could be replaced by a single function of h, then the resulting form of (4) would be a useful statement involving only scalar irradiance. It is at this point that the customary appeal to *Fick's law* of diffusion is made. This law states that, for some nonnegative valued function D, on X:

$$\boxed{\mathbf{H}(x,t) = -\, D(x,t)\ \nabla h(x,t)} \tag{5}$$

for every t in some time interval. In other words, at each point x and time t, the vector $\mathbf{H}(x,t)$ has the direction of the negative of the gradient of the scalar irradiance field h. In still other terms, $\mathbf{H}$ has the direction from the greatest to the smallest values of h in the neighborhood of a point. The spatial and temporal variation of D is required to be quite mild, and for essentially all practical applications D is assumed constant. The types of media for which Fick's law is a reasonably good description of the state of affairs between $\mathbf{H}$ and h are those for which the scattering attenuation ratio ρ is large, say on the order of 0.6 and above. All other things being equal the closer ρ is to 1 (i.e., the larger the proportion of scattering compared to absorption), the closer does Fick's law describe $\mathbf{H}$ in terms of h. Furthermore, Fick's law, all other things being equal, increases in accuracy with distance from the boundaries and highly directional or concentrated sources of the medium until the effects of these boundaries and sources have disappeared. Any physical breakdown of a formula of the resultant theory is eventually traceable to a marked inapplicability of Fick's law. Using (5) in (4), we have:

$$\boxed{\frac{1}{v} \cdot \frac{\partial h(x,t)}{\partial t} - \nabla \cdot (D(x,t)\ \nabla h(x,t)) = -\, a(x,t)\ h(x,t) + h_\eta(x,t)} \tag{6}$$

Equation (6) is the desired *scalar diffusion equation* for scalar irradiance h. D is the *diffusion function* (or *constant,* as the case may be), a is the volume absorption function, and h_η the *emission* or *source* term for the equation. The diffusion theory based on (6) is the classical (*scalar*) *diffusion theory.* When D is assumed constant over the space

X and a given time interval, an assumption which henceforth shall be in force, (6) may be written:

$$\boxed{\frac{1}{v}\frac{\partial h}{\partial t} - D\,\nabla^2 h = -\,ah + h_\eta} \qquad (7)$$

Equation (7) has the Gestalt of the diffusion equation of classical heat conduction and other diffusion phenomena with source term (h_η) and annihilation term ($-ah$), hence the mathematics of the diffusion of photons as governed by (7) is identical to that of the diffusion of heat and other classical diffusion phenomena, the theory of which is thoroughly understood. Therefore (7) may possibly be applied to such problems as describing the transient light field set up by pulsed sources. Equation (7) and related equations are studied further in Table 1 below, and in Sec. 6.6.

The Approach via Spherical Harmonics

The next approach to diffusion theory we shall describe is that via the spherical harmonic theory developed in Sec. 6.4. It will be seen that the approach can take place on several levels of generality and in an infinite number of directions on each level. We shall begin our discussion with one of the simpler directions of approach on a very practical level, the goal being once again the classical scalar diffusion equation (7). However, now awaiting us at the goal is the added bonus of a theoretical representation for the diffusion constant D and a formula describing the radiance function in a general diffusing medium in terms of the vector and scalar irradiances.

In our present approach to diffusion theory we shall be guided by the following two special principles concerning the components F_a^b of the spherical harmonic representation of the radiance function:

(i) All components F_a^b other than F_0^0, F_1^{-1}, F_1^1 are set equal to zero in the system (27) of Sec. 6.3. All components of $F_{\eta,a}^b$ other than $F_{\eta,0}^b$ are zero.

(ii) All time derivatives of the components F_a^b other than F_0^0 are set equal to zero in the system (27) of Sec. 6.3.

The reason for these two special principles stems ultimately from our intuitive conception of a diffusive flow of material (or light) particles: (i) the amount of diffusive flow about a point varies mildly from direction to direction, and (ii) the overall directional structure of the flow itself varies mildly from moment to moment. With this intuitive conception in mind, the rules of action stated in (i) and (ii) above are arrived at by pairing F_0^0 with h and by identifying the components F_1^{-1}, F_1^0, F_1^1 as the first three of an infinite set of components describing the overall directional flow of radiant energy at a point. The basis of this pairing of F_0^0

with h is as follows. By (6) and (25) of Sec. 6.3 we have the definitional identity:

$$F_0^0(x,t) = \int_{\Xi} N(x,\xi,t)\ \overline{\Phi_0^0(\xi)}\ d\Omega(\xi)$$

$$= A_0^0 \int_{\Xi} N(x,\xi,t)\ P_0^0(\xi)\ d\Omega(\xi)$$

$$= A_0^0\ h(x,t) = (4\pi)^{-1/2}\ h(x,t) \quad . \qquad (8)$$

The fact that the three components F_1^{-1}, F_1^0, F_1^1 are associated with the overall directional structures of the radiant flux is established by first noting that:

$$\mathbf{H}(x,t) = \int_{\Xi} N(x,\xi,t)\ \xi d\Omega(\xi)$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_n^m(x,t) \int_{\Xi} \Phi_n^m(\xi)\ \xi d\Omega(\xi) \qquad (9)$$

Furthermore, we have (cf. Fig. 2.4):

$$\xi = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \qquad (10)$$

If we could now express the quantities sin θ cos θ, sin θ sin φ and cos φ as linear combinations of the Φ_n^m, then we could directly evaluate the integral in (9) using the orthonormality properties of the Φ_n^m. Toward this end we recall that $\sin\theta = (1-\cos^2\theta)^{1/2} = (1-\mu^2)^{1/2}$. Furthermore, an examination of any list of associated Legendre functions reveals that:

$$L_1^1(\mu) = -\,2P_1^{-1}(\mu) = (1-\mu^2)^{1/2} \quad .$$

Then:

$$\sin\theta\ (\cos\phi + i\sin\phi) = P_1^1(\mu)\ e^{i\phi}$$

$$= (A_1^1\ P_1^1(\mu)\ e^{i\phi})/A_1^1$$

$$= \Phi_1^1(\xi)/A_1^1$$

Similarly:

$$
\begin{aligned}
\sin\theta\,(\cos\phi - i\sin\phi) &= -\,2P_1^{-1}(\mu)\,e^{-i\phi} \\
&= (-2\,A_1^{-1}\,P_1^{-1}(\mu)\,e^{-i\phi})/A_1^{-1} \\
&= -\,2\,\Phi_1^{-1}(\xi)/A_1^{-1} \\
&= -\,\Phi_1^{-1}(\xi)/A_1^{1}
\end{aligned}
$$

From these expressions we deduce that:

$$\sin\theta\cos\phi = \frac{1}{2A_1^1}\left(\Phi_1^1(\xi) - \Phi_1^{-1}(\xi)\right) \qquad (11)$$

$$\sin\theta\sin\phi = \frac{1}{2iA_1^1}\left(\Phi_1^1(\xi) + \Phi_1^{-1}(\xi)\right) \qquad (12)$$

Finally, we observe that:

$$
\begin{aligned}
\cos\theta &= \mu = P_1(\mu) = P_1^0(\mu) \\
&= A_1^0\,P_1^0(\mu)\,e^{io\phi}A_1^0 = \Phi_1^0(\xi)/A_1^0 \qquad (13)
\end{aligned}
$$

Using (11) to (13) in (10), we have the requisite representation of ξ as a linear combination involving only members Φ_n^m of the orthonormal family. The conjugates of Φ_n^m are obtained using (8) of Sec. 6.3. As a result, (9) reduces immediately to:

$$
\boxed{
\begin{aligned}
\mathbf{H}(x,t) = {} & \frac{1}{2A_1^1}\left[F_1^1(x,t) - F_1^{-1}(x,t)\right]\mathbf{i} \\
& - \frac{1}{2iA_1^1}\left[F_1^1(x,t) + F_1^{-1}(x,t)\right]\mathbf{j} \\
& + \frac{1}{A_1^0}\,F_1^0(x,t)\,\mathbf{k}
\end{aligned}
}
\qquad (14)
$$

This is the desired representation of the vector irradi- $\mathbf{H}(x,t)$ in terms of the spherical harmonic components F_a^b of the radiance function N. The representation reveals the rôle played by the three components F_1^{-1}, F_1^0, F_1^1 in the description of the overall directional structure of the light field (see also (29) below).

With the basis for the two special principles (i) and (ii) now reasonably well established, we next apply these special principles to the system (27) of Sec. 6.3. According to principle (i), we need consider only the cases a = 0, 1. According to principle (ii), all time derivatives, except that of F_0^0, vanish. The resultant set of four equations is:

$$\frac{1}{v}\,\frac{\partial F_0^0}{\partial t} + C(1,0)\frac{\partial F_1^0}{\partial x_3} - \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)B(1,1)F_1^{-1} + \frac{1}{2}\left(\frac{\partial}{\partial x_1}\overset{+i}{+i}\,\frac{\partial}{\partial x_2}\right)B(1,1)\,F_1^1$$

$$= (-\alpha + \sigma_0)\;F_0^0 + F_{\eta,0}^0 \qquad (a = 0,\ b = 0 \text{ in } F_a^b)$$

(15)

$$-\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)B(1,1)\;F_0^0 = (-\alpha\ +\sigma_1)\;F_1^{-1}$$

$$(a = 1,\ b = -1 \text{ in } F_a^b) \qquad (16)$$

$$C(1,0)\;\frac{\partial F_0^0}{\partial x_3} = (-\alpha + \sigma_1)\;F_1^0$$

$$(a = 1,\ b = 0 \text{ in } F_a^b) \qquad (17)$$

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)B(1,1)\;F_0^0 = (-\alpha\ +\sigma_1)\;F_1^1$$

$$(a = 1,\ b = 1 \text{ in } F_a^b) \qquad (18)$$

Our present goal is to obtain a single diffusion equation for h(x,t) from the system (15) to (18). In view of the connection between F_0^0 and h stated in (8), we see that the goal will be in sight if we use (16) to (18) to replace each occurrence of F_1^{-1}, F_1^0, F_1^1 in (15) in terms of F_0^0. Thus the term:

$$C(1,0)\;\frac{\partial F_1^0}{\partial x_3}$$

in (15), with the help of (17), becomes:

$$\frac{C^2(1,0)}{(-\alpha+\sigma_1)}\;\frac{\partial^2 F_0^0}{\partial x_3^2} = \frac{1}{3(-\alpha+\sigma_1)}\;\frac{\partial^2 F_0^0}{\partial x_3^2} \qquad (19)$$

Further the term:

$$-\frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) \quad B(1,1)\ F_1^{-1}$$

in (15), with the help of (16), becomes:

$$\frac{1}{4}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)\frac{B^2(1,1)}{(-\alpha+\sigma_1)}\,F_0^0 = \frac{1}{6}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\frac{1}{(-\alpha+\sigma_1)}\quad F_0^0$$

In a similar way the term:

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) \quad B(1,1)\ F_1^{1}$$

in (15), with the help of (18), becomes:

$$\frac{1}{6}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\frac{1}{(-\alpha+\sigma_1)}\quad F_0^0$$

Combining these terms in (15), the result is:

$$\frac{1}{v}\,\frac{\partial F_0^0}{\partial t} + \frac{1}{3(-\alpha+\sigma_1)}\left[\frac{\partial^2 F_0^0}{\partial x_1^2} + \frac{\partial^2 F_0^0}{\partial x_2^2} + \frac{\partial F_0^0}{\partial x_3^2}\right] = (-\alpha+\sigma_0)\ F_0^0 + F_{n,0}^0 \quad . \tag{20}$$

We are now ready to pair off the terms in (20) with their correspondents in (7). Multiplying each side of (20) by $(4\pi)^{1/2}$ and using (8), we can replace each occurrence of "F_0^0" in (20) by "h". Next, by (15) of Sec. 6.3, we have:

$$\sigma_0(x;t) = 2\pi\int_{-1}^{1} \sigma(x;\mu;t)\ P_0(\mu)\ d\mu$$

$$= \int_{\Xi} \sigma(x;\xi';\xi;t)\ d\Omega(\xi)$$

$$= s(x,t) \quad .$$

In other words, σ_0 in (20) is the volume total scattering coefficient. Hence:

$$-\alpha+\sigma_0 = -\ a$$

by virtue of (2). Finally, from (29) of Sec. 6.3 and the definition of h_η in (4), we have:

$$F^0_{\eta,0} = h_\eta \quad .$$

In view of these observations, we may say that the structure of equation (20) is identical with that of (7). Therefore the diffusion coefficient D in (7) is represented by the relation:

$$\boxed{D = \frac{1}{3(\alpha-\sigma_1)}} \tag{21}$$

where α is the volume attenuation coefficient and σ_1 is defined as in (15) of Sec. 6.3 (setting $j = 1$). This representation of D rests on the basis of the spherical harmonic decomposition of the equation of transfer *subject to the special principles (i) and (ii) stated above which fix the level of approximation of the spherical harmonic decomposition.* In sum, then, the left side of (21) arises when we approach diffusion theory via Fick's law; the right side arises when we approach diffusion theory via the spherical harmonic method. At the point where the twain shall meet, we generate (21).

There are several alternate but equivalent forms of (21) arising in practice. For example, if we write

$$\text{"}\bar{\mu}(x,t)\text{"} \quad \text{for} \quad \frac{2\pi}{s(x,t)} \int_{-1}^{1} \sigma(x;\mu;t)\ \mu d\mu \tag{22}$$

Then, by (15) of Sec. 6.3, we have:

$$\sigma_1(x;t) = \bar{\mu}(x,t)\ s(x,t) \tag{23}$$

Thus we see that $\bar{\mu}(x,t)$ is a mean value of the cosine $\mu = \cos\theta = \xi \cdot \xi'$ of the scattering angle θ. Another way of writing (22) to see this more clearly is to note that, when isotropy holds:

$$2\pi \int_{-1}^{1} \sigma(x;\mu;t)\ \mu d\mu = \int_{\Xi} \sigma(x;\xi';\xi;t)\ \xi'\cdot\xi\ d\Omega(\xi) \tag{24}$$

Hence (22) becomes:

$$\bar{\mu}(x,t) = \frac{\int_{\Xi} \sigma(x;\xi';\xi;t)\ \xi'\cdot\xi\, d\Omega(\xi)}{\int_{\Xi} \sigma(x;\xi';\xi;t)\ d\Omega(\xi)} \tag{25}$$

and from this the mean value property of $\bar{\mu}(x,t)$ is quite clear; and by a mean value theorem of integral calculus,

$$-1 \leq \bar{\mu}(x,t) \leq 1 \tag{26}$$

For optical media with large forward scattering values for σ, the values of $\bar{\mu}$ are near 1. For media with uniform scattering, i.e., σ independent of ξ' and ξ, the value of $\bar{\mu}$ is 0. For media with predominant backward scattering values, $\bar{\mu}$ has negative values. Thus, in this sense, $\bar{\mu}$ is a measure of the relative amount of the forward or backward scattering occurring in a beam of flux within the medium. Returning now to (21) we use (23) to obtain:

$$\boxed{\begin{aligned} D &= \frac{1}{3(\alpha-\bar{\mu}s)} \\ &= \frac{1}{3\alpha(1-\bar{\mu}\rho)} \\ &= \frac{L_\alpha}{3(1-\bar{\mu}\rho)} \end{aligned}} \tag{27}$$

where ρ is the scattering-attenuation ratio and where "L_α" denotes the *attenuation length* for the medium; that is, we have written "L_α" for $1/\alpha$. Hence the diffusion coefficient has the dimensions of length and in particular is equal to the attenuation length of the medium divided by the factor $3(1-\bar{\mu}\rho)$.

Radiance Distribution in Diffusion Theory

We conclude the discussion of the present approach by deriving the characteristic form of the radiance distribution $N(x,\cdot,t)$ at a point x about which exists a diffusion process with the properties (i) and (ii). Thus, the radiance $N(x,\xi,t)$ at x at time t in the direction ξ is of the general form:

$$\begin{aligned} N(x,\xi,t) = {} & F_0^0(x,t)\ \Phi_0^0(\xi) + F_1^{-1}(x,t)\ \Phi_1^{-1}(\xi) \\ & + F_1^0(x,t)\ \Phi_1^0(\xi) + F_1^1(x,t)\ \Phi_1^1(\xi) \end{aligned} \tag{28}$$

This form follows by using the present diffusion properties (i) and (ii) in (24) of Sec. 6.3. By evaluating each of the eight factors in the four terms of (28), and simplifying, we obtain:

$$\boxed{N(x,\xi,t) = \frac{1}{4\pi}\,[h(x,t) + 3\xi\cdot\mathbf{H}(x,t)]} \tag{29}$$

Equation (29) displays the relatively mild structure of the radiance distribution associated with a classical diffusion process in an arbitrary optical medium. The greatest radiance occurs in the direction of $\mathbf{H}(x,t)$. In directions ξ perpendicular to $\mathbf{H}(x,t)$ the radiance is simply $h(x,t)/4\pi$. Observe that the overall graphical structure of $N(x,\cdot,t)$ at a point is simply that of a cardioid of revolution with axis along the direction of $\mathbf{H}(x,t)$. Using (5) we may cast (29) into radiometric terms involving $h(x)$ only:

$$\boxed{N(x,\xi,t) = \frac{1}{4\pi}\,[h(x,t) - 3D\,\xi\cdot\nabla h(x,t)]} \tag{30}$$

As a representative indication of the details of the derivation of (29) from (28), observe that by (8):

$$F_0^0(x,t) = (4\pi)^{-1/2}\,h(x,t)$$

and that:

$$\Phi_0^0(\xi) = A_0^0 P_0^0(\mu)\,e^{i0\phi} = (4\pi)^{-1/2}$$

Hence:

$$F_0^0(x,t)\;\Phi_0^0(\xi) = h(x,t)/4\pi \quad . \tag{31}$$

Furthermore, by (16):

$$F^{-1}(x,t) = -\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)\cdot\left(\frac{2}{3}\right)^{1/2}\cdot F_0^0(x)\cdot\frac{1}{(-\alpha+\sigma_1)}$$

$$= \frac{1}{2}\left(\frac{6}{4\pi}\right)^{1/2} D\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) h(x,t)$$

Also: $$\Phi_1^{-1}(\xi) = A_1^{-1}\,P_1^{-1}(\mu)\,e^{-i\phi} \quad .$$

Hence:

$$\left.\begin{aligned}
&F_1^{-1}(x,t)\;\Phi_1^{-1}(\xi) = -\frac{3}{2}\cdot\frac{1}{4\pi}\cdot D\cdot\sin\theta\,(\cos\phi - i\sin\phi)\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) h(x,t)\\
&\text{In a similar way it can be found that:}\\
&F_1^{1}(x,t)\;\Phi_1^{1}(\xi) = -\frac{3}{2}\cdot\frac{1}{4\pi}\cdot D\cdot\sin\theta\,(\cos\phi + i\sin\phi)\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) h(x,t)
\end{aligned}\right\}$$

$$F_1^0(x,t)\ \Phi_1^0(\xi) = -\ 3 \cdot \frac{1}{4\pi} \cdot D \cdot \cos\phi\ \frac{\partial h(x,t)}{\partial z} \qquad (33)$$

Note that the two expressions in (32) are complex conjugates; so that, upon addition, the imaginary terms cancel. On adding together (31) to (33), equation (30) is obtained. Then using (5), equation (29) is obtained.

Equation (29) constitutes an effective means of verifying empirically whether a given light field satisfies the conditions (i) and (ii) for a diffusion approximation. All three radiometric concepts, N, h, and **H** in (29) are readily measurable in practice. Hence if an empirical radiance distribution comes to within an accepted interval of approximation of a cardioid of revolution, then the classical diffusion equation may be used to describe such a light field. We note a rather interesting near-confirmation of the steady state form of (29) in the case of heavily overcast skies. Empirical measurements reported in [186] show that the radiance of the underside of a heavy cloud overcast has essentially the form of (29), i.e., the cardioidal form.

Approaches via Higher Order Approximations

We pause in our description of the three main approaches to diffusion theory to place the discussion of the preceding paragraphs into perspective. We wish to show in particular how the classical diffusion equation (20) (or its equivalent form (7)) takes its place somewhere near the bottom of an infinitely high ladder of successively more detailed diffusion-type equations, each obtainable by following well-defined principles of modification, such as (i) and (ii) above, of the basic system (27) of Sec. 6.3.

In order to facilitate the classification of the various approaches possible via the system (27) of Sec. 6.3, let us write:

$$\text{"}\mathbf{F}_a\text{"} \quad \text{for} \quad (F_a^{-a},\ F_a^{-a+1},\ \ldots,\ F_a^{-1},\ F_a^0,\ F_a^1,\ \ldots,\ F_a^a)$$

Thus, e.g., "$\mathbf{F}_0$" denotes (F_0^0), "F_1" denotes (F_1^{-1}, F_1^0, F_1^1), and so on. In other words F_a is a (2a+1) component vector centered on the component F_0^0. When we say F_a is zero, we mean that each of its 2a+1 components is zero. Further, when we write "$\partial\mathbf{F}_a/\partial t$" we shall mean $(\partial F_a^{-a}/\partial t, \ldots, \partial F_a^a/\partial t)$. In a similar way we can define $\mathbf{F}_{\eta,a}$.

Now the two principles (i) and (ii) used above to arrive at the classical diffusion equation (20) (or its equivalent (7)) may be recast into the following equivalent forms:

(i) (if $a > 1$, then $\mathbf{F}_a = 0$) and (if $a > 0$, then $\mathbf{F}_{\eta,a} = 0$).

(ii) if $a > 0$, then $\partial\mathbf{F}_a/\partial t = 0$.

This relatively succinct way of describing the modification of the system (22) of Sec. 6.3 may form the basis of classifying various diffusion processes. Thus in the following list, let the vectors $\mathbf{F}_a$, $\mathbf{F}_{\eta,a}$ and their derivatives appearing there be the only vectors not set equal to zero in the indicated approximation derived from (27) of Sec. 6.3. The symbol in the "process type" column to the left of the nonzero vectors is a succinct way of denoting the numerical classification of the approximation; some suggestive names for the approximations are given to the right of the vectors. Thus the approximation [1/0] is that giving rise to the classical scalar diffusion equation derived earlier by setting to zero all terms in (27) of Sec. 6.3 except those of $\mathbf{F}_0, \partial\mathbf{F}_1/\partial t$, $\mathbf{F}_1$, $\mathbf{F}_{\eta,0}$.

TABLE 1

A short list of diffusion processes

Process type	Nonzero terms in (27) of Sec. 6.3	Name of associated diffusion process
[0/1]	$\mathbf{F}_0$; $\mathbf{F}_{\eta,0}$	Equilibrium
[0/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_{\eta,0}$	Monotonic
[1/0]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$; $\mathbf{F}_{\eta,0}$	Scalar
[1/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$; $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_{\eta,1}$	Wave
[2/0]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$, $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_2$; $\mathbf{F}_{\eta,1}$	Tensor
[2/t]	$\mathbf{F}_0$, $\partial\mathbf{F}_0/\partial t$; $\mathbf{F}_1$, $\partial\mathbf{F}_1/\partial t$; $\mathbf{F}_2$, $\partial\mathbf{F}_2/\partial t$; $\mathbf{F}_{\eta,2}$	Wave-tensor

The present classification of diffusion processes places two theories below the scalar diffusion theory ("below" in the sense of "less complex"). The first of these, the equilibrium diffusion theory, merely serves to describe the radiometric state of affairs in an equilibrium situation by means of the equation:

$$-\alpha F_0^0 + \sigma_0 F_0^0 + F_{\eta,0}^0 = 0$$

which may be written:

$$h(x,t) = \frac{h_\eta(x,t)}{a} \qquad (34)$$

Thus (34) holds for a uniform, steady light field in equilibrium with its emission sources distributed throughout a medium X. The term h_η/a is reminiscent of Kirchhoff's law in radiometry, or of the equilibrium radiance N_q (see (2) of Sec. 4.3). A slightly more detailed description is given by the monotonic diffusion equation:

$$\frac{1}{v}\frac{\partial h}{\partial t} = -ah + h_\eta \quad . \tag{35}$$

Thus the diffusion process [0/t] described in (35) gives rise to a light field whose scalar irradiance h at a point generally grows or decays monotonically with time. The scalar diffusion process [1/0] was discussed in detail above.

We next encounter the processes [1/t], which is one step more accurate and complex than the classical diffusion process [1/0]. This new process is called the *wave diffusion process* by virtue of the fact that its associated equation (derived from (27) of Sec. 6.3 in the general manner illustrated for the case of [1/0]) is a wave equation of the form

$$\boxed{A\frac{\partial^2 h}{\partial t^2} + B\frac{\partial h}{\partial t} - D\,\nabla^2 h = -ah + h_\eta} \tag{36}$$

where we have written:

$$\text{"A" for } 3D/v^2 \quad , \quad \text{"B" for } (1 + 3Da)/v \qquad (37),(38)$$

Comparing (36) with (7), we see that the process [1/t] adds the next higher derivative term to the equation for the process [1/0], plus slightly modifying the coefficients of the derivatives of the latter's equation. The physical processes corresponding to (36) and to (7) differ markedly: (36) describes a general damped wave-like process which propagates outward from any epicenter at the finite speed $v/\sqrt{3}$. Indeed, (36) is the well-known *telegrapher's equation*, which describes in another context the propagation of wave signals through a resistive wave-conducting medium. Equation (7), on the other hand, is the classical diffusion equation which describes a general monotonic decaying (or growing) diffusion process (with absorption and emission of the diffusing entities) propagating with infinite speed from a given epicenter. Equation (7) may be essentially obtained from (36) by letting v become so large that the second-derivative term in (36) becomes negligible, i.e., so that A is small compared to B.

The next higher diffusion process beyond wave diffusion is the process [2/0]. A new entity enters the picture here with $\mathbf{F}_2$. Whereas $\mathbf{F}_1$ describes the vectorial properties of the radiant flux (see the description of the vector irradiance **H** in terms of the components of $\mathbf{F}_1$, in (14)), $\mathbf{F}_2$ describes the tensorial properties of the radiant flux, properties very much like those described by the stress tensor in fluid dynamics.

Our present goal has essentially been reached; we have shown the place of the classical diffusion theory in the hierarchy of diffusion theories possible in radiative transfer theory. It is seen that the classical diffusion equation (7) is neither the beginning nor the end of the possibilities of describing diffusive transport of photons in an optical medium. However, equation (7) is on the borderline between those theories which, on the one hand, are too crude to admit useful descriptions, and those which, on the other hand, are more accurate in their descriptive powers, but which are relatively complex and intractable in the light of current mathematical techniques. It is because of this convenient middling ground straddled by the diffusion equation (7) that it has been so popular with researchers looking for easily handled, reasonably accurate quantitative accounts of natural light fields. Some of the simple models arising from (7) will be considered in Sec. 6.6.

The Approach via Isotropic Scattering

The third and final main approach to diffusion theory we shall consider in this section is that via the assumption of the isotropic scattering property for an optical medium. The nature of this assumption is quite different from those used in the preceding two approaches. The earlier approaches, via Fick's law and via the spherical harmonic method, were gotten under way by first tampering with the directional structure of the light field, i.e., by reducing its awesome directional complexity to some relatively innocuous, mildly varying form (see, e.g., (29)) so that, for example, either Fick's law or the [1/0] process defined in Table 1 above could cope with the resultant weakened field. The nature of the assumption we shall adopt the present discussion is such that it leaves inviolate the intricate geometric structure of the radiance field; but in order to inculcate a semblance of manageability into the field, it is to be hypothesized that the volume scattering function σ is independent of ξ' and ξ throughout the medium. The resultant light field belonging to such a σ is a relatively tame analytic object by natural light field standards--so tame, in fact, that some quite elegant mathematical analyses of the classical mold can be employed to carry to completion the exact solution of the resulting equations for scalar irradiance. The associated theory is called *exact diffusion theory*. The "exactness" of the theory resides in its mathematical procedures, and not necessarily in its fidelity as a physical theory.

The manner in which we shall approach exact diffusion theory will be such as to show the necessity of the isotropic scattering assumption in the construction of the theory. By holding back the invocation of the isotropic scattering assumption until the last stage of the main analysis, it shall become quite clear that this is the essential physical concession made by an otherwise elegant, powerful theory which in principle is applicable to arbitrary (finite or infinite) inhomogeneous media with both internal and external sources.

To begin, let the optical medium X be of arbitrary spatial extent (in Fig. 6.3 it is shown as being finite), generally inhomogeneous, with arbitrary volume scattering function σ and volume scattering attenuation function α, and with arbitrary emission function N_η defined throughout X, and boundary radiance distribution N_0. For simplicity of exposition,

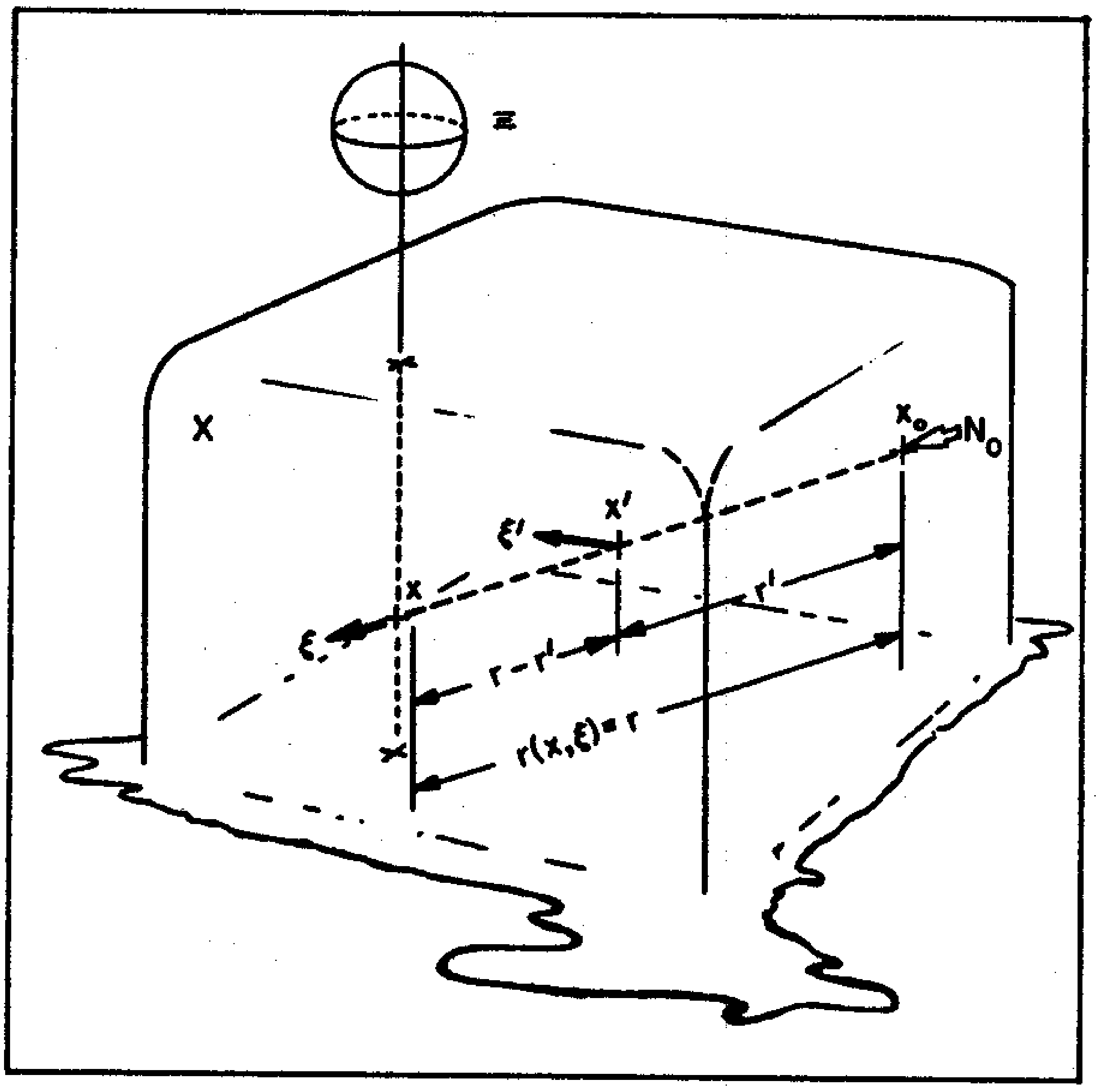

FIG. 6.3 Setting up the exact diffusion theory.

we postulate a steady-state radiance field N through X x Ξ. The corresponding formulation for the time-dependent field is obtained by simple modifications of the steady-state case. (See, e.g., (12) of 7.14.) The present discussion will be facilitated if at the outset we define certain integral operators. First, there is the path function operator **R** of Sec. 3.17:

$$\mathbf{R} = \int_{\Xi} [\,]\ \sigma(x;\xi';\xi)\ d\Omega(\xi') \quad .$$

The path radiance operator **T** of Sec. 3.17 will also be needed:

$$\mathbf{T} = \int_0^{r(x,\xi)} [\ ]\ T_{r-r'}(x'.\xi)\ dr' \quad .$$

The variables occurring in these operators are depicted in Fig. 6.3. Further, we shall write:

$$\text{"}\mathbf{U}\text{"} \quad \text{for} \quad \int_{\Xi} [\ ]\ d\Omega(\xi) \qquad (39)$$

This operator maps radiance distributions $N(x,\cdot)$ at a point x into their associated scalar irradiances h(x), thus:*

$$h(x) = N\mathbf{U}(x) = \int_{\Xi} N(x,\xi)\; d\Omega(\xi) \tag{40}$$

or simply:

$$h = N\mathbf{U} = vu$$

for short, where vu is an alternate form of h (Sec. 2.7) involving radiant density u, and the speed of light, v. We shall also need the following two compositions of operators. First, the scattering operator $\mathbf{S}^1$ of Sec. 5.1:

$$\mathbf{S}^1 = \mathbf{RT}$$

and the composition **V**, where we have written:

$$\text{"}\mathbf{V}\text{"} \quad \text{for} \quad \mathbf{TU} \tag{41}$$

The reader may verify directly from its definition that **V** has the representation:

$$\mathbf{V} = \int_X [\;]\, K_\alpha(x',\cdot)\; dV(x') \tag{42}$$

which is the iteration of the integral operators **T** and **U**, where for every x' and x in the medium we have written:

$$\text{"}K_\alpha(x',x)\text{"} \quad \text{for} \quad \frac{T_{r-r'}(x',\xi)}{|r-r'|^2} \tag{43}$$

and where $\xi = (x-x')/|r-r'|$; $|r-r'|$ is the distance $|x-x'|$ from point x' to point x as measured along the path of direction ξ. (As usual, "x" denotes a point of E_3, and as such is an ordered triple of real numbers.) The integration in **V** is with respect to the volume measure V. Thus $dV(x) = r^2 dr d\Omega(\xi)$, where $x = x_0 + r\,\xi$.

With all this machinery securely in place, we can go on to obtain the requisite equations so as to keep easily in view at all times the essential physical and mathematical features of the derivation.

The integral form of the equation of transfer ((2) of Sec. 3.15) with emission function N_η is:

---

*The notation "$N\mathbf{U}(x)$" denotes the value at x of the function $N\mathbf{U}$, and $N\mathbf{U}$ in turn is the result of operating on the function N with the operator **U**.

$$N(x,\xi) = (N_o + N_\eta)\,\mathbf{T}(x,\xi) + N\mathbf{S}^1(x,\xi) \qquad (44)$$

where* N_0 is the initial radiance function within the medium due to boundary radiances, i.e., where we have written:

$$\text{"}N_o(x,\xi)\text{"} \quad \text{for} \quad N_o(x_o,\xi)\,\delta(x-x_o)$$

and where $N_0(x_0,\cdot)$ is the given incident radiance distribution at an arbitrary point x_0 of X. By writing:

$$\text{"}N^o_\eta(x,\xi)\text{"} \quad \text{for} \quad (N_o + N_\eta)\,\mathbf{T}(x,\xi)$$

(44) becomes:

$$N(x,\xi) = N^o_\eta(x,\xi) + N\mathbf{S}^1(x,\xi)$$

Applying $\mathbf{U}$ to each side, we have

$$N\mathbf{U}(x) = N^o_\eta\mathbf{U}(x) + N\mathbf{S}^1\mathbf{U}(x)$$

whence:

$$h(x) = h^o_\eta(x) + (N\mathbf{R})\mathbf{T}\mathbf{U}(x)$$

$$= h^o_\eta(x) + N_*\mathbf{T}\mathbf{U}(x)$$

Hence

$$\boxed{h(x) = h^o_\eta(x) + N_*\mathbf{V}(x)} \qquad (45)$$

where we have written:

$$\text{"}h^o_\eta(x)\text{"} \quad \text{for} \quad N^o_\eta\mathbf{U}(x) \qquad (46)$$

Equation (45) is but one step away from being an integral equation for scalar irradiance h. On first sight it might appear promising to use the operator $\mathbf{U}$ on N_* to obtain the product of the volume total scattering function s(x) and scalar irradiance as follows:

$$N_*\mathbf{U}(x) = s(x)\,h(x)$$

Toward this end, the N_* term in (45) may have the identity operator I in the form of $\mathbf{U}\mathbf{U}^{-1}$ slipped between N_* and $\mathbf{V}$, thus:

---

*The notation: "$(N_o + N_\eta)\mathbf{T}(x,\xi)$" denotes the value at (x,ξ) of the function $(N_o + N_\eta)\mathbf{T}$.

$$N_*UU^{-1}V(x) = sh(U^{-1}V)(x)$$

so that (45) could be written:

$$h(x) = h^o_\eta(x) + sh(U^{-1}V)(x)$$

which is an operator equation in the unknown h. Unfortunately the inverse U^{-1} to the operator U does not generally exist, for the reason that there are many distinct radiance distributions at a point x giving rise to the same scalar irradiance h(x). This shows the necessity for assuming isotropic scattering for the medium if we are to obtain an integral equation for h. For then we have:

$$N_*(x,\xi) = NR(x,\xi) = \frac{s(x)}{4\pi} h(x) \qquad (47)$$

where we have assumed that:

$$s(x;\xi';\xi) = s(x)/4\pi \qquad (48)$$

Using $N_*(x,\xi)$ in (45) as given by (47) we have:

$$\boxed{h(x) = h^o_\eta(x) + \frac{1}{4\pi}(hs)\, V(x)} \qquad (49)$$

This is the requisite general form of the basic equation of exact diffusion theory.

The natural solution of (49) is obtained by rearranging it as follows:

$$\begin{aligned} h^o_\eta(x) &= h(x) - \frac{1}{4\pi}(hs)\, V(x) \\ &= h[I - V_*](x) \end{aligned} \qquad (50)$$

where we have written:

$$\text{"}V_*\text{"} \quad \text{for} \quad \frac{1}{4\pi}\int_X [\ ]s(x')\, K_\alpha(x',\cdot)\, dV(x') \qquad (51)$$

It is easily shown that the inverse $[I - V_*]^{-1}$ of $I - V_*$ generally exists, i.e., that V_* has the contraction property (cf. Sec. 5.14). Hence (44) yields:

$$\boxed{h(x) = h^o_\eta[I - V_*]^{-1}(x)} \qquad (52)$$

where generally:

$$[I - V_*]^{-1} = I + V_* + V_*^2 + V_*^3 + \dots \qquad (53)$$

Here V_*^2 is V_*V_*, i.e., the operator V_* followed by V^*. In general V_*^j is the operator V_*^{j-1} followed in application by V_*. This solution procedure is quite general. The operator V_*, which depends on the space X and its optical properties α and s, requires only the contraction property to be verified before it can be used in theory or practice.

An alternate form of (49), the form most often used in the classicial solution procedures, is obtained by rewriting (45) as:

$$\begin{aligned} h(x) &= (N_o + N_\eta)TU(x) + N_*V(x) \\ &= N_oV(x) + (N_\eta + N_*)V(x) \end{aligned}$$

so that:

$$\boxed{h(x) = h^o(x) + (N_\eta + N_*)\, V(x)} \qquad (54)$$

In order to obtain an equation in h only (all other terms being given functions) it follows, for the same reasons as those leading to (49), that the isotropic scattering assumption (48) must be adopted. In addition, if we are to retain the particular grouping of terms exhibited in (54), we may (though it is not strictly necessary to do so) also assume that N_η is of uniform directional structure, i.e., we assume:

$$N_\eta(x,\xi) = h_\eta(x)/4\pi \qquad (55)$$

where h_η is defined in (4). Under these conditions, (54) reduces to:

$$\boxed{h(x) = h^o(x) + \frac{1}{4\pi}(h_\eta + hs)\, V(x)} \qquad (56)$$

If the space X is infinite in all directions about x, and α generally is not zero, then $h^o(x) = 0$, and (56) becomes:

$$\boxed{h(x) = \frac{1}{4\pi}(h_\eta + hs)\, V(x)} \qquad (57)$$

which is the somewhat special but customary form of the integral equation on which the exact diffusion theory is based.

We now sketch the customary method of solution of (57). The medium is assumed homogeneous, so that $s(x)$ is independent of x and so that $K_\alpha(x',x)$ depends only on the difference $|x-x'|$. This assumption of homogeneity is necessary if the Fourier transform method (the usual method used) is to be applied to (57). Thus, if "$\mathcal{F}$" denotes the three-dimensional spatial Fourier transform operator for functions on X (which is now all of euclidean three space) we have, applying $\mathcal{F}$ to each side of (57):

$$(\mathcal{F}h)(k) = \frac{1}{4\pi}\ \mathcal{F}[(h_\eta + hs)\ \mathbf{V}]\ (k)$$

where k is the spatial frequency variable associated with the spatial variable x. The value of $\mathcal{F}[h]$ at k is written as "$\mathcal{F}[h;k]$", "$(\mathcal{F}h)(k)$", or "$\hat{h}(k)$", similarly with the inverse transform. Using the convolution theorem for Fourier transforms, (see, e.g., (6) of Sec. 7.14) this becomes:

$$\hat{h}(k) = \frac{1}{4\pi}\ (\hat{h}_\eta(k) + s\hat{h}(k))\ \hat{K}_\alpha(k) \qquad (58)$$

where for brevity we also write:

$$\text{"}\hat{K}_\alpha(k)\text{"} \quad \text{for} \quad \mathcal{F}[K_\alpha;k]$$

The carat over the letter "h" denotes, e.g., that h is the Fourier transform of h. The beauty and power of the Fourier transform method is now strikingly evident in (58): the integral operator equation (57) has been reduced to an algebraic equation in $\hat{h}(k)$ so that (58) may be directly solved for $\hat{h}(k)$:

$$\hat{h}(k) = \frac{\hat{h}_\eta(k)}{(4\pi - s\hat{K}_\alpha(k))}$$

Taking the inverse Fourier transform of each side, we have:

$$\boxed{h(x) = \mathcal{F}^{-1}\left[\frac{\hat{h}_\eta}{(4\pi - s\hat{K}_\alpha)}\right](x)} \qquad (59)$$

which rivals the natural solution (52) in simplicity and elegance (but evidently not in power and scope). The solutions of (57) will be discussed in more detail in Sec. 6.7.

The present discussion is concluded with the observation of how the radiance distribution $N(x,\cdot)$ is obtained from knowledge of scalar irradiance $h(x)$ when using exact diffusion theory. Once the scalar irradiance field h has been obtained from either (52) or (59), we use the representation of N_*, as given by (47), in the general relation (44):

$$N(x,\xi) = (N_0 + N_\eta)\ \mathbf{T}(x,\xi) + N_*\mathbf{T}(x,\xi)$$

Thus:

Thus:

$$\boxed{N(x,\xi) = \left[N_o + N_\eta + \frac{hs}{4\pi}\right] T(x,\xi)} \tag{60}$$

If the medium is source-free, so that $N_\eta = 0$, then

$$N(x,\xi) = \left[N_o + \frac{hs}{4\pi}\right] T(x,\xi) \tag{61}$$

If the medium is in addition infinite, so that $N_o = 0$ at all interior points of X then

$$N(x,\xi) = \left[\frac{hs}{4\pi}\right] T(x,\xi) \tag{62}$$

If the medium is also homogeneous, then

$$N(x,\xi) = (s/4\pi)\ [hT(x,\xi)] \quad . \tag{63}$$

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6.6 Solutions of the Classical Diffusion Equations

In this and the following section we shall exhibit some of the more useful general solutions of the classical and exact diffusion equations introduced in the preceding section. We begin with the classical diffusion equation in its simplest context.

Plane-Parallel Case

Consider an homogeneous plane-parallel source-free optical medium with a steady, stratified light field generated by incident flux at its upper boundary. For example, natural light fields in the seas, lakes, and harbors can supply such instances. Further instances may be found in heavy fogbanks and thick cloud layers. Suppose that the conditions for the diffusion equations hold in such media. What are the resultant forms of the light field--say the radiance distribution and associated scalar irradiance function--that the classical diffusion theory predicts for such media? We now seek the answers to these questions.

Starting with equation (7) of Sec. 6.5, and imposing the source-free, steady light field condition, we have:

$$D\,\nabla^2 h - ah = 0 \tag{1}$$

Recall that in a three-dimensional Cartesian coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$