Thus:

$$N(x,\xi) = \left[N_o + N_\eta + \frac{hs}{4\pi}\right] T(x,\xi) \tag{60}$$

If the medium is source-free, so that $N_\eta = 0$, then

$$N(x,\xi) = \left[N_o + \frac{hs}{4\pi}\right] T(x,\xi) \tag{61}$$

If the medium is in addition infinite, so that $N_o = 0$ at all interior points of X then

$$N(x,\xi) = \left[\frac{hs}{4\pi}\right] T(x,\xi) \tag{62}$$

If the medium is also homogeneous, then

$$N(x,\xi) = (s/4\pi)\ [hT(x,\xi)] \quad . \tag{63}$$

6.6 Solutions of the Classical Diffusion Equations

In this and the following section we shall exhibit some of the more useful general solutions of the classical and exact diffusion equations introduced in the preceding section. We begin with the classical diffusion equation in its simplest context.

Plane-Parallel Case

Consider an homogeneous plane-parallel source-free optical medium with a steady, stratified light field generated by incident flux at its upper boundary. For example, natural light fields in the seas, lakes, and harbors can supply such instances. Further instances may be found in heavy fogbanks and thick cloud layers. Suppose that the conditions for the diffusion equations hold in such media. What are the resultant forms of the light field--say the radiance distribution and associated scalar irradiance function--that the classical diffusion theory predicts for such media? We now seek the answers to these questions.

Starting with equation (7) of Sec. 6.5, and imposing the source-free, steady light field condition, we have:

$$D\,\nabla^2 h - ah = 0 \tag{1}$$

Recall that in a three-dimensional Cartesian coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$

Since the light field is stratified, the x and y derivatives in $\nabla^2 h$ will be zero. Thus (1) reduces to:

$$D \frac{d^2h}{dz^2} - ah = 0 \quad . \qquad (2)$$

Therefore, in its simplest guise, the classical diffusion equation (7) of Sec. 6.5 takes the form of a linear, second-order differential equation whose general solution for $a \neq 0$ is of the form:

$$h(z) = c_+ \, e^{\kappa z} + c_- \, e^{-\kappa z} \qquad (3)$$

where we have written:

$$\text{"}\kappa\text{"} \quad \text{for} \quad \sqrt{\frac{a}{D}} \qquad (4)$$

We call κ, as defined in (4), the (classical) *diffusion coefficient*. Recalling (27) of Sec. 6.5, we can express κ alternatively as:

$$\kappa = \sqrt{3a(\alpha - \bar{\mu}s)}$$

$$= \sqrt{3a(a + (1-\bar{\mu})s)}$$

The diffusion coefficient κ is the physical core of the solution (3) and, indeed, of all of the solutions of the classical diffusion equation. There may be variations in the geometry of a medium--spherically symmetric, cylindrically symmetric, plane parallel, as in the present case--and corresponding variations in the forms of solutions, as we shall see, but running through these cases, and common to them all, is the notion of the diffusion coefficient κ. Observe how κ depends jointly on the volume absorption coefficient a, the total volume scattering coefficient s, and on the mean cosine $\bar{\mu}$, which is a measure of the anisotropic scattering property of the medium.

As a special solution of (3) let the plane-parallel medium be infinitely deep, so that on physical grounds $c_+ = 0$ in (3) (see (12)). Then (3) can be shown to reduce to:

$$\boxed{h(z) = h(0)e^{-\kappa z}} \qquad (6)$$

This is at once the most useful and representative example of the analytic form of light fields in natural optical media. The models for light fields in natural optical media come in all orders of complexity and power of representation, but in the final analysis all exhibit, in greater or lesser degree, and with an accuracy that generally increases with increasing depth, the overall *exponential* structure of natural light fields. The simplest of models of light fields in natural media--namely (2)--already exhibits this exponential structure of the light fields. More sophisticated models will give correspondingly more detail on the structure of h(z) as a function of z; and still other models may sharpen the dependence of κ on a and s. Yet for all its simplicity, (2) has captured the salient analytic property of the light in natural hydrosols: that of exponentiality.

How does the magnitude of the diffusion coefficient compare with that of the volume attenuation coefficient α? We note first of all that these quantities are indeed comparable, both having dimensions of inverse length. From the representation (5) of κ we can build up the following chain of inequalities leading to α:

$$\kappa = \sqrt{3a(a + (1-\bar{\mu})s)} \le \sqrt{3a(a+s)} \le \sqrt{3(a+s)(a+s)} = \sqrt{3}\,\alpha \qquad (7)$$

A more instructive inequality can be deduced provided that some explicit relation between s and a is hypothesized. Such a relation has already been observed in connection with the validity of diffusion theory. In the remarks following Fick's law (5) of Sec. 6.5, it was noted that the law holds when, among other things, the scattering-attenuation ratio ρ is at least 0.6. This condition on ρ in turn requires that: $s \ge (10/4)\,a > 2a$. It therefore seems reasonable to be able to use this inequality between s and a whenever diffusion theory itself is being used. Therefore, starting the chain of inequalities in (7) once again, we are now led to:

$$\kappa = \sqrt{3a(a + (1-\bar{\mu})s)} \le \sqrt{3a(a+s)} < \sqrt{(a+s)(a+s)} = \alpha \quad .$$

Hence we see that, whenever diffusion theory is applicable, we must have:

$$\boxed{\kappa < \alpha} \qquad (9)$$

The physical interpretation of (9) is clear: since κ is generally smaller than α, we have, depth-for-depth:

$$e^{-\alpha z} < e^{-\kappa z}$$

This means that transmitted radiant flux undergoing diffusion along a path of length z is greater at the end of the path than that having undergone pure attenuation. This may be seen also by direct appeal to the intuitive meaning of diffusion and attenuation in their technical senses used in transport theory: a stream of photons undergoing attenuation, loses photons under the joint action of absorption and scattering. Once a set of photons is scattered out of the beam, they are no longer considered part of the beam even though some of them may reenter the beam. A stream of photons undergoing diffusion, on the other hand, may scatter out of and back into the beam and be recounted upon rejoining the main stream. Thus the main loss mechanism for diffusion is absorption. Therefore, length for length, a packet of diffusing photons will have fewer loss casualties than a packet of attenuating (beam transmitted) photons. This relation between κ and α may be alternatively stated by means of the attenuation length L_κ, where we have written:

$$"L_\kappa" \quad \text{for} \quad 1/\kappa \tag{10}$$

Then an equivalent statement to (9) is:

$$L_\alpha < L_\kappa \tag{11}$$

This inequality may be interpreted in a dual fashion to (9) as follows: The length of path in a medium over which a packet of photons undergoes a fixed fraction r of loss by means of diffusion is greater than the length of travel over which the packet undergoes the same fraction of loss by means of attenuation. In other words, a packet of diffusing photons will travel farther before incurring a given loss than it would travel before it incurred the same loss by pure attenuation.

If the plane-parallel medium is of finite depth d, then in general both c_+ and c_- in (3) are not zero. In fact c_+ and c_- are determined, for example, by specifying the scalar irradiances at any two depths in the medium. It is customary and convenient to specify h(z) for z = 0 and z = d. Thus, supposing h(0) and h(d) given, we have from (3):

$$h(0) = c_+ + c_-$$

$$h(d) = c_+ \, e^{\kappa d} + c_- \, e^{-\kappa d}$$

We treat these two equations as linear algebraic equations in the unknowns c_+ and c_-, and find that:

$$c_\pm = \pm \frac{h(d) - h(0) \, e^{\mp\kappa d}}{e^{\kappa d} - e^{-\kappa d}} \tag{12}$$

We observe from these representations of c_+ and c_- that, for very deep media, $c_+ \cong 0$ and $c_- \cong h(0)$, so that in the limit of infinitely deep media, we return to the solution (6).

We consider next the specific form of the radiance distribution in the plane-parallel diffusion case. By (30) of Sec. 6.5 we know the general shape of the radiance distributions. But with a specific depth dependence of h(z) now known, say in the case of (6) for an infinitely deep medium, the gradient of h(z) is readily estimable, and so a specific estimate of $N(z,\xi)$ is possible. Since the light-field is stratified, we have

$$\nabla h = - \mathbf{k} \frac{dh(z)}{dz} \tag{13}$$

where **k** is the unit outward normal to the medium at its upper boundary. The medium has the standard terrestrially based coordinate system for hydrologic optics (Sec. 2.4). Hence for infinitely deep media:

$$N(z,\xi) = \frac{h(z)}{4\pi}\,[1 - 3\,\kappa\,D\,\xi \cdot \mathbf{k}] \tag{14}$$

where h(z) is given in (6). A similar formula for N(z,ξ) can be developed for finitely deep media using (3) with c_+ and c_- as given in (12).

Finally, we consider the upward and downward irradiances associated with the diffusion field in an infinitely deep optical medium. Using the ideas of Sec. 2.4 in which the properties of irradiance were described at length, let "H(z,+)" and "H(z,-)" denote the upward and downward irradiances in the medium. That is, in the terminology of (9), (10) of Sec. 2.4, we have written:

"H(z,+)" for $H(z,\mathbf{k})$

"H(z,-)" for $H(z,-\mathbf{k})$.

Then:

$$H(z,+) = \int_{\Xi_+} N(z,\xi)\;\xi \cdot \mathbf{k}\;d\Omega(\xi)$$

and:

$$H(z,-) = \int_{\Xi_-} N(z,\xi)\;\xi \cdot (-\mathbf{k})\;d\Omega(\xi)$$

which are based on (8) of Sec. 2.5. H(z,±) can be explicitly evaluated using (14) for N(z,ξ). Thus:

$$H(z,+) = \frac{h(z)}{4\pi} \int_{\Xi_+} (1 - 3\,\kappa\,D\,\xi \cdot \mathbf{k})\;\xi \cdot \mathbf{k}\;d\Omega(\xi)$$

$$= \frac{h(z)}{4}\,(1 - 2\kappa D) \tag{15}$$

In a similar manner:

$$H(z,-) = \frac{h(z)}{4}\,(1 + 2\kappa d) \quad . \tag{16}$$

From this we can estimate the ratio of downward to upward irradiance at each depth z in the medium. Writing:

$$"R(z,-)" \quad \text{for} \quad \frac{H(z,+)}{H(z,-)} \quad , \tag{17}$$

we have:

$$\boxed{R(z,-) = \frac{1 - 2\kappa D}{1 + 2\kappa D}} \tag{18}$$

for the *reflectance* R(z,-) associated with an infinitely deep plane-parallel homogeneous medium as described by the concepts of classical diffusion theory. Observe that R(z,-) in the present case is independent of z.

It is interesting to note that from (15), (16) and the concepts of vector irradiance (Sec. 2.8):

$$|\mathbf{H}(z)| = H(z,-) - H(z,+) = \kappa D h(z) \tag{19}$$

so that:

$$\boxed{\mathbf{H}(z) = -\kappa D h(z)\, \mathbf{k}} \tag{20}$$

Furthermore:

$$H(z,+) + H(z,-) = h(z)/2 \quad . \tag{21}$$

Relations (15) through (21) will be reconsidered in the light of the exact two-flow theory in plane-parallel media, as developed in Chapter 8.

Point Source Case

Consider an infinite homogeneous optical medium with an isotropic point source at the origin generating a steady light field throughout the medium. For example, a bright flare of uniform directional output deep in the ocean far from surface and bottom effects would generate such a light field. Flares deep within foggy atmospheric media such as in fogbanks and clouds also offer real instances of the present case. Errata the plane-parallel case, we are interested in the scalar irradiance field and the radiance field set up by the point as in source in the surrounding medium. In particular, we now study these fields as predicted by classical diffusion theory.

At all points of the medium other than at the position of the point source, equation (7) of Sec. 6.5 governs the resultant scalar irradiance field:

$$D\nabla^2 h - ah = 0 \tag{22}$$

The appropriate coordinate frame at present would be a spherical polar coordinate frame with origin at the point source. For then $\nabla^2 h$ takes a particularly simple form because of the spherical symmetry of the field about the point source. Thus, in general for spherical coordinates in which $x = (r,\theta,\phi)$:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial\phi^2} \tag{23}$$

By spherical symmetry we now have:

$$\nabla^2 h = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dh}{dr}\right)$$

$$= \frac{d^2 h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{1}{r}\frac{d^2(rh)}{dr^2} \tag{24}$$

Hence (22) becomes in the present case:

$$D\,\frac{1}{r}\frac{d^2(rh)}{dr^2} - ah = 0 \tag{25}$$

If we write, *ad hoc:*

$$\text{"F" for } rh \quad , \tag{26}$$

then (25) becomes:

$$D\,\frac{d^2F}{dr^2} - aF = 0 \tag{27}$$

and we are back, mathematically, to the case described by (2). Hence the general solution of (27) is:

$$F(r) = c_+ e^{\kappa r} + c_- e^{-\kappa r} \quad , \tag{28}$$

or, in view of (26):

$$h(r) = \frac{1}{r}\left(c_+ e^{\kappa r} + c_- e^{-\kappa r}\right) \tag{29}$$

In view of the spherical symmetry, the values of $h(x)$ depend only on r, where $x = (r,\theta,\phi)$, and we therefore have written for brevity "h(r)" instead of "h(x)".

For the presently considered setting, namely an infinite medium, we can, for physical reasons, immediately set c_+ to zero. The exact mathematical procedure for this is completely analogous to that used to obtain (12). Therefore the scalar irradiance about a point source generally behaves in the manner described by the following equation:

$$h(r) = \frac{c_- \, e^{-\kappa r}}{r} \tag{30}$$

That is to say, h(r) falls off jointly as the inverse first power of r and exponentially with r. The constant c_- can be evaluated if we use the connection between vector irradiance **H** and scalar irradiance h given in Fick's law (5) of Sec. 6.5:

$$\begin{aligned} |\mathbf{H}(r)| &= |D\nabla h(r)| \\ &= |D \frac{dh(r)}{dr}| \\ &= \frac{Dc_- \, e^{-\kappa r}}{r^2} [1 + \kappa r] \end{aligned} \tag{31}$$

Here we have used the fact that **H**(r) is directed radially outward from the source (again a consequence of spherical symmetry). The magnitude of **H**(r) is the net outward irradiance at each point of a spherical surface of radius r. Hence:

$$4\pi r^2 \, |H(r)| = 4\pi Dc_- \, e^{-\kappa r} \, [1 + \kappa r] \tag{32}$$

is the total net outward radiant flux, call it "P_r", across the spherical surface of radius r. For general radii r we do not know *a priori* the magnitude of this net outward flow. Even if we knew the radiant flux output, say P_o, of the point source at the origin, there is no *a priori* connection between P_o and P_r. However, if one measures P_r for some r, then (27) yields up at once an empirical estimate of c_-. On further examination of (27) it appears possible to devise a theoretical means of finding c_- by considering P_r for very small values of r. In such cases the spherical volume enclosing the point source is so small that the *net* outward flow across the boundary due to the field flux is zero, or very nearly so, for the reason that there is very small chance for a packet of photons diffusing into and then out of the spherical volume to lose any members by absorption during the traversal of the volume (the main loss mechanism which affects diffusing particles). At any rate, it is clear *a priori* that this chance goes to zero in magnitude with the radius of the sphere. Hence in the limit of zero radius the net outward flow across the spherical surface is due solely to the point source's output P_o. Thus from (32) we find:

$$\begin{aligned} P_o = \lim_{r\to o} P_r &= \lim_{r\to o} 4\pi Dc_- \, e^{-\kappa r} \, [1 + \kappa r] \\ &= 4\pi Dc_- \quad , \end{aligned}$$

whence:

$$\boxed{h(r) = \frac{P_o \, e^{-\kappa r}}{4\pi Dr}} \tag{34}$$

Equation (34) describes the scalar irradiance at distance r from a point source of isotropic radiant flux output P_o. The flux is evolving in a diffusing medium with diffusion constant D, and diffusion coefficient κ. Equation (34) may be phrased in terms of the radiant intensity J_o of the point source. Thus, using (17) of Sec. 2.9,

$$\boxed{h(r) = \frac{J_o e^{-\kappa r}}{Dr}} \tag{35}$$

where we have written:

$$"J_o" \quad \text{for} \quad \frac{P_o}{4\pi} \tag{36}$$

The radiance distribution associated with the point source diffusion problem is obtained at each distance r from the point source by means of (30) of Sec. 6.5, now using as gradient:

$$\nabla h(x) = -\mathbf{r}\,\frac{dh(r)}{dr} \quad , \tag{37}$$

where $\mathbf{r}$ is the unit radial vector directed *toward* the point source. The gradient (37) was evaluated in (31), so that with the aid of (34):

$$\nabla h(x) = \frac{P_o e^{-\kappa r}(1+\kappa r)}{4\pi D r^2}\,\mathbf{r} \tag{38}$$

Hence

$$\boxed{\mathbf{H}(r) = -D\nabla h = -\frac{Dh(r)(1+\kappa r)}{r}\,\mathbf{r}} \tag{39}$$

Therefore, by means of (30) of Sec. 6.5 we have:

$$\boxed{N(r,\xi) = \frac{h(r)}{\pi}\left[1 - 3\,\frac{D(1+\kappa r)}{r}\,\xi\cdot\mathbf{r}\right]} \tag{40}$$

where h(r) is given in (34). Equation (40) represents the radiance function in an infinite medium with isotropic point source under the usual conditions for classical diffusion theory (see process [1/0], Table 1, Sec. 6.5). A similar formula can be developed for finite spherical media. However, in this case care must be taken to see that the basic diffusion conditions hold, in particular so that Fick's law (5) of Sec. 6.5 is applicable. Observe that at great distances r from the source, the expression for $N(r,\xi)$ as given in (40) approaches (14) of the plane-parallel case. Thus the radiance distribution at great distances from the point source settles down to become the product of a spatial factor and a directional factor. In other words, the spatial and directional dependences of $N(r,\xi)$ eventually multiplicatively uncouple at great distances from the point source. This fact was used as a motivation for the spherical harmonic method in Sec. 6.1, and will be discussed in Sec. 10.6 as a special case of the general asymptotic radiance theorem (Sec. 10.5).

We conclude the discussion of the point source case by deriving the expressions for the outward and inward irradiances $H(r,\pm)$, where we have written:

$$\text{"}H(r,\pm)\text{"} \quad \text{for} \quad H(r,\pm\mathbf{r}) \tag{41}$$

on the basis of the general irradiance (11) of Sec. 2.6. Thus, in a manner similar to that used to find (15) and (16), we have for the point source context:

$$H(r,\pm) = \frac{h(r)}{4\pi}\left[1 \mp \frac{2D(1+\kappa r)}{r}\right] \tag{42}$$

so that, analogously to (18), we have:

$$R(r,-) = \frac{1 - [2D(1+\kappa r)]/r}{1 + [2D(1+\kappa r)]/r} \tag{43}$$

for the reflectance $R(r,-)$ of the medium at distance r from the point source, where we have written:

$$\text{"}R(r,-)\text{"} \quad \text{for} \quad \frac{H(r,+)}{H(r,-)} \tag{44}$$

Unlike the reflectance $R(z,-)$ obtained in the plane-parallel case, the present reflectance $R(r,-)$ varies with the distance r. In the limit of increasing r, however, $R(r,-)$ approaches the form of $R(z,-)$. Observe also how the values of r cannot be arbitrarily small and still have formulas such as (40) and (43) physically meaningful. The reason for this breakdown of the diffusion theory formulas is traceable to the eventual inapplicability of the original Fick's law hypothesis. In the presence of the highly varying directional structure of radiance distributions that occur near point sources, the simple cardioidal structure of radiance distributions, characteristic of diffusion theory, simply does not hold. It is at this point that the spherical harmonic approach to diffusion theory, on which the cardioidal radiance law is based, shows the inapplicability of Fick's law assumption. See, e.g., (14), (20), and (29) of Sec. 6.5.

Discrete Source Case

We take up once again the setting described in the point source case, just concluded. Now we imagine a set of point sources distributed throughout the infinite homogeneous medium. This set may be finite or infinite. In either case we assume the "points" to be disjoint, small regular-shaped volumes of given minimum size, the centers at points x_j. The definition of point source adopted in the present case is that given in Sec. 2.9. Our present purpose is to derive the equations for the scalar irradiance and vector irradiance fields associated with such sets of point sources. From these representations, the radiance field follows at once using (29) of Sec. 6.5.

Suppose the set of point sources is located at the points x_1, x_2, ..., in the medium and that point x_j has isotropic radiant flux output $P_o(x_j)$. It follows from (34) and the interaction principle (which now assures superimposability of effects) that the total irradiance $h(x)$ generated at x by the point sources at each x_j is given by:

$$h(x) = \sum_{j=1}^{\infty} \frac{P_o(x_j)\, e^{-\kappa|x-x_j|}}{4\pi\, D|x-x_j|} \tag{45}$$

where, as usual, "$|x-x_j|$" denotes the distance between point x and point x_j. In case only a finite number n of point sources are present, we set $P_o(x_j) = 0$ in (45) for every j such that $j > n$. There is no question about the convergence of the infinite series in (45) since we have assumed that each x_j is embedded in a small but finite volume of given minimum size. Hence the points x_i cannot all cluster in any finite region of space. The exponential factors in (45) then assure convergence of the infinite series, since the distances $|x-x_j|$ increase regularly with j, in the limit.

The relation (45) has a deceptive amount of generality. We could, if required, partition all of euclidean three space (except some arbitrarily small neighborhood of x) into cubes of varying sizes if need be. Then each cube with center x_i is assigned an output $P_o(x_j)$. Equation (45) then gives the total scalar irradiance at x generated by these discrete sources throughout space.

As an example of the preceding observation, suppose that small, finite, contiguous volumes are used to similate a thin cylindrical region with a straight-line segment in space as axis and along which sources are distributed. Such cylinders may simulate narrow beams of radiant flux sent out by highly directional sources, for example laser sources. In this case $P_o(x_j)$ is generated by the scattering, within the jth volume segment, of the residual flux of the beam reaching the jth volume. Thus, suppose a laser source is at point x_o and directed along the path $P_r(x_o,\xi)$ with initial point x_o and direction ξ, as in Fig. 6.4. Partition the beam, which has initial radiance N_o, into n parts, each a cylinder of length r/n and initial point x_j (= $x_o + (jr/n)$). Finally, suppose the volume scattering function σ is independent of ξ', ξ, i.e., that isotropic scattering prevails throughout the medium. Then it is clear that:

$$N_o\, e^{-jr\alpha/n}$$

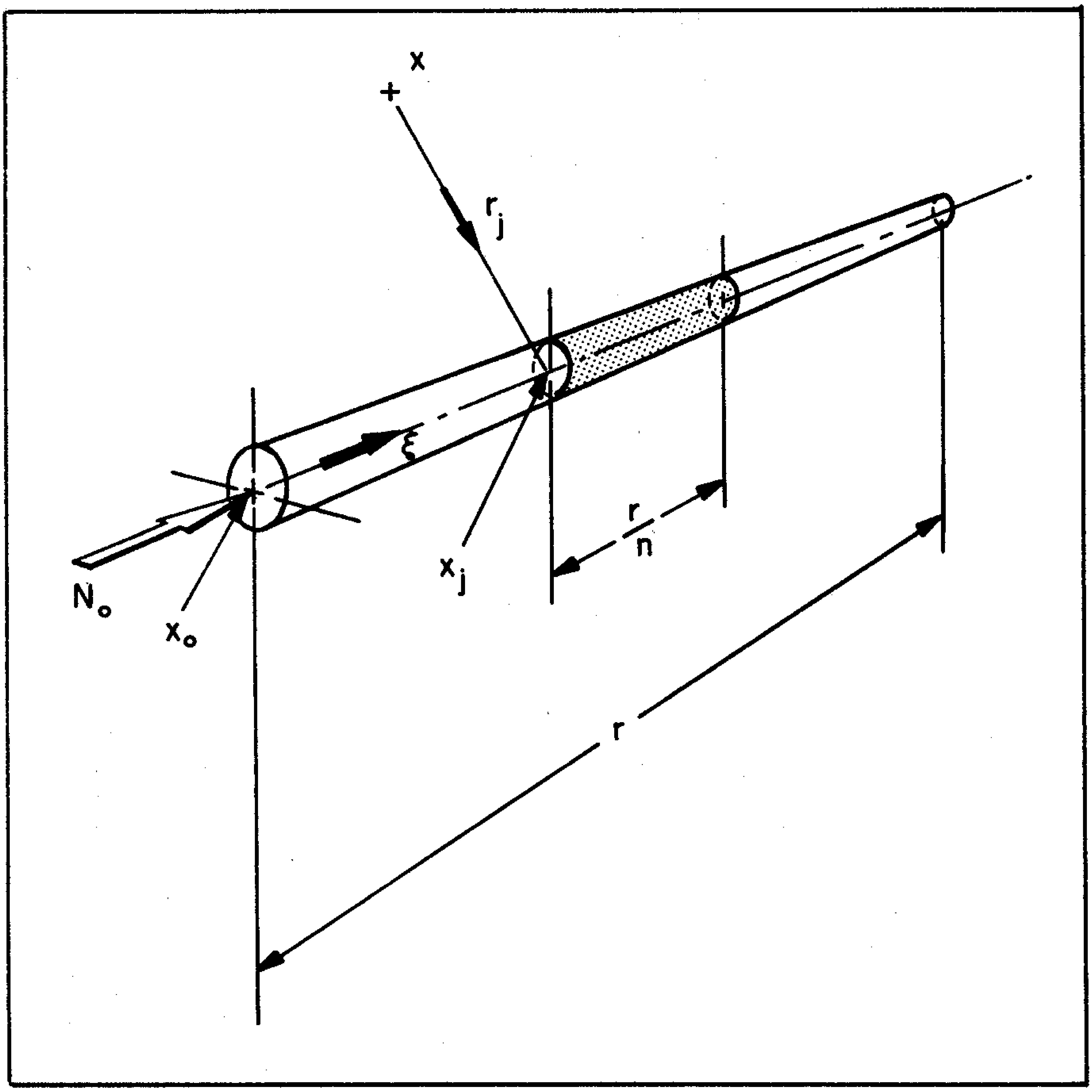

FIG. 6.4 Geometry for a narrow cylindrical beam source of radiant flux in diffusion theory.

is the residual radiance reaching the initial point x_j of the jth cylindrical part of the beam. From this and the definition of path function it follows that:

$$N_o\ e^{-jr\alpha/n}\ (s/4\pi)$$

is the path function value at the initial point x_j of the jth cylindrical part of the beam. Because scattering is isotropic, this value is assigned to each direction about x_j. Since path function values have the dimension of intensity per unit volume (e.g., see note (h) for Table 3 in Sec. 2.12), we can make the following assignation: To

$$P(x_i)/4\pi\ (= J(x_i))$$

in (45), we assign:

$$N_o\, e^{-jr\alpha/n}\, (sV(x_j)/4\pi) \quad ,$$

where $V(x_j)$ is the volume of the jth part of the beam, so that (45) now becomes:

$$h(x) = \frac{N_o s}{4\pi D} \sum_{j=1}^{n} \frac{e^{-jr\alpha/n}\, e^{-\kappa|x-x_j|}\, V(x_j)}{|x-x_j|} \tag{46}$$

This shows how the discrete-source case can simulate important internal source problems in natural optical media, provided, of course, that the basic diffusion point source model is valid for the given medium.

The radiance distribution associated with a discrete source scalar irradiance field given by (45) is obtained by appeal to the interaction principle, so that by simply adding together terms of the form shown in (40), the desired radiance distribution is obtained. An alternate representation of $N(x,\xi)$ is obtainable as follows: From (39) and the interaction principle it is clear that the vector irradiance generated by the point sources at x_1, x_2, ... is:

$$\mathbf{H}(x) = -\sum_{j=1}^{\infty} \frac{D\, h_j(x)\, (1+\kappa|x-x_j|)}{|x-x_j|}\, \mathbf{r}_j \tag{47}$$

where we have written:

$$\text{"}h_j(x)\text{"} \quad \text{for} \quad \frac{P_o(x_i)\, e^{-\kappa|x-x_j|}}{4\pi D|x-x_j|}$$

and where $\mathbf{r}_j$ is the unit vector directed from the observation point x to the jth source point x_j (see Fig. 6.4). Then using $\mathbf{H}(x)$ and $h(x)$ as given, respectively, by (47) and (45), the radiance $N(x,\xi)$ at x in the direction ξ is given once again by (29) of Sec. 6.5.

Continuous Source Case

We now make the transition from the discrete source case, just concluded, to the continuous source case. We begin with the finite version of (45) in which we have partitioned a subset X_n of the infinite medium into a set of n small volumes X_j ("small" in the sense of less than one attenuation length in diameter) each of which has a radiant flux output of $P_o(x_j)$, where x_j is a point of X_j. Hence the radiant flux output per unit volume about x_j is very nearly $P_o(x_j)/V(X_j)$, where $V(x_j)$ is the volume of X_j. We assume that the radiant flux output of X_j is uniform in all directions about x_j. Then the radiant intensity per unit volume:

$$\frac{P_o(x_j)}{4\pi\ V(X_j)}$$

may be represented by an emission radiance distribution $N_\eta(x_j,\xi)$ which is independent of direction ξ. (Recall that N_η has the same dimensions as path function N_*, and that the latter's dimensions may be characterized as radiant intensity per unit volume). Therefore, using the definition of h_η in (4) of Sec. 6.5, we may write:

$$\frac{P_o(x_j)}{4\pi\ V(x_j)} = N_\eta(x_j,\xi) = h_\eta(x_j)/4\pi \quad , \tag{48}$$

so that:

$$h_\eta(x_j) = \frac{P_o(x_j)}{V(X_j)} \quad . \tag{49}$$

With this meaning of $h_\eta(x_j)$, the finite version of (45) may be rewritten as:

$$h(x) = \sum_{j=1}^{n} \frac{h_\eta(x_j)\ e^{-\kappa|x-x_j|}\ V(x_j)}{4\pi D|x-x_j|} \tag{50}$$

By letting the partition of X_η become finer, so that in the limit the associated Riemann integral over X_η is obtained, (50) becomes:

$$\boxed{h(x) = \int_{X_\eta} \frac{h_\eta(x')\ e^{-\kappa|x-x'|}}{4\pi D|x-x'|}\ dV(x')} \tag{51}$$

This is the desired representation of the scalar irradiance $h(x)$ generated by isotropic point sources of strength $h_\eta(x)$ watts per unit volume, at points x' throughout a region X_η of the medium X. In analogy to (43) of Sec. 6.5 we write:

$$\text{"}K_\kappa(x',x)\text{"} \quad \text{for} \quad \frac{e^{-\kappa|x-x'|}}{4\pi D|x-x'|} \tag{52}$$

and

$$\text{"}\mathbf{W}\text{"} \quad \text{for} \quad \int_{X_\eta} [\ ]\ K_\kappa(x',x)\ dV(x') \tag{53}$$

so that (51) may be written:

$$h(x) = \int_{X_\eta} h_\eta(x')\ K_\kappa(x',x)\ dV(x') = h_\eta \mathbf{W}(x) \tag{54}$$

Finally, the vector irradiance $\mathbf{H}(x)$ in the continuous source case can be obtained by starting with (47) and going to the Riemann integral counterpart of that sum. Thus, suppose initially the sum is finite and that the sources are confined to a part X_η of the medium. Then, as before the set X_η is partitioned and "$h_\eta(x_j)$" introduced to denote the unit volume output of the medium at point x_j in X_η. Thus (47) becomes:

$$\mathbf{H}(x) = \sum_{j=1}^{n} \frac{D\ h_\eta(x_j)\ K_\kappa(x_j,x)(1+\kappa|x-x_j|)(-\mathbf{r}_j)\ V(x_j)}{4\pi|x-x_j|}$$

in which (52) is used. Observe that $-\mathbf{r}_j$ is $(x-x_j)/|x-x_j|$ so that as the partition of X_η is made suitably fine, the sum has the limit:

$$\mathbf{H}(x) = \int_{X_\eta} \frac{D\ h_\eta(x')\ K_\kappa(x',x)(1+\kappa|x-x'|)}{4\pi|x-x'|^2}\ (x-x')\ dV(x') \tag{55}$$

When h(x) and H(x), as given by (51) and (55), are used in (29) of Sec. 6.5, we obtain the appropriate radiance function for the diffusing light field generated by a continuous distribution of sources in X_η. The limitations of the point source case are as considered above. Indeed, since the point source case fails for points of observation too near the point source, it follows that points of observation x in (51) and (55) should not be in X_η, and preferably at some distances from X_η. We must impose this limitation on all diffusion integrals in practice. This problem of the proximity of the sources of the diffusing field will be examined in the following paragraphs.

Primary Scattered Flux as Source Flux

Time and again in the preceding illustrations of the diffusion method, precautionary observations were required on the use of the various derived equations because of possible inapplicability of Fick's law. For example, when an observation point x is too near a point source point x_o in an otherwise suitably diffusing medium, the radiance distribution about x may depart too markedly from the cardioidal distribution indigenous to classical diffusion theory. This departure is due principally to the highly directional residual radiance originating at x_0 and arriving at x. It would therefore seem desirable to improve the radiometric conditions prior to applying the classical diffusion theory by first computing the primary scattered radiance field generated by the given sources and using this radiance field as the source field in the continuous diffusion case considered above. We shall explore this possibility and its generalization in this and the subsequent paragraph.

In order to correctly implement the present discussion it seems best to return directly to the basic equation of transfer for scalar irradiance, (1) of Sec. 6.5. Our immediate task is to decompose the steady-state scalar irradiance h(x) into its residual component h^0 and its diffuse component h^*, where the basis for these concepts were defined in (15) and (22) of Sec. 5.1. Thus, using the operator **U** in (39) of Sec. 6.5, we write:

$$\text{"}h^*(x)\text{"} \quad \text{for} \quad N^*(x,\cdot)\,\mathbf{U} \tag{56}$$

so that:

$$h(x) = h^0(x) + h^*(x) \tag{57}$$

and

$$h^*(x) = \sum_{j=1}^{\infty} h^j(x) \quad . \tag{58}$$

In other words, the scalar irradiance h(x) consists of the sum of all scalar irradiances $h^n(x)$ associated with n-ary radiance distributions $N^n(x,\cdot)$ at x. Hence $h^*(x)$ consists of radiant flux having undergone one or more scattering operations. Clearly, (57) may be obtained immediately from (4) of Sec. 5.4 by applying the operator **U** (cf. (39) of Sec. 6.5). That is, from

$$N = N^0 + N^* \tag{59}$$

we obtain

$$N\mathbf{U} = (N^0 + N^*)\,\mathbf{U} = N^0\mathbf{U} + N^*\mathbf{U}$$

that is:

$$h = h^0 + h^* \tag{60}$$

We now use this mode of decomposition of h in the steady state version of (1) of Sec. 6.5. The details are as follows, starting with:

$$\xi \cdot \nabla N = -\alpha N + \int_{\Xi} N\sigma\, d\Omega + N_\eta$$

we first decompose N as in (59) to obtain:

$$\xi \cdot \nabla (N^o + N^*) = -\alpha(N^o + N^*) + \int_{\Xi} (N^o + N^*) \, \sigma d\Omega + N_\eta$$

Hence:

$$\xi \cdot \nabla N^* = -\alpha N^* + \int_{\Xi} N^* \sigma d\Omega + \int_{\Xi} N^o \sigma d\Omega \qquad (61)$$

where we have used the relation:

$$\xi \cdot \nabla N^o = -\alpha N^o + N_\eta$$

which follows from (2) of Sec. 5.8. Recalling the definition of N_*^1 ((2) of Sec. 5.1), (61) can be cast into the form:

$$\boxed{\xi \cdot \nabla N^* = -\alpha N^* + \int_{\Xi} N^* \sigma d\Omega + N_*^1} \qquad (62)$$

This is the equation of transfer which governs the diffuse radiance field N^* consisting of primary and higher order scattered flux. An alternate derivation of (62) was performed in (7) of Sec. 5.2. The source for the field N^* is the first order path function N_*^1. Because the residual radiance N^o coming in from the boundaries of the medium, and emission radiance N_η are now absent from N^*, the directional structure of N^* is considerably milder than that of N_1 so that Fick's law is more likely to hold for N^* than N.

It is to the scalar irradiance h^* induced by N^* that we now direct attention and derive from (62) the required diffusion equation for h^*. Thus, applying the operator **U** to (62) we have:

$$\nabla \cdot \mathbf{H}^* = -ah^* + h_*^1 \qquad (63)$$

where:

$$h_*^1 = h^o s$$

and where we write:

$$\text{"}\mathbf{H}^*\text{"} \quad \text{for} \quad \int_{\Xi} N^* \xi \, d\Omega \qquad (64)$$

Assuming Fick's law to hold between H^* and h^* (cf. (5) of Sec. 6.5), i.e., assuming:

$$\boxed{H^* = -D \nabla h^*} \tag{65}$$

(63) becomes:

$$\boxed{-D \nabla h^* + ah^* = h^1_*} \tag{66}$$

This is the requisite steady-state diffusion equation for h^* in which the primary scattered scalar irradiance h^1_* serves as an auxiliary source to the basic emission sources h_η in the medium. The assumption of Fick's law for h^* in (65) has a better chance of being valid than for h, since h has h^0 as a component which can be associated with highly directional flows from boundaries and internal sources.

The theory of the continuous source developed above and summarized in (51) and (55) may now be applied to the case where h_η in those equations is replaced by h^1_*. The proof of this procedure is based on the fact that the derivation of (51) and (55) ultimately rests on the steady-state version of (7) of Sec. 6.5; and this has just been shown to be identical with (66) in which h_η in the earlier equation is now replaced by h^1_*.

We now illustrate the use of (66) by means of a simple example. We consider an isotropic point source in an infinite homogeneous medium which scatters isotropically (i.e., is independent of ξ' and ξ). The source is at the origin and in reality constitutes a very small, essentially transparent sphere of radius r_o which has a uniform surface radiance N_o. Thus the radiant emittance of the spherical surface is πN_o and therefore the total flux output is $4\pi^2 r_o^2 N_o$. The average flux per unit volume of the spherical source is $4\pi^2 r_o^2 N_o/(4\pi r_o^3/3) = 3\pi N_o/r_o$. It is this output which would customarily be used in the estimate of h_η in the continuous case (cf. (49)). However, now the source is allowed first to generate a primary scattered flux field h^1_* in the space surrounding it. In principle this primary scattered flux is generated at every point of the medium and may be estimated as follows at a point x' a distance $r' > r_o$ from the center of the spherical source. First note that $r' = |x'|$. Then let $\Omega(|x'|)$ $(=\Omega(r'))$ be the magnitude of the solid angle subtended by the sphere at vantage point x'. Then very nearly:

$$N^1_*(x',\xi) = \int_\Xi N^o \sigma(x';\xi';\xi)\, d\Omega(\xi') = N^o \Omega(r') s/4\pi = N_o e^{-\alpha r'} \Omega(r') s/4\pi$$

for every ξ. Hence:

$$h_*^1(x') = \int_{\Xi} N_*^1(x',\xi)\, d\Omega(\xi)$$

$$= N_o\, e^{-\alpha r'} \Omega(r') s \quad . \tag{67}$$

This representation is not exact because the integration over the set of directions from the emitting sphere assumed the distances from the point x' to the various points on the spherical surface were all equal to the fixed distance r'. However (67) should give excellent estimates of $h_*^1(x')$ for points x' when the sphere is viewed as a point source. We shall adopt (67) as a working basis in the present example.

We now use equation (51) with $h_n(x')$ in that equation replaced by $h_*^1(x')$ as given in (67). Here r' is the distance from x' to the origin; hence $r' = |x'|$. With these observations (51) now lets us write:

$$h^*(x) = N_o s \int_X \frac{e^{-[\kappa|x-x'| + \alpha|x'|]}\, \Omega(|x'|)\, dV(x')}{4\pi D|x-x'|} \tag{68}$$

Finally, the residual scalar irradiance $h^o(x)$, was essentially evaluated in arriving at (67); that is, the scalar irradiance induced by the small sphere is:

$$h^o(x) = N_o\, e^{-\alpha|x|}\, \Omega(|x|) \quad . \tag{69}$$

The full scalar irradiance h(x) for the present problem is, according to (57), the sum of $h^o(x)$ and $h^*(x)$ as they are given in (68) and (69). A generalization of (68) is readily effected by letting N^o vary in direction. All this means formally is that "N^o" goes under the integral sign in (68). In this case, the approximation of $h^o(x)$ by $N_o\Omega(|x|)\, e^{-\alpha|x|}$ must be examined. This will not be attempted here.

Higher Order Scattered Flux as Source Flux

The preceding example of the use of primary scattered radiant flux as source flux in the classical diffusion equation seems sufficiently useful to encourage carrying out the underlying idea of the example to its logical conclusion. Toward this end, suppose that it is possible to compute the first n+1 scattering orders for radiance: N^j, j = 0, 1, ..., n. We then supplement this exact calculation by estimating the radiance function

$$\sum_{j=n+1}^{\infty} N^j$$

using diffusion theory. Clearly this procedure includes that of the preceding discussion as a special case; in fact it is the case n = 0.

As in the special investigation for the case n = 0, we begin with the steady-state equation of transfer:

$$\xi \cdot \nabla N = -\alpha N + \int_{\Xi} N \sigma d\Omega + N_\eta$$

and now write N as:

$$N = \sum_{j=0}^{\infty} N^j = \sum_{j=0}^{n} N^j + \sum_{j=n+1}^{\infty} N^j$$

$$= N^{(n)} + N^{(n,*)} \qquad (70)$$

where the definitions of the two terms $N^{(n)}$ and $N^{(n,*)}$ are implicit in (70). Thus in particular $N^{(o)} = N^o$ and $N^{(o,*)} = N^*$.

Using this decomposition in the equation of transfer, we have:

$$\xi \cdot \nabla(N^{(n)} + N^{(n,*)}) = -\alpha(N^{(n)} + N^{(n,*)}) + \int_{\Xi} (N^{(n)} + N^{(n,*)}) \sigma \, d\Omega + N_\eta \qquad (71)$$

Now, from (1) of Sec. 5.2 we have for every $j \geq 1$

$$\xi \cdot \nabla N^j = -\alpha N^j + \int_{\Xi} N^{j-1} \sigma \, d\Omega \qquad (72)$$

and from (2) of Sec. 5.8:

$$\xi \cdot \nabla N^o = -\alpha N^o + N_\eta \qquad (73)$$

By adding equations (72) and (73) together from j = 1 up to j = n, we obtain:

$$\xi \cdot \nabla N^{(n)} = -\alpha N^{(n)} + \int_{\Xi} N^{(n-1)} \sigma \, d\Omega + N_\eta \qquad (74)$$

This equation is now used with (71) to reduce the latter to:

$$\xi \cdot \nabla N^{(n,*)} = -\alpha N^{(n,*)} + \int_{\Xi} N^{(n,*)} \sigma \, d\Omega + N_*^{n+1} \quad (75)$$

This equation is the direct generalization of (62), the latter being obtained by setting n = 0 in (75).

Next the operator **U** ((39) of Sec. 6.5) is applied to each side of (75); the result is:

$$\nabla \cdot \mathbf{H}^{(n,*)} = -a\, h^{(n,*)} + h_*^{n+1} \quad (76)$$

The final step is to hypothesize that Fick's law holds between $\mathbf{H}^{(n,*)}$ and $h^{(n,*)}$:

$$\mathbf{H}^{(n,*)} = -D\nabla h^{(n,*)} \quad (77)$$

so that (76) becomes:

$$\boxed{-D\,\nabla h^{(n,*)} + ah^{(n,*)} = h_*^{n+1}} \quad (78)$$

This is the requisite diffusion equation for $h^{(n,*)}$. It is a direct generalization of (66) which is the case n = 0. The source term for the flux $h^{(n,*)}$ is (n+1)-ary scattered flux, which should have relatively mild direction structure, so that (77) has a good chance of holding in practice. In general, the greater the n, the more likely--on intuitive grounds--(77) would seem to hold. (See the discussion following (13) of Sec. 5.12.)

Once $h^{(n,*)}$ is obtained by solving (78) with the continuous source h_*^{n+1}, using, e.g., (51) with h_n replaced by h_*^{n+1}, we then find the complete scalar irradiance h by noting that

$$\boxed{h(x) = h^{(n)}(x) + h^{(n,*)}(x)} \quad . \quad (79)$$

where we write:

$$\text{"}h^{(n)}\text{"} \quad \text{for} \quad N^{(n)}\,\mathbf{U} \quad (80)$$

and:

$$\text{"}h^{(n,*)} \quad \text{for} \quad N^{(n,*)}\,\mathbf{U} \quad (81)$$

From $h^{(n,*)}(x)$ we can then find $\mathbf{H}^{(n,*)}(x)$ using (77) and so, in turn, $N^{(n,*)}(x,\xi)$ using the diffusion equation

(29) of Sec. 6.5 as a model. This diffusion-based estimate of $N^{(n,*)}(x,\xi)$ is then added to the known radiance $N^{(n)}(x,\xi)$.

Time-Dependent Diffusion Problems

Time-dependent radiative transfer problems arise, for example, whenever extremely short pulses of radiant energy are released in scattering-absorbing media, and when the evolution of the subsequent scattered radiant energy of the pulse is to be described or predicted in detail. We study now a particularly simple and useful model of time-dependent light fields based on classical diffusion theory, in particular, equation (7) of Sec. 6.5.

Consider an infinite homogeneous optical medium with a single point source at x' which at time t' emits a single Dirac-delta pulse of unit radiant energy. That is, we assume h_η in (7) of Sec. 6.5 to have the form: $h_\eta(x,t) = U_\eta \delta(x-x') \cdot \delta(t-t')$, where at present $U_\eta = 1$, and U_η in general has the dimensions of radiant energy.

It may be verified directly from (7) of Sec. 6.5 (by performing the indicated differentiations and simplifying) that the resultant scalar irradiance $h(x,t)$, $t > t'$, varies in space and time according as $K_\kappa(x',x;t',t)$, where we have written:

$$\text{"}K_\kappa(x',x;t',t)\text{" for } \frac{v}{[4\pi vD(t-t')]^{3/2}} \exp\left\{ -\frac{|x-x'|^2}{4vD(t-t')} - av(t-t') \right\} \tag{82}$$

That is, for fixed x' and t', the function $K_\kappa(x',\cdot\,;t',\cdot)$ defined by (82) satisfies (7) of Sec. 6.5 at every space-time point (x,t), such that $x' \neq x$ and $t > t'$. The function $K_\kappa(x',\cdot\,;t',\cdot)$ first arose in the theory of transient heat conduction.

In general, with a continuous source distribution $h_\eta(x',t')$ defined throughout a part X_η of the medium for all times $t > t'$, we have, by means of the interaction principle, the resultant scalar irradiance field given by:

$$\boxed{h(x,t) = \int_{X_\eta} \int_{-\infty}^{t} h_\eta(x',t')\, K_\kappa(x',x;t',t)\, dt\, dV(x')} \tag{83}$$

Of course, h_η may be set equal to zero for all times t' earlier than some fiducial time t_0', so that $h_\eta(x',t')$ in (83) represents the general source condition (7) of Sec. 6.5. Therefore the resultant scalar irradiance field h defined by

(83) is the general solution of (7) of Sec. 6.5, as may be established by a direct appeal to (7) of Sec. 6.5.

It is of interest to connect (83) with two results obtained earlier in the present work. First we will show that if a steady point source condition subsists for all time, i.e., $h_\eta(x',t')$ is independent of time t' for all t' < t and is zero for all points x other than a given point x' on the medium, then:

$$\boxed{K_\kappa(x',x) = \int_{-\infty}^{t} K_\kappa(x',x;t',t)\, dt'} \tag{84}$$

so that (83) reduces to the steady state case (54). To see this we note that $K_\kappa(x',x;t',t)$ has the general Gestalt of:

$$a\,\frac{e^{\left(-\frac{b}{t} - ct\right)}}{t^{3/2}}$$

where we have written, *ad hoc*:

$$\text{"a"} \quad \text{for} \quad \frac{v}{[4\pi vD]^{3/2}}$$

$$\text{"b"} \quad \text{for} \quad \frac{|x-x'|^2}{4vD}$$

and:

$$\text{"c"} \quad \text{for} \quad av$$

and have replaced occurrences of "(t-t')" by "t". Then it is clear that on setting $t = u^2$:

$$\int_{-\infty}^{t} K_\kappa(x',x;t',t)\, dt' = 2a\int_{0}^{\infty} \frac{e^{\left(-\frac{b}{u^2} - cu^2\right)}}{u^2}\, du$$

$$= \frac{a\sqrt{\pi}}{\sqrt{b}}\, e^{-2\sqrt{bc}}$$

$$= \frac{e^{-\kappa|x-x'|}}{4\pi D|x-x'|} = K_\kappa(x',x) \quad .$$

The second connection we can make is that between (83) and the earlier result which describes the behavior of radiant energy under standard decay conditions, namely, property

8 of Sec. 5.10. To establish this connection we now assume that $h_\eta(x,t) = U_\eta\ \delta(x)\ \delta(t)$. This simulates the instantaneous localized introduction of an amount U_η of radiant energy into the medium. However, the actual manner of introduction is immaterial for the present discussion. With this condition on h_η, (83) yields:

$$h(x,t) = U_\eta\ K_\kappa(0,x;0,t) \quad ,$$

so that the radiant energy content of the medium at time t is:

$$U(t) = \frac{1}{v}\int_X h(x,t)\ dV(x)$$

$$= \frac{U_\eta}{v}\int_X K_\kappa(0,x;0,t)\ dV(x)$$

$$= \frac{U_\eta\ e^{-avt}}{[4\pi vDt]^{3/2}}\int_X \exp\left\{-\frac{|x|^2}{4vDt}\right\} dV(x)$$

Hence:

$$U(t) = U_\eta\ e^{-avt} \quad ,$$

which is precisely the analytic content of property 8 of Sec. 5.10. This most interesting result shows that the classical diffusion theory is globally exact and thereby may be used to help fill, in a consistent manner, the general gap in our knowledge about the *local* radiance distributions within a time-dependent radiant field. That is, we may use (83) to supplement the exact theory of the time-dependent radiant energy field studied in Chapter 5, by giving approximate but useful estimates of the radiant density throughout the medium.

To implement the program just outlined of supplementing the exact radiant energy theory of Chapter 5 by diffusion theory, we construct the basic diffusion equations for n-ary scalar irradiance from the time-dependent equation of transfer (19) of Sec. 5.8. Thus, by applying the operator **U** to the equation of transfer for n-ary radiance, we have for $n \geq 1$:

$$\boxed{\frac{1}{v}\frac{\partial h^n}{\partial t} + \nabla \cdot \mathbf{H}^n = -\alpha h^n + s h^{n-1}} \qquad (85)$$

where for every $n \geq 1$ we have written:

$$"\mathbf{H}^n" \quad \text{for} \quad \int_{\Xi} N^n \, \xi \, d\Omega(\xi)$$

and: (86)

$$"h_*^n" \quad \text{for} \quad \int_{\Xi} N_*^n \, d\Omega(\xi)$$

Assuming Fick's law holds between $\mathbf{H}^n$ and h^n, for every n, $n \geq 1$, i.e., assuming:

$$\mathbf{H}^n = -D \nabla h^n \quad , \tag{87}$$

then (85) yields the time-dependent diffusion equation for n-ary scalar irradiance, $n \geq 1$:

$$\boxed{\frac{1}{v}\frac{\partial h^n}{\partial t} - D \nabla^2 h^n = -\alpha h^n + s h^{n-1}} \tag{88}$$

One immediate application of (88) is the direct generalization, to the time-dependent setting, of the results (68) and (79) of the continuous source cases with all the analytic advantages of those results now transferred to the time-dependent context. In particular, we can replace $h_\eta(x',t;)$ in (83) by $h_*^1(x',t')$ which is computed exactly as in (67), but with suitable time lag to account for the travel of the initial pulse of the source from the source to x'. Then we compute $h^*(x,t)$ as follows:

$$h^*(x,t) = \int_{X_\eta} \int_{-\infty}^{t} h_*^1(x',t') \, K_\kappa(x',x;t',t) \, dt' \, dV(x') \tag{89}$$

so that:

$$h(x,t) = h^o(x,t) + h^*(x,t) \tag{90}$$

where $h^o(x,t)$ is the residual scalar irradiance computed from the given source condition, which may be discrete or finite.

The theoretical basis for (89) is the time-dependent counterpart to (66). This time-dependent counterpart is obtained, e.g., by adding up all equations in (88) for $n = 1, 2, \ldots$.

The result is:

$$\boxed{\frac{1}{v}\frac{\partial h^*}{\partial t} - D\nabla^2 h^* = -\, a h^* + h_*^1} \qquad (91)$$

Observe how the infinite number of Fick's laws in (87) imply (65). On the basis of (91), the representation (89) is established by simply repeating the arguments leading to (83). Finally, the generalization of (91) to the time-dependent version of (78), and the derivation of the corresponding representation of (79), is readily made following the patterns of derivation established in that steady-state case.

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6.7 Solutions of the Exact Diffusion Equations

The exact diffusion equation on which we base the discussion of the present section is (57) of Sec. 6.5. In full notation, this equation is of the form:

$$h(x) = \frac{1}{4\pi}\,(h_\eta + sh)\,V(x)$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,K_\alpha(x',x)\,dV(x')$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,\frac{T_{r-r'}(x',\xi)}{|x-x'|^2}\,dV(x') \qquad (1)$$

The current settings in which this integral equation is to describe the scalar irradiance field h are infinite and semi-infinite homogeneous media with arbitrary sources described by h_η within X. Once a solution h is found for a space X, the associated radiance distribution throughout X is obtained by means of (60) of Sec. 6.5. The first of our two main goals in this section is to solve (1) for a point source in an infinite medium and arrange the solution in such a manner as to be directly applicable to problems of finding radiance distributions associated with general source conditions in X. It will be seen that by judiciously tabulating the point source solution of (1), all solutions of (1) corresponding to the possible source conditions within X, are obtainable in principle by relatively straightforward numerical procedures based on the tabulated solution. The second main goal is to discuss the solutions of (1) for semi-infinite media (infinitely deep, plane-parallel media) with arbitrary internal sources.