The result is:

$$\boxed{\frac{1}{v}\frac{\partial h^*}{\partial t} - D\nabla^2 h^* = -ah^* + h^1_*} \qquad (91)$$

Observe how the infinite number of Fick's laws in (87) imply (65). On the basis of (91), the representation (89) is established by simply repeating the arguments leading to (83). Finally, the generalization of (91) to the time-dependent version of (78), and the derivation of the corresponding representation of (79), is readily made following the patterns of derivation established in that steady-state case.

6.7 Solutions of the Exact Diffusion Equations

The exact diffusion equation on which we base the discussion of the present section is (57) of Sec. 6.5. In full notation, this equation is of the form:

$$h(x) = \frac{1}{4\pi}(h_\eta + sh)\,\mathbf{V}(x)$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,K_\alpha(x',x)\,dV(x')$$

$$= \frac{1}{4\pi}\int_X \Big(h_\eta(x') + s(x')\,h(x')\Big)\,\frac{T_{r-r'}(x',\xi)}{|x-x'|^2}\,dV(x') \qquad (1)$$

The current settings in which this integral equation is to describe the scalar irradiance field h are infinite and semi-infinite homogeneous media with arbitrary sources described by h_η within X. Once a solution h is found for a space X, the associated radiance distribution throughout X is obtained by means of (60) of Sec. 6.5. The first of our two main goals in this section is to solve (1) for a point source in an infinite medium and arrange the solution in such a manner as to be directly applicable to problems of finding radiance distributions associated with general source conditions in X. It will be seen that by judiciously tabulating the point source solution of (1), all solutions of (1) corresponding to the possible source conditions within X, are obtainable in principle by relatively straightforward numerical procedures based on the tabulated solution. The second main goal is to discuss the solutions of (1) for semi-infinite media (infinitely deep, plane-parallel media) with arbitrary internal sources.

Infinite Medium with Point Source

We begin with (1) for the case of an infinite homogeneous medium X with a point source at the origin. The homogeneity assumption frees $\alpha(x)$ and $s(x)$ of dependence on x throughout X and lets us write:

$$\frac{T_{r-r'}(x',\xi)}{|r-r'|^2} = \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \tag{2}$$

where, as usual "x" denotes a point in X, and where $|x-x'|$ is the distance between points x and x'. The point source condition is represented by:

$$h_\eta(x') = P_\eta \; \delta(x') \tag{3}$$

where P_η is the quantity of radiant flux emitted steadily in time and uniformly in all directions by the point source at the origin. We may leave the nature of this source quite arbitrary throughout the discussion. As a result, we shall be able to adapt various solutions of (1) for the point source case, by means of integration, and in such a manner that the actual nature of the source may vary from true emission processes, through transpectral scattering processes, on through elastic scattering processes. This will be illustrated later in the discussion. For the present we go on to investigate the case of (1) with a single point source. The requisite form of (1) is:

$$h(x) = \frac{1}{4\pi} \int_X \left[P_\eta \; \delta(x') + sh(x') \; \frac{e^{-\alpha|x-x'|}}{|x-x'|^2} \right] dV(x') \tag{4}$$

The theory of the solution of (4) is thoroughly understood; a representative detailed development of the solution of (4) may be found, e.g., in [40]. Therefore, beyond the general observations leading from (39) to (59) of Sec. 6.5, we shall not need to discuss the details of the solution procedure of (4) in the present work. However, we wish to display the solution of (4) in such a manner that the results of [40] may be readily adapted to the radiative transfer context. Such an adaptation requires the preliminary transition to a certain class of dimensionless geometric parameters, which we now define.

Throughout this section we shall write:

$$\text{"}\tau(x,x') \quad \text{for} \quad \int_0^r \alpha(x'') \, dr''\text{"} \tag{5}$$

where α is the volume attenuation function for the medium. The integral is a line integral along a path $\mathcal{P}_\tau(x,\xi)$ with initial point x and terminal point x'. Since the medium X is isotropic and homogeneous, paths are straight-line segments and

$$\tau(x,x') = \alpha|x-x'| \tag{6}$$

When no confusion will result, we will simply write:

$$"\tau" \quad \text{for} \quad \tau(x,x') \quad ,$$

with x, and x' thereby being understood.

The quantity τ assigned to the distance $|x-x'|$ between x and x' is dimensionless, and by virtue of (6) may be viewed as the number of attenuation lengths L_α between x and x'.

Next, for every subset Y of X we write:

$$"V_\alpha(Y)" \quad \text{for} \quad \int_Y \alpha^3(x')\, dV(x') \tag{7}$$

The quantity $V_\alpha(Y)$ is dimensionless. Throughout this section, both $\tau(x,x')$ and V (Y) may be thought of and referred to as *optical lengths* and *optical volumes*, respectively, without fear of confusion with the classical notions of the same names.

With definitions (5) and (7) in mind, (4) may be rewritten as:

$$h(x) = \frac{1}{4\pi}\int_X \left[\frac{P_\eta}{\alpha}\,\delta(x') + \rho\, h(x')\,\frac{e^{-\tau(x,x')}}{\tau^2(x,x')}\right] dV_\alpha(x') \tag{8}$$

where ρ is the scattering-attenuation ratio s/α. Equation (8) is the required dimensionless version of (4); and for purposes of a solution tabulation, we now impose the *unit source condition* in the context of (8):

$$\frac{P_\eta}{\alpha} = 1 \tag{9}$$

provided that the Dirac-delta function δ with dimensions L^{-3} (to go with the volume measure V) is retained. Otherwise, if a dimensionless Dirac-delta function δ (to go with the optical V_α) is adopted, in (3) we write $h_\eta\,\delta(x')$ and the unit source condition is

$$\frac{h_\eta}{\alpha} = 1 \quad . \tag{9a}$$

The scalar irradiance field h governed by (8) is clearly spherically symmetric about the point source so that h depends only on radial distance r or (now that the transition to dimensionless parameters has been made) on τ. Let us denote the solution of (8), *under the unit source condition (9a)*, by "K_ε". Then it can be shown (cf. [40]) that the scalar irradiance at optical distance τ from the origin is $K_\varepsilon(\tau)$, where:

$$K_\varepsilon(\tau) = A(\rho,\tau)\, K_\alpha(\tau) + B(\rho,\tau)\, K_\kappa(\tau) \tag{10}$$

and where, in turn we have written:

$$\text{"}A(\rho,\tau)\text{"} \quad \text{for} \quad \frac{1}{4\pi}\, \varepsilon(\rho,\tau) \tag{11}$$

and

$$\text{"}B(\rho,\tau)\text{"} \quad \text{for} \quad D_o \frac{\partial k_o^2}{\partial \rho} \tag{12}$$

to point up the fact that $K_\varepsilon(\tau)$ is simply a linear combination of the dimensionless diffusion kernel $K_\kappa(\tau)$ (cf. (52) of Sec. 6.6) where now we write:

$$\text{"}K_\kappa(\tau)\text{"} \quad \text{for} \quad \frac{e^{-\kappa_o \tau}}{4\pi D_o \tau} \tag{13}$$

and the dimensionless beam transmittance kernel $K_\alpha(\tau)$ (cf. (43) of Sec. 6.5) where now we write:

$$\text{"}K_\alpha(\tau)\text{"} \quad \text{for} \quad \frac{e^{-\tau}}{\tau^2} \quad . \tag{14}$$

It remains to specify the terms $\varepsilon(\rho,\tau)$, κ_o, $\partial k_o^2/\partial\rho$, and D_o. The latter term is simply αD, where D is the diffusion constant (cf. (27) of Sec. 6.5) for the classical diffusion theory. The remaining three terms form the heart of the exact solution and are tabulated in Tables 1 and 2 below for various values of ρ and τ.

Thus from (10), we have

$$K_\varepsilon(\tau) = \frac{\varepsilon(\rho,\tau)}{4\pi\tau^2}\, e^{-\tau} + \frac{\partial k_0^2}{\partial\rho}\, \frac{1}{4\pi\tau}\, e^{-\kappa_0 \tau} \tag{15}$$

TABLE 1

The function $\varepsilon(\rho,\tau)$

τ	$\rho = 0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	1.0210	1.0418	1.0542	1.0526	1.0420
0.2	1.0000	1.0382	1.0773	1.1000	1.0962	1.0756
0.3	1.0000	1.0532	1.1088	1.1409	1.1346	1.1046
0.4	1.0000	1.0667	1.1375	1.1781	1.1692	1.1301
0.5	1.0000	1.0790	1.1640	1.2126	1.2008	1.1529
0.6	1.0000	1.0904	1.1888	1.2448	1.2300	1.1736
0.7	1.0000	1.1010	1.2121	1.2752	1.2571	1.1926
0.8	1.0000	1.1109	1.2342	1.3038	1.2826	1.2100
0.9	1.0000	1.1202	1.2552	1.3311	1.3066	1.2262
1.0	1.0000	1.1291	1.2753	1.3571	1.3293	1.2412
1.5	1.0000	1.1674	1.3644	1.4724	1.4273	1.3034
2.0	1.0000	1.1990	1.4402	1.5699	1.5068	1.3504
2.5	1.0000	1.2258	1.5068	1.6551	1.5738	1.3874
3.0	1.0000	1.2494	1.5667	1.7311	1.6314	1.4171
3.5	1.0000	1.2704	1.6213	1.8000	1.6818	1.4415
4.0	1.0000	1.2895	1.6718	1.8630	1.7265	1.4617
4.5	1.0000	1.3070	1.7188	1.9214	1.7665	1.4786
5.0	1.0000	1.3231	1.7630	1.9757	1.8026	1.4928
6.0	1.0000	1.3521	1.8443	2.0745	1.8654	1.5147
7.0	1.0000	1.3779	1.9182	2.1630	1.9182	1.5304
8.0	1.0000	1.4010	1.9863	2.2432	1.9634	1.5412
9.0	1.0000	1.4222	2.0497	2.3169	2.0024	1.5486
10.0	1.0000	1.4417	2.1094	2.3851	2.0366	1.5531
11.0	1.0000	1.4599	2.1659	2.4499	2.0667	1.5554
12.0	1.0000	1.4770	2.2196	2.5086	2.0933	1.5559
13.0	1.0000	1.4931	2.2710	2.5652	2.1172	1.5550
14.0	1.0000	1.5084	2.3204	2.6188	2.1385	1.5529
15.0	1.0000	1.5230	2.3682	2.6700	2.1578	1.5498
16.0	1.0000	1.5370	2.4141	2.7190	2.1752	1.5459
17.0	1.0000	1.5503	2.4586	2.7658	2.1910	1.5413
18.0	1.0000	1.5632	2.5019	2.8109	2.2055	1.5361
19.0	1.0000	1.5757	2.5439	2.8543	2.2186	1.5304
20.0	1.0000	1.5877	2.5849	2.8963	2.2307	1.5243

Now that it is clear how $K_\varepsilon(\tau)$ depends on the diffusion kernel K_κ ((52) of Sec. 6.6) and the attenuation kernel K_α ((43) of Sec. 6.5) we write (10) in its explicit form:

TABLE 1--Concluded

The function $\varepsilon(\rho,\tau)$.

τ	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 1.0$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0269	1.0099	0.9921	0.9745	0.9564
0.2	1.0474	1.0162	0.9843	0.9528	0.9222
0.3	1.0643	1.0206	0.9767	0.9341	0.8934
0.4	1.0786	1.0236	0.9693	0.9173	0.8683
0.5	1.0909	1.0257	0.9621	0.9019	0.8460
0.6	1.1017	1.0271	0.9551	0.8878	0.8260
0.7	1.1113	1.0279	1.9483	0.8747	0.8077
0.8	1.1198	1.0282	0.9417	0.8625	0.7910
0.9	1.1275	1.0282	0.9353	0.8510	0.7755
1.0	1.1343	1.0278	1.9290	0.8402	0.7612
1.5	1.1601	1.0229	0.9002	0.7936	0.7019
2.0	1.1763	1.0149	0.8748	0.7562	0.6568
2.5	1.1866	0.0054	0.8519	0.7250	0.6207
3.0	1.1929	0.9952	0.8313	0.6982	0.5908
3.5	1.1963	0.9847	0.8124	0.6749	0.5655
4.0	1.1978	0.9742	0.7951	0.6543	0.5437
4.5	1.1976	0.9637	0.7791	0.6358	0.5246
5.0	1.1963	0.9534	0.7643	0.6191	0.5076
6.0	1.1912	0.9334	0.7374	0.5901	0.4788
7.0	1.1838	0.9144	0.7137	0.5654	0.4550
8.0	1.1749	0.8964	0.6926	0.5440	0.4349
9.0	1.1651	1.8793	0.6734	0.5253	0.4175
10.0	1.1547	0.8631	0.6560	0.5086	0.4024
11.0	1.1438	0.8477	0.6400	0.4936	0.3890
12.0	1.1327	0.8330	0.6252	0.4800	0.3769
13.0	1.1215	0.8190	0.6114	0.4676	0.3661
14.0	1.1102	0.8055	0.5985	0.4562	0.3562
15.0	1.0989	0.7926	0.5864	0.4456	0.3471
16.0	1.0876	0.7802	0.5750	0.4357	0.3387
17.0	1.0764	0.7683	0.5643	0.4265	0.3310
18.0	1.0653	0.7568	0.5540	0.4178	0.3238
19.0	1.0542	0.7457	0.5443	0.4096	0.3170
20.0	1.0433	0.7349	0.5349	0.4019	0.3107

TABLE 2

The functions κ_0 and $dk_0^2/d\rho$

ρ	κ_0	$dk_0^2/d\rho$
0.0	1.000000	0.000000
0.1	1.000000	0.164892(-5)*
0.2	0.999909	0.009094
0.3	0.997414	0.116201
0.4	0.985624	0.373272
0.5	0.957504	0.731896
0.6	0.907332	1.145954
0.7	0.828635	1.590033
0.8	0.710412	2.051119
0.9	0.525430	2.522370
0.92	0.474002	2.617473
0.94	0.413976	2.712805
0.96	0.340829	2.808348
0.98	0.242983	2.904085
0.99	0.172511	2.952020
1.00	0.000000	3.000000

*Note: "(-5)" means "multiply by 10^{-5}."

In this way we can see that, for computation purposes, the scalar irradiance $K_\varepsilon(\tau)$ at optical distance τ from the origin consists of two terms, one which may be attributed to residual flux (the first term) and the other which may be attributed to scattered flux. This type of partitioning of the exact representation of h(x) into a residual part (h^o) and a scattered part (h^*) was already encountered in the classical diffusion theory, e.g., in (7) of Sec. 1.5, in (57) of Sec. 6.6, and more generally in (79) of Sec. 6.6. Also, in the time-dependent case, this partition was encountered in (90) of Sec. 6.6.

A tabulation of $4\pi\,\tau^2 K_\varepsilon(\tau)$ is given in Table 3 for two cases of ρ and for a range of τ from 0 to 10 units. These choices of ρ are representative orders of magnitude for ρ in the case of the ocean ($\rho = 0.3$) and the atmosphere ($\rho = 0.9$) for wavelengths around 500 mμ, for the middle of the visible spectrum. For the determination of $K_\varepsilon(\tau)$ for values of ρ other than $\rho = 0.3$, 0.9, Tables 1 and 2 may be used. It must be kept in mind that these tabulations are for the unit source condition (9a).

TABLE 3

The function $4\pi\tau^2 K_\varepsilon(\tau)$

τ	$\rho = 0.3$	$\rho = 0.9$
0.0	1.0000	1.0000
0.1	0.9644	1.1211
0.2	0.9196	1.2343
0.3	0.8710	1.3384
0.4	0.8209	1.4326
0.5	0.7708	1.5168
0.6	0.7215	1.5914
0.7	0.6737	1.6567
0.8	0.6277	1.7130
0.9	0.5838	1.7607
1.0	0.5421	1.8006
1.5	0.3675	1.8974
2.0	0.2441	1.8660
2.5	0.1599	1.7547
3.0	0.1037	1.5992
3.5	0.0668	1.4239
4.0	0.0427	1.2454
4.5	0.0272	1.0742
5.0	0.0173	0.9158
6.0	0.0069	0.6483
7.0	0.0028	0.4467
8.0	0.0011	0.3018
9.0	0.0004	0.2007
10.0	0.0002	0.1318

Infinite Medium with Arbitrary Sources

We now develop a procedure whereby Table 3, and more generally (15), may be used to compute scalar irradiance fields generated by arbitrary sources. Suppose the source term $h_\eta(x)$ is given throughout an infinite medium X; $h_\eta(x)$ may be associated with plane sources, finite volume sources of flux, etc., and may be of quite arbitrary spatial dependence throughout X. It is clear either intuitively or formally (from the interaction principle using the theorems of Sec. 3.16) that the scalar irradiance h(x) associated with $h_\eta(x)$ is given by:

$$h(x) = \frac{1}{\alpha} \int_X h_\eta(x') \; K_\varepsilon(x',x) \; dV_\alpha(x') \qquad (16)$$

where we have written:

$$\text{"}K_\varepsilon(x',x)\text{"} \quad \text{for} \quad K_\varepsilon(\tau(x,x')) \qquad (17)$$

The reason for the presence of "α" in (16) may be found by tracing back through the unit source condition (9a) and ultimately to (3) and (4). If h_η is given in watts per cubic meter, and α in per meter, then h is given in units of watts per square meter.

A practical computation scheme for h(x) may be based on the following procedure: given $h_\eta(x)$ throughout a subset X_η of X, divide X_η into n small cubes $C(x_i)$ (or any other conveniently shaped regions) over each of which both $\tau(x,x')$ and $h_\eta(x)$ vary only slightly. Thus each cube $C(x_i)$ is representative of the radiometric properties of X around x_i, where x_i is the cube's centerpoint. Then (16) may be replaced by the approximating finite sum:

$$h(x) = \frac{1}{\alpha} \sum_{i=1}^{n} h_\eta(x_i) \; K_\varepsilon(x_i,x) \; V_\alpha(C(x_i)) \qquad (18)$$

The evaluation of h(x) using (18) is facilitated by using Table 3 for optical distances $\tau(x,x')$ up to 10. More generally, (15) would be used with Tables 1 and 2.

As a specific example of a setting in which (18) may be applied, consider the problem of determining the irradiance field generated in an infinite homogeneous medium by a beam-type source, such as that associated with powerful search lights or laser beams. The geometrical relations of the present example are summarized in Fig. 6.5. The source may be represented as a small sphere of radius r_o with surface radiance N_o and which is allowed to emit uniformly over a conical set Ξ_o of directions with central direction ξ_o. Thus Ξ_o may be all directions ξ such that $\xi \cdot \xi_o \geq \cos\theta_o$ where θ_o is the half angle opening of Ξ_o. By varying θ_o, the cone can represent everything from narrow beams (small θ_o) to uniform point sources ($\theta_o = \pi$).

With these geometrical preliminaries fixed, we now return to the discussion in Sec. 6.6 which developed the theory of primary scattered flux as source flux and which culminated in the formulas (67) through (69) of Sec. 6.6. We can

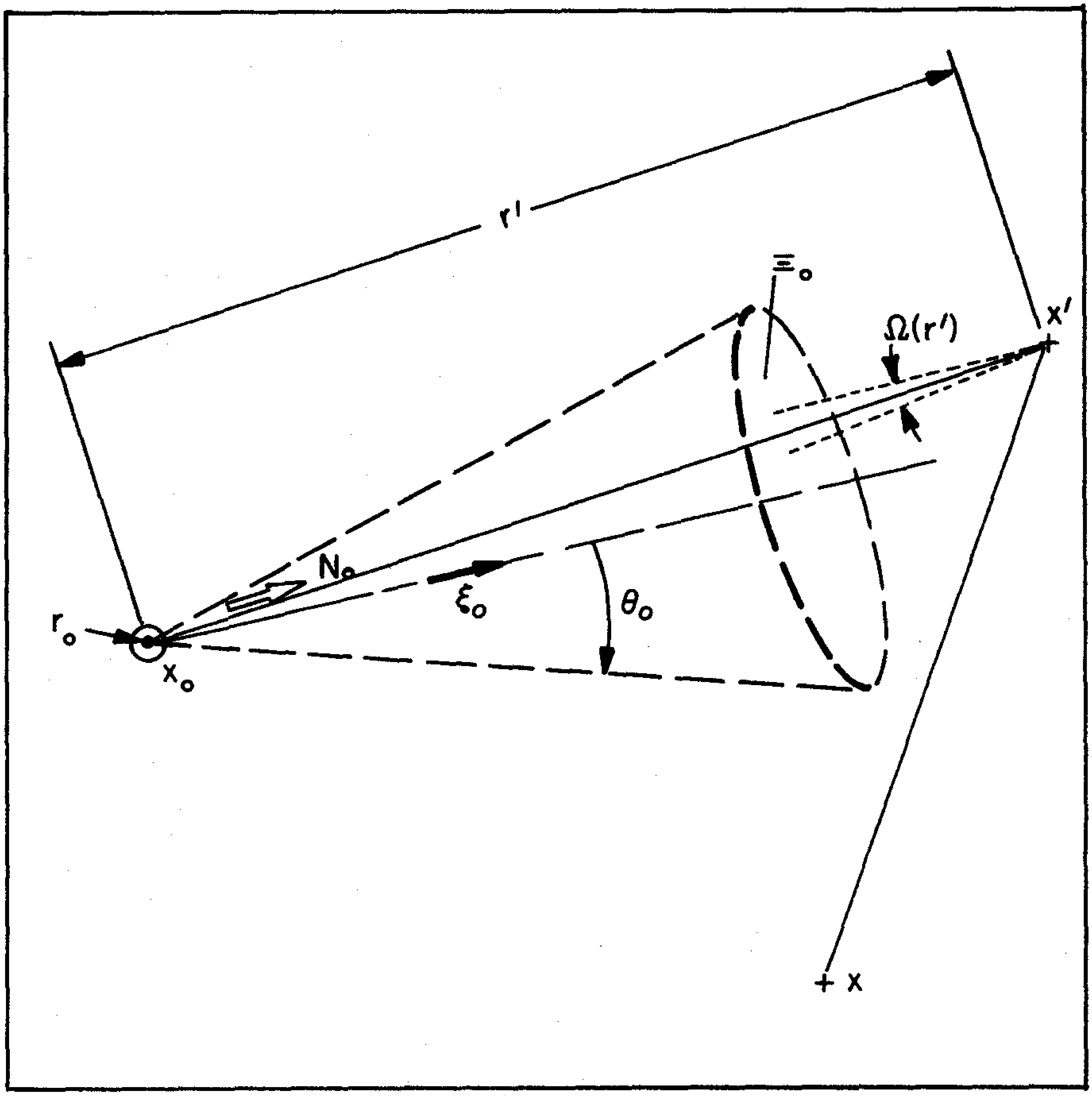

FIG. 6.5 Geometry for a nonisotropic point source of radiant flux in diffusion theory.

immediately adopt for our present purposes the formula (67) of Sec. 6.6 which describes the primary scalar irradiance $h^1_*(x')$ in terms of the inherent radiance N_0, the total scattering coefficient s, the beam transmittance $e^{-\alpha r'}$, and the solid angle $\Omega(r')$ subtended by the point source at point x'. (See Fig. 6.5.) Now $h^1_*(x')$ replaces $h_\eta(x')$ in (16) or $h_\eta(x_i)$ in (18). Thus (16) becomes:

$$h(x) = \rho N_0 \int_{X_0} e^{-\alpha|x'|} \, \Omega(|x'|) \, K_\varepsilon(x',x) \, dV_\alpha(x') \qquad (19)$$

and (18) becomes:

$$h(x) = \rho N_o \sum_{i=1}^{n} e^{-\alpha|x_i|} \Omega(|x_i|) K_\varepsilon(x_i,x) V_\alpha(C(x_i)) \quad (20)$$

In (19) the integration may be limited to the subset X_o of X defined by the cone Ξ_o of directions. Thus point x' is in X_o if and only if $x'/|x'|$ is in Ξ_o. In (20) the sum is over all cells $C(x_i)$ which partition X_o. Because of the exponentials and the solid angles $\Omega(|x_i|)$ in (20), the sums (for a given N_o) need not be extended over very many attenuation lengths within X_o before good estimates of h(x) can be made.

Semi-Infinite Medium with Boundary Point Source

The exact diffusion solution (16) holds for media which extend indefinitely far in all directions about the point source. Such a situation will hold more or less in natural waters when the source and observer are at relatively great depths (several attenuation lengths, say). However, if the source is relatively near the surface, the reflectance properties of the remaining thin layer of medium above the source would differ noticeably from that of an infinitely deep layer above the source, so that the scalar irradiance $h(\tau)$ at shallow depths in a light field induced by a point source near the boundary would differ markedly from that predicted by (16). Similar observations may be made for fogs and cloud banks in the atmosphere. In the present example, we summarize some results of exact diffusion theory which can predict $h(\tau)$ for relatively shallow depths in natural waters (or for points near flat cloud or fog boundaries) when the point source is on the boundary. The reflection effects of the air-water surface are not included in the present analysis and must be accounted for separately. In the second example below the results will be extended to the case of internal point sources. Both examples are based on the results by Elliott given in Ref. [88]. A generalization of the equations developed below and their appropriate place in the general theory of radiative transfer in media with internal sources, will be given in Sec. 7.13.

The starting point for the present discussion is equation (8) in which the medium X is now an infinitely deep homogeneous plane-parallel medium exhibiting isotropic scattering and with a point source of small positive radius r_o at depth $x = c \geq 0$. We shall use the terrestrially based reference system for natural hydrosols (cf. Sec. 2.4). Furthermore we use the unit source condition (9a) in (8).

Thus we start with (8), now in the form:

$$h(x) = \frac{1}{4\pi} \int (\delta(x'-x_o) + \rho h(x')) \frac{e^{-\tau(x,x')}}{\tau^2(x,x')} dV_\alpha(x') \quad (21)$$

where X_+ is the set of all x $(=(x_1,x_2,x_3))$ in the terrestrial coordinate frame such that $x_3 = z \geq 0$. The Dirac-delta function ξ in (21) is dimensionless, and is centered on the point x_0 $(=(0,0,c))$, $c \geq 0$. Furthermore, it is to be explicitly noted that for the remainder of this section all coordinates x_1, x_2, x_3 (hence all distances, areas, and volumes) are to be measured in units of optical length (cf. (5), (7)).

Now the procedure in Ref. [88] is to take the Fourier transform of (21) with respect to the variables x_1 and x_2 over an arbitrary horizontal plane at depth x_3 (=z). Thus let ω_1 and ω_2 be the spatial frequencies along the x_1 and x_2 directions and let us write:*

$$\text{"}f_0(z;\omega_1,\omega_2)\text{"} \quad \text{for} \quad \int_{X_z} h(x)\, e^{i(x_1\omega_1 + x_2\omega_2)}\, dA(x) \tag{22}$$

where X_z is the horizontal plane at depth z, and A is the area measure over X_z. Thus f_0 is the Fourier transform of h over X_z, and f_0 has the same dimensions as h. Therefore, applying the operator:

$$\int_{X_z} [\quad]\, e^{i(x_1\omega_1 + x_2\omega_2)}\, dA(x)$$

to each side of (21), we obtain:

$$f_0(z;\omega) = \frac{\rho}{2} \int_0^\infty \left[\frac{1}{\rho}\,\delta(z'-a) + f_0(z';\omega)\right] I(|z-z'|,\omega)\, dz' \tag{23}$$

where we have written:

$$\text{"}I(|z-z'|,\omega)\text{"} \quad \text{for} \quad \int_1^\infty \frac{e^{-|z-z'|t}}{t}\, J_0\left(\sqrt{\omega_1^2+\omega_2^2}\;|z-z'|\;\sqrt{t^2-1}\right) dt \tag{24}$$

---

*In the present exposition, we retain the Fourier transform conventions used in [88] in order to facilitate the study of the results therein.

where J_0 is a zero-order Bessel function, and where, for brevity, we have written:

$$\text{"}f_0(z;\omega)\text{"} \quad \text{for} \quad f_0(z,\omega_1,\omega_2) \tag{25}$$

The next step in the solution procedure is the observation that (23) can be solved using the Wiener-Hopf technique provided that $c = 0$, i.e., that the source is at the boundary. This solution procedure is quite intricate and beyond the immediate interests of the present work; therefore the interested reader is referred to Ref. [88] for details and further references. The main results of the present example may be understood without recourse to the solution details. We need only observe that the required scalar irradiance is obtained from the solution $f_0(\cdot;\omega)$ of (23) by means of the following integration which is the inverse Fourier transformation to that in (22):

$$h(x) = \frac{1}{2\pi} \int_0^\infty f_0(z,\omega)\ \omega\ J_0(\omega r)\ d\omega \tag{26}$$

in which:

$$x = (x_1, x_2, z)$$

and:

$$\omega^2 = \omega_1^2 + \omega_2^2 \tag{27}$$

$$r^2 = x_1^2 + x_2^2 \tag{28}$$

Since $h(x)$ depends only on depth z and the radial distance r, we agree to write:

$$\text{"}h(z,r)\text{"} \quad \text{for} \quad h(x) \tag{29}$$

Figure 6.6 depicts the geometrical details of the case where the point source is at the boundary. Observe that the medium is divided into region A (shaded) and conical region B (unshaded). It is found that $h(z,r)$ for points $x = (x_1, x_2, z)$ in region A is approximated by the relation:

$$h(z,r) = \frac{\sqrt{3}\ h_\eta\ \psi_1(z)}{2\pi\alpha r^3}\ e^{-\kappa_0 r}(1 + \kappa_0 r) \tag{30}$$

(Valid in region A, Fig. 6.6.)

where in turn $\psi_1(z)$ is evaluated in [172] and is tabulated in Table 4, and κ_0 is given in Table 2. Table 4 may be extended,

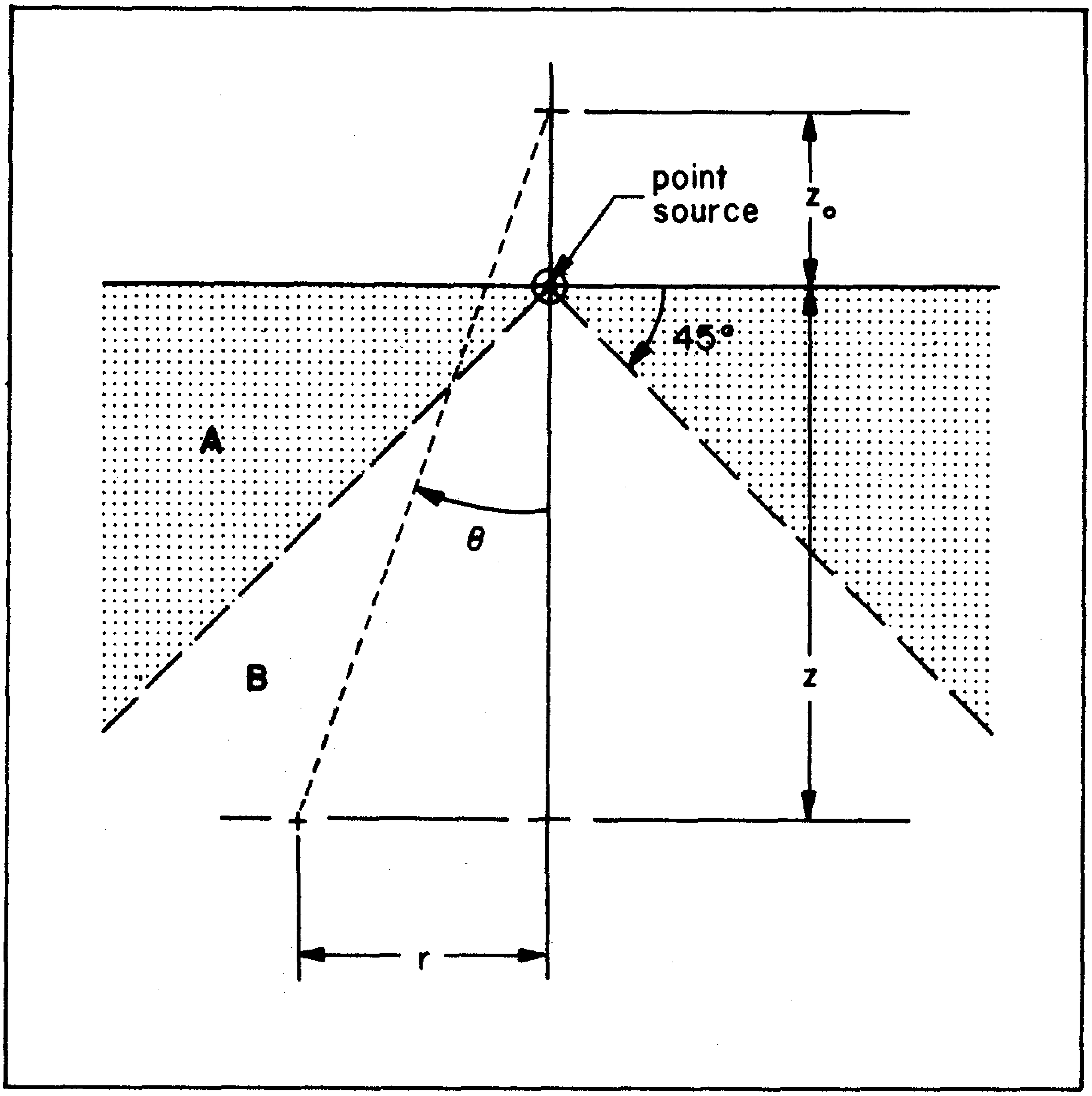

FIG. 6.6 Domains of validity of approximate solutions (30) and (31).

if necessary, using the eddingtonian approximation to ψ_1:

$$\psi_1(z) = \left(z + \frac{17}{24}\right)\left[\frac{2 + 3z - (E_2(z) - \frac{3}{2}\, E_3(z))}{2 + 3z - 3\,(E_4(z) - \frac{3}{2}\, E_5(z))}\right]$$

The functions $E_n(z)$ are the exponential integrals

$$\int_1^\infty u^{-n}\, e^{-zu}\, du \quad ,$$

and are tabulated. The farther the point x $(=(x_1, x_2, x_3))$ in region A is from the dashed dividing lines between regions A and B, the better the approximation (30).

TABLE 4

Evaluation of $\psi_1(z)$

z	$z + z_o$	$\psi_1(z)$
0	0.7104	0.5773
0.01	0.7204	0.5982
0.02	0.7304	0.6154
0.03	0.7404	0.6312
0.05	0.7604	0.6607
0.1	0.8104	0.7279
0.2	0.9104	0.8495
0.3	1.0104	0.9633
0.4	1.1104	1.0731
0.5	1.2104	1.1803
0.6	1.3104	1.2858
0.7	1.4104	1.3901
0.8	1.5104	1.4935
0.9	1.6104	1.5963
1.0	1.7104	1.6985
1.2	1.9104	1.9019
1.5	2.2104	2.2051
2.0	2.7104	2.7079
2.5	3.2104	3.2092
3.0	3.7104	3.7098
3.5	4.2104	4.2101
4.0	4.7104	4.7102

The error of the approximation by (30) is of the order of magnitude of $|z^3/r^5|$ and (30) is applicable when ρ is 0.6 or more.

Furthermore, it is found that $h(z,r)$ for points x $(=(x_1,x_2,z))$ in region B is approximated by the relation:

$$h(z,r) = \frac{\sqrt{3}\, h_\eta \cos\theta}{2\pi\alpha d^2} e^{-\kappa_o d}(1 + \kappa_o d) \tag{31}$$

(Valid in region B, Fig. 6.6.)

where we have written:

$$\text{"d" for } \sqrt{r^2 + (z+z_o)^2} \tag{32}$$

and where:

$$\tan\theta = \frac{r}{z + z_o} \tag{33}$$

and:

$$z_o = 0.7104 \tag{34}$$

This approximation improves with the distance of x ($=x_1, x_2, z$)) in region B from the dashed dividing lines between regions A and B. The error of approximation by (31) is of the order of magnitude of $|1/d^5|$ and (31) is applicable when ρ is 0.6 or more.

A study of (30) and (31) readily shows the effect on h(x) of the presence of the boundary at depth z = 0. Suppose for the moment that $\kappa_o = 0$ (no absorption case). Then in region A of Fig. 6.6, and for fixed z, the scalar irradiance falls off as the inverse cube of the distance r from the symmetry axis of the field, whereas in region B, which is relatively farther removed from the boundary than region A, the scalar irradiance falls off only as the inverse square of the distance d. The fixed number z_o (known as the "extrapolation length") in (34) arises in the correct adjustment of boundary conditions of the present problem.

Semi-Infinite Medium with Internal Point Source

The results of the preceding example will now be extended to the case of a semi-infinite homogeneous medium with a point source at $x_o = (0,0,c)$, $c \geq 0$, i.e., with a point source in the interior of the medium rather than on the boundary. Let us denote the solution of (21) for this case by "$f_c(z;\omega)$". Hence, when c = 0 we are to have $f_o(z;\omega)$ of (23) back once again, and f_c is to be a proper generalization of f_o. Now assume a general point source condition h_η/α (cf. (9a)). Then the functional relations connecting f_c and f_o, as derived by Elliott [88] are of the form:

$$f_c(z,\omega) = f_o(|z-c|,\omega) + \frac{s}{h_\eta}\int_0^z f_o(t,\omega)\, f_o(t+c-z,\omega)\, dt, \quad z \leq c \tag{35}$$

$$f_c(z,\omega) = f_o(|z-c|,\omega) + \frac{s}{h_\eta}\int_0^c f_o(t,\omega)\, f_o(t-c+z,\omega)\, dt, \quad z \geq c \tag{36}$$

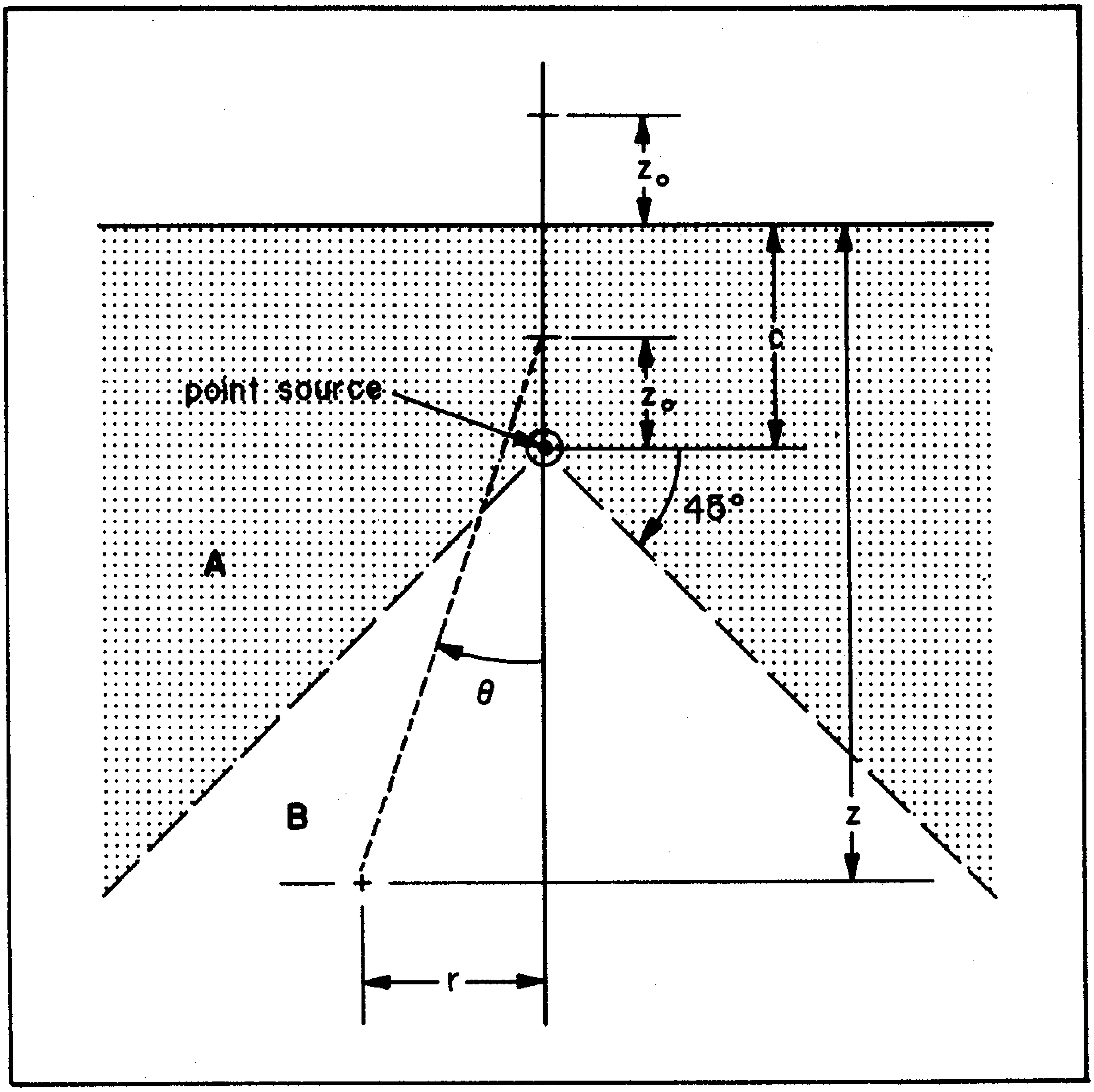

FIG. 6.7 Domains of validity of approximate solutions (38) through (40).

Once $f_c(z,\omega)$ is obtained using (35) or (36), h(z,r) can be obtained by means of the inversion formula:

$$h(z,r) = \frac{1}{2\pi}\int_0^\infty f_c(z,\omega)\,\omega J_o(\omega r)\,d\omega \tag{37}$$

which is simply (26) now with f_c in place of f_o. A few observations on these functional relations will be made below, but for the present we go on to their immediate consequences. Figure 6.7 depicts the semi-infinite medium with point source at (0,0,c). The medium is divided into two regions with the shaded region A and the conical region B, exactly analogously to the partition depicted in Fig. 6.6. Corresponding to (30) we now have the approximate solution:

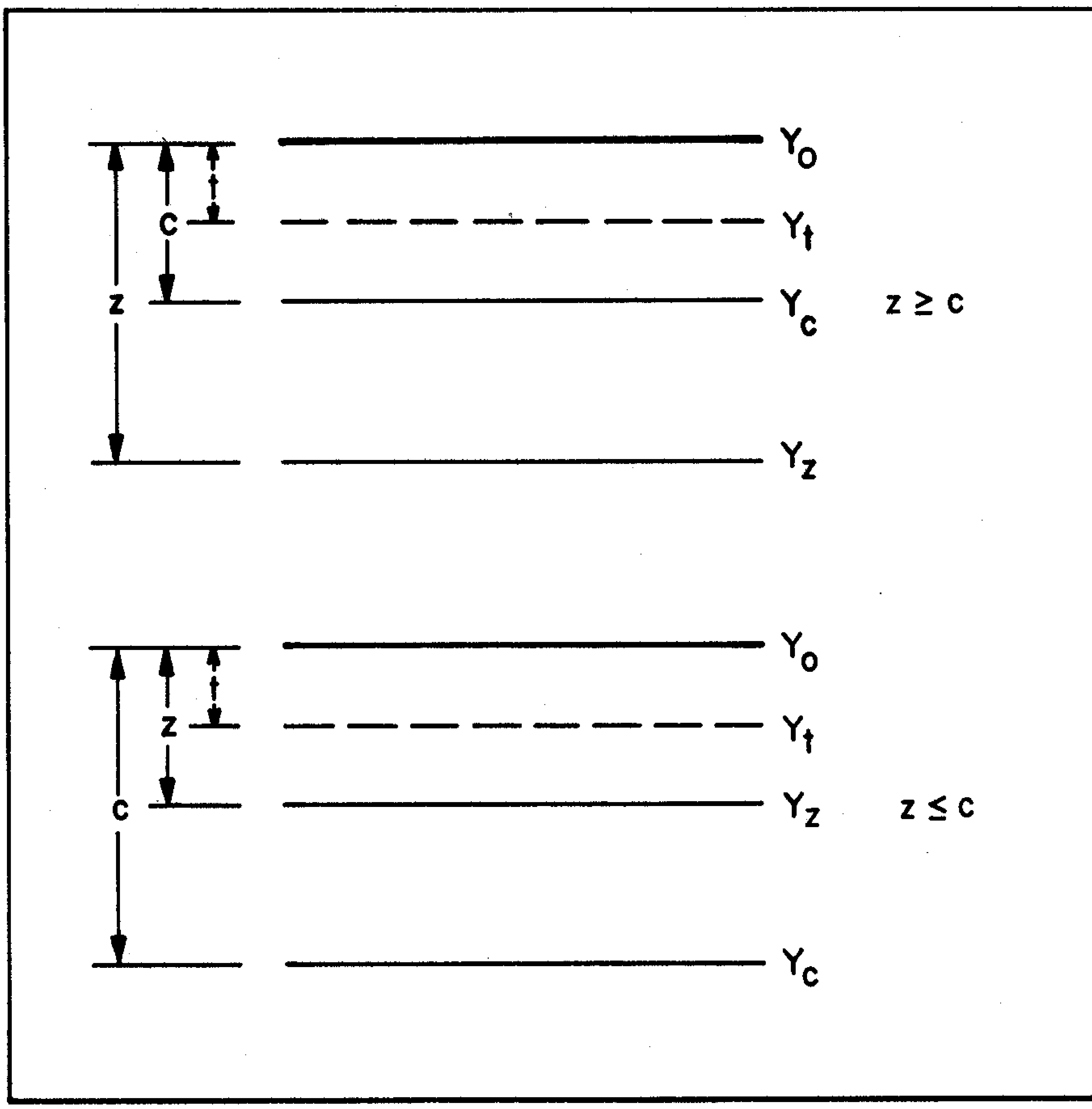

FIG. 6.8 Relative placement of source (c) and observation (z) levels in (35) and (36).

$$h(z,r) = \frac{\sqrt{3}h_\eta}{2\pi\alpha r^3}\left[\psi_1(c-z) + \sqrt{3}\int_0^z \Big(\psi_1(t) + \psi_1(t+c-z)\Big)\,dt\right] e^{-\kappa_o r}(1+\kappa_o r) \quad , \quad \text{for } z \leq c \quad . \tag{38}$$

$$h(z,r) = \frac{\sqrt{3}h_\eta}{2\pi\alpha r^3}\left[\psi_1(z-c) + \sqrt{3}\int_0^c \Big(\psi_1(t) + \psi_1(t+z-c)\Big)\,dt\right] e^{-\kappa_o r}(1+\kappa_o r) \quad , \quad \text{for } z \geq c \quad . \tag{39}$$

(Valid in region A, Fig. 6.7.)

All the terms occurring in (38) and (39) were defined in (30). The ranges of integration may be visualized with the help of Fig. 6.8. Observe how (39) reduces to (30) when $c = 0$. The errors of approximation are on the order of $|c^3/r^5|$ for (38) and $|z^3/r^5|$ for (39). The approximations (38), (39) are applicable for media with $\rho = 0.6$ or more.

Corresponding to (31) we now have:

$$h(z,r) = \frac{\sqrt{3}h_\eta}{2\pi\alpha d^2}(1 + c\sqrt{3})\cos\theta\, e^{-\kappa_0 d}(1 + \kappa_0 d) \tag{40}$$

(Valid in region B, Fig. 6.7.)

Observe in this instance, also, how (40) reduces to its limiting case (31) for $c = 0$, where now in (40) we have written:

$$\text{"}d\text{" for } \sqrt{r^2 + (z + z_0 - c)^2} \tag{41}$$

and also where

$$\tan\theta = \frac{r}{z + z_0 - c} \tag{42}$$

The approximation (40) holds for large $|z-c|$ and has an error on the order of magnitude of $|c/d^3|$, for media with $\rho = 0.6$ or more.

Observations on the Functional Relations for f_c and f_o

The various solutions displayed above for $h(z,r)$ in a semi-infinite medium are of great interest for two reasons. The first reason is clear enough: They supply additional information on the behavior of $h(x)$ in deep plane-parallel media in which there are point sources near the boundaries. The second reason for interest in these solutions does not exist so much on a practical level as on a theoretical or conceptual level. This interest centers on the *form of the functional relations* (35) and (36) which seem to hold considerable importance for radiative transfer theory. These two remarkable relations show how to connect the point source solution for the case $c = 0$ with that for the case $c > 0$. The general form of the functional relations (35) and (36) are those of the relations usually found by the techniques of invariant imbedding, the techniques growing out of the classical invariance principles of Chandrasekhar. It will be shown in Sec. 7.13 how the general counterparts of (35) and (36) for radiance fields may be deduced from the invariant imbedding relations (cf. also examples 2, 3, 5 of Sec. 3.9). As a result of the derivations in Sec. 7.13, there will be a unified set of analytical techniques for solving internal-source problems in general optical media.

<< Table of Contents << Title Page >> Next Page >>

6.8 Bibliographic Notes for Chapter 6

The discussions of Sec. 6.1 leading to (36) of that section are based on some elementary properties of complete orthonormal families of functions, which in turn find their rightful place in Hilbert space theory, or general vector space theory. For an exposition of these ideas, see, e.g., [104]. The isolation of the two properties, namely: the *finite recurrence property* of the orthonormal family and the *isotropy property* of the medium led to the finite forms (26) of the abstract harmonic equations in Sec. 6.2. This explicit delineation of the necessary properties to be held jointly by orthonormal families and optical media, which lead to the abstract harmonic equations (26) of Sec. 6.2, appears to be new.

The exposition of the classical spherical harmonic method in Sec. 6.3 is based on that of Refs. [175] and [314]. The solution procedures of the classical spherical harmonic equations for plane-parallel media in Sec. 6.4 are based on modern algebraic methods in differential equation theory, such as those in [47]. Some innovations in numerical procedures in the spherical harmonic method may be found in [323] and [325]. The manner of approach to diffusion theory in Sec. 6.5 is dictated by the specific needs and outlook of geophysical radiative transfer theory. The classification of diffusion processes in Sec. 6.5 is of course only partially complete; a systematic investigation of such classified processes appears to be of some interest to radiative transfer theory, and offers interesting physically based problems in partial differential equation theory.

The general solutions of the classical diffusion equations in the opening paragraphs of Sec. 6.6 are widely known, useful formulas for scalar irradiance. The various primary scattered flux source methods and those based on higher ordered scattered flux sources in the latter part of Sec. 6.6 offer some novelty in the otherwise quite thoroughly formed classical method of treatment of the diffusion of light through scattering media. Furthermore, the particular needs of hydrologic optics and meteorologic optics has caused some emphasis to be placed on the representation of the radiance distribution $N(x,\cdot)$ throughout diffusing media. This resulted in derivations of formulas for $N(x,\xi)$ in general diffusion contexts, such as (29) of Sec. 6.5; and (14) and (40) of Sec. 6.6, which do not appear to be too widely known.

The solutions of the exact diffusion equations in Sec. 6.7 for the case of infinite media are based on the work in [40]. This work also contains many useful tables and graphs of associated solutions. The theory of semi-infinite media with point sources is relatively unexplored. However, reference [88] forms a definitive beginning of such a theory, and the latter half of the discussions in Sec. 6.7 are based on the results of [88].