7.6-7.8. An example of an actual numerical computation of the R and T operators based on the functional relations of Section 7.1 is given in Section 7.9 for the case of homogeneous source-free plane-parallel media with isotropic scattering. This numerical method is generalized in Section 7.10. In Section 7.11 the preceding results are consolidated and generalized. Section 7.12 is concerned with the conditions of homogeneity and isotropy and related ideas, which will help simplify theoretical and numerical work and help classify optical media in general. Section 7.13 develops some deep connections among the various standard and invariant imbedding operators within media with internal sources. Finally, in Section 7.14, it is observed how the Laplace and Fourier transform techniques, which have proved so useful in the classical formulation of the transport phenomena, can be combined with the invariant imbedding approach to simplify the functional relations of the latter approach and to encourage their applications to time-dependent problems, point source problems, and other transport problems which ordinarily involve higher numbers of variables.

7.1 Differential Equations Governing the Steady State, R and T Operators

In Sections 3.6 and 3.7 we saw how the R and T operators of plane-parallel (and other) media were used in both theory and practice to determine light fields in natural optical media. In this section we show how the four R and T operators generally associated with stratified plane-parallel media may be determined from knowledge of the volume scattering and volume attenuation functions within the medium. This will be done by deriving the differential equations governing the operators as a function of the thickness of the medium. Thus, if we know the R and T operators for a given layer of material the differential equation will show how the operators change by addition of a very thin layer of the material to the given layer. By letting the given layer grow continuously from some given thickness, we will therefore know how its R and T operators evolve from their given values, and how they may be computed in both theory and practice. We turn now to the details of the derivations.

Local Forms of the Principles of Invariance

We begin the derivations by casting the equation of transfer for a stratified plane-parallel medium into a pair of equations which are strongly reminiscent of the two main principles of invariance for such media (Ex. 3, Section 3.7); the main difference being the presence of derivatives of N in the new equations. Thus under the assumption that all functions (radiance distributions and optical properties) depend only on depth y in the medium (cf. Fig. 7.1) Equation (3) of Sec. 3.15 becomes:

$$(-\xi\cdot\mathbf{k})\ \frac{dN(y,\xi)}{dy} = -\alpha(y)N(y,\xi) + N_*(y,\xi) \tag{1}$$

where

$$N_*(y,\xi) = \int_{\Xi} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') \quad . \tag{2}$$

Here $\mathbf{k}$ is the unit outward normal to the medium, ξ is an arbitrary direction in Ξ, and $a \le y \le b$.

To obtain the requisite form of the equation of transfer we restrict the radiance distribution $N(y,\cdot)$ to the two halves Ξ_+ and Ξ_- of Ξ, (cf. Fig. 7.1, and Sec. 2.4). We denote the restriction of $N(y,\cdot)$ to Ξ_+ as usual by "$N_+(y)$", and the restriction of $N(y,\cdot)$ to Ξ_- by "$N_-(y)$"; $a \le y \le b$. Next we write:

$$\text{"}\rho(y)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ \ ]\sigma(y;\xi';\xi)\ d\Omega(\xi') \tag{3}$$

in which ξ is in Ξ_-; and :

$$\text{"}\tau(y)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ \ ]\sigma(y;\xi';\xi)\ d\Omega(\xi') - \frac{1}{|\xi\cdot\mathbf{k}|}\alpha(y) \tag{4}$$

in which ξ is in Ξ_+, and in both of which $a \le y \le b$. Furthermore, we assume the medium to be isotropic, so that $\sigma(y;\xi';\xi)$ depends only on the value $\xi'\cdot\xi$ for each choice of ξ' and ξ. Hence for each ξ,y, the values of the integrals in each definition in (3) and (4) are unchanged if Ξ_+ is replaced by Ξ_-, and Ξ_- by Ξ_+ throughout. The operator $\rho(y)$ is the *local reflectance* operator and $\tau(y)$ is the *local transmittance* operator. In discussions where it is necessary to consider the possibility of anisotropic media, the operators $\rho(y)$ and $\tau(y)$ must be defined with specific reference to the domains of integration in (3) and (4). Thus "$\rho(y)$" in (3) becomes "$\rho_+(y)$" and "$\rho_-(y)$" denotes the same kind of integral but over Ξ_- with ξ in Ξ_+. Similarly (4) will define what we will call "$\tau_+(y)$" and a similar integral over Ξ_- with ξ in Ξ_- will be denoted by "$\tau_-(y)$". (See, Ref. [251], and Sec. 7.13 below.)

The local operators $\rho(y)$ and $\tau(y)$ are used as follows: In (1) let ξ be in Ξ_+, so that $N(y,\xi) = (N_+(y))(\xi)$. Furthermore, writing $N_*(y,\xi)$ in (1) as:

$$N_*(y,\xi) =$$

$$= \int_{\Xi_+} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') + \int_{\Xi_-} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') \ ,$$

we divide through each side of (1) by $|\xi\cdot\mathbf{k}|$, apply (3) and (4), and end up with:

$$-\frac{dN_+(y)}{dy} = N_+(y)\tau(y) + N_-(y)\rho(y) \quad . \tag{5}$$

Similarly, now with ξ in Ξ_-, so that $N(y,\xi) = (N_-(y))(\xi)$, we obtain:

$$\frac{dN_-(y)}{dy} = N_-(y)\tau(y) + N_+(y)\rho(y) \quad . \tag{6}$$

Equations (5) and (6) are the desired *local (or integrodifferential) forms* of the principles of invariance for plane-parallel media. The striking similarity between the pair (5), (6) and the pair I, II of Ex. 3, Sec. 3.7, is evident once "T" is paired with "τ" and "R" with "ρ". These equations can be put into a more compact form by first writing:

$$\text{"}\mathcal{K}(y)\text{"} \quad \text{for} \quad \begin{pmatrix} -\tau(y) & \rho(y) \\ -\rho(y) & \tau(y) \end{pmatrix} \tag{7}$$

and

$$\text{"}N(y)\text{"} \quad \text{for} \quad (N_+(y), N_-(y)) \quad . \tag{8}$$

Then (5), (6) become:

$$\frac{dN(y)}{dy} = N(y)\mathcal{K}(y) \tag{9}$$

which is an alternate and equivalent rendition of the equation of transfer (1) via the local forms (5), (6) of the principles of invariance. We shall return to this form of the equation of transfer in subsequent sections, wherein it will play an important role in determining the radiance functions. For the present, we continue the derivation of the desired functional relations for the R and T operators.

The Differential Equations for R and T

The main step in the derivation of the differential equations for the R and T operators will now be taken. We begin with the operator R(y,b) for an arbitrary subslab X(y,b) of the plane-parallel medium X(a,b), $a \leq y \leq b$. (We now are using the notation of Section 3.7). We let $N_-(a)$ be an arbitrary incident radiance function over the plane upper boundary of X(a,b) at level a. We set $N_+(b) = 0$, and assume that no sources of radiant flux are within X(a,b). The two main principles of invariance for an arbitrary subslab X(x,z), $a \leq x \leq y \leq z \leq b$, of X(a,b) are as given in Ex. 3, Sec. 3.7:

$$\text{I.} \quad N_+(y) = N_+(z)T(z,y) + N_-(y)R(y,z)$$

$$\text{II.} \quad N_-(y) + N_-(x)T(x,y) + N_+(y)R(y,x) \quad .$$

We next set $z = b$ in principle I, which with the present boundary lighting conditions becomes:

$$N_+(y) = N_-(y)R(y,b) \quad . \tag{10}$$

Equation (10) states that the upward radiance distributions at level y in $X(a,b)$ consist of the reflected flux from $X(y,b)$ induced by the downward radiance distributions entering $X(y,b)$ at level y. We next take the derivative of each side of (10) with respect to y, thus:

$$\frac{dN_+(y)}{dy} = \frac{d}{dy}\left(N_-(y)R(y,b)\right)$$

$$= \frac{dN_-(y)}{dy} R(y,b) + N_-(y) \frac{dR(y,b)}{dy} \tag{11}$$

where we have written:

$$\text{"}\frac{dR(y,b)}{dy}\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ ]\frac{d}{dy}R(y,b;\xi';\xi)\,d\Omega(\xi') \tag{12}$$

and where $R(y,b;\xi';\xi)$ is defined in Example 3 of Sec. 3.7 (cf. also (8)-(11) of Sec. 3.6). Therefore $dR(y,b)/dy$ in (11) is an integral operator acting on $N_-(y)$. Further, $R(y,b)$ in (11) acts on the function $dN_-(y)/dy$. Thus all terms of (11) are well defined. Now, we are interested in $R(a,b)$, which we may envision as the limit of $R(y,b)$ as $y \to a$. Hence in (11) we let y approach a. Thus we are led to consider

$$\lim_{y\to a} \frac{dN_+(y)}{dy}$$

which by (5) is given as:

$$\lim_{y\to a} \frac{dN_+(y)}{dy} = -\left[N_+(a)\tau(a) + N_-(a)\rho(a)\right] \quad . \tag{13}$$

By principle III of Example 3, Sec. 3.7, which we repeat here for convenience:

$$\text{III.} \quad N_+(a) = N_+(b)T(b,a) + N_-(a)R(a,b) \quad ,$$

equation (13) becomes:

$$\lim_{y\to a} \frac{dN_+(y)}{dy} = -N_-(a)\left[R(a,b)\tau(a) + \rho(a)\right] \tag{14}$$

where we have used the boundary condition that $N_+(b) = 0$. In a similar manner, we find that the limit of the remaining derivative of $N_-(y)$ in (11) can be represented via (6) and principle III as:

$$\lim_{y\to a} \frac{dN_-(y)}{dy} = N_-(a)\left[\tau(a) + R(a,b)\rho(a)\right] . \qquad (15)$$

Let us agree to write:

$$\text{"}\frac{\partial R(a,b)}{\partial a}\text{"} \quad \text{for} \quad \lim_{y\to a} \frac{dR(y,b)}{dy} . \qquad (16)$$

Then applying the limit operation, $\lim_{y\to a}$, to each side of (11), we have:

$$-N_-(a)\left[R(a,b)\tau(a)+\rho(a)\right] =$$

$$= N_-(a)\left[\tau(a)+R(a,b)\rho(a)\right]R(a,b) + N_-(a)\frac{\partial R(a,b)}{\partial a} . \qquad (17)$$

This equation holds for every incident radiance function $N_-(a)$. Hence we can formally cancel "$N_-(a)$" from each side. After rearranging the resultant operator equation, we have:

I'

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a)+\tau(a)R(a,b)+R(a,b)\tau(a)+R(a,b)\rho(a)R(a,b) \qquad (18)$$

Equation I' is the requisite differential equation for $R(a,b)$ as a function of the depth parameter a. Observe that I' has the form of a Riccati equation for the operator $R(a,b)$ with known operators $\rho(a)$ and $\tau(a)$. Thus, (18) is in principle solvable for $R(a,b)$ with the initial condition: $R(a,b) = 0$ whenever $a = b$, (cf. (30) of Sec. 3.7). Hence from I' we have:

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a) \qquad (19)$$

for $a = b$, showing that the initial rate of growth of $R(a,b)$ is given directly by the local reflectance operator, $\rho(a)$, i.e., the integral operator with the volume scattering function as kernel.

The determination of the differential equation for $T(a,b)$ may be made next, starting with principle II in which $x = a$, the result being:

$$N_-(y) = N_-(a)T(a,y) + N_+(y)R(y,a) . \qquad (20)$$

Taking the derivative of each side with respect to y:

$$\frac{dN_-(y)}{dy} = N_-(a)\,\frac{dT(a,y)}{dy} + \frac{dN_+(y)}{dy}\,R(y,a) + N_+(y)\,\frac{dR(y,a)}{dy} \tag{21}$$

Here we have written:

$$\text{"}\frac{dT(a,y)}{dy}\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot k|}\int_{\Xi_-}[\;]\frac{d}{dy}\,T(a,y;\xi';\xi)\;d\Omega(\xi') \tag{22}$$

and where $T(a,y;\xi';\xi)$ is defined in Example 3 of Sec. 3.7 (cf. also (8)-(11) of Sec. 3.6). Thus $dT(a,y)/dy$ in (21) is an integral operator acting on $N_-(a)$. Now in (21) we consider:

$$\lim_{y\to b}\frac{dN_-(y)}{dy}$$

which by (6) is given as:

$$\lim_{y\to b}\frac{dN_-(y)}{dy} = [N_-(b)\tau(b) + N_+(b)\rho(b)] \quad . \tag{23}$$

From principle IV of Example 3 Sec. 3.7:

$$\text{IV} \quad N_-(b) = N_-(a)T(a,b) + N_+(b)R(b,a) \;,$$

which, applied to (23) yields:

$$\lim_{y\to b}\frac{dN_-(y)}{dy} = N_-(a)T(a,b)\tau(b) \quad . \tag{24}$$

In a similar way we obtain for the derivative of $N_+(y)$ in (21):

$$\lim_{y\to b}\frac{dN_+(y)}{dy} = -N_-(a)T(a,b)\rho(b) \quad . \tag{25}$$

Writing:

$$\text{"}\frac{\partial T(a,b)}{\partial b}\text{"} \quad \text{for} \quad \lim_{y\to b}\frac{dT(a,y)}{dy} \;, \tag{26}$$

we here apply the limit operator $\lim_{y\to b}$ to each side of (21), the result being:

$$\boxed{\text{II}' \quad \frac{\partial T(a,b)}{\partial b} = T(a,b)[\tau(b) + \rho(b)R(b,a)]} \tag{27}$$

in which we have used the fact that $N_-(a)$ is arbitrary. This shows that once $R(a,b)$ is known, the operator $T(a,b)$ is obtainable by a simple quadrature.

The pattern of derivation of the differential equations is now clear. By using next the second versions of principles III and IV in Example 3 of Sec. 3.7, together with (5) and (6) we arrive at:

$$\text{III}' \quad \frac{\partial R(a,b)}{\partial b} = T(a,b)\rho(b)T(b,a) \tag{28}$$

$$\text{IV}' \quad -\frac{\partial T(a,b)}{\partial a} = [\tau(a)+R(a,b)\rho(a)]T(a,b) \tag{29}$$

Discussion of the Differential Equations

Statements I'-IV' above are the desired differential equations for the R and T operators associated with the plane-parallel medium X(a,b). Observe how I' is autonomous with respect to R(a,b). Thus, as we already observed, I' in principle can yield R(a,b), starting with the initial condition R(a,b) = 0. By reversing "a" and "b" in I', (i.e., by literally turning X(a,b) upside down in the given coordinate system) we can also obtain R(b,a). Observe that if $\rho(y)$ and $\tau(y)$ vary with depth y in X(a,b), we generally will have $R(a,b) \neq R(b,a)$, i.e., the R operator will exhibit *polarity*. If X(a,b) is homogeneous, then R(a,b) = R(b,a) and clearly depends only on the difference b-a of the depth parameters. (Recall, we have assumed at the outset that X(a,b) is isotropic.) Once R(a,b) and R(b,a) have been found, T(a,b) and T(b,a) both follow from II' using first R(a,b) then R(b,a) by reversing "a" and "b" in II'. If polarity is the case for R-operators, then generally, the T operator will possess polarity also. Thus I' and II' are in principle sufficient to determine the four R and T operators. However, it is interesting to note that I'-IV' are sufficient, as they stand, to determine in principle all four operators R(a,b), T(a,b), T(b,a), R(b,a) in that order, by successively using I', IV', III', II', in corresponding order. Alternatively, the equations may be solved in the order I', IV', II', III'. For the general forms of these observations, in the context of general media the reader may consult section 25 and other relevant sections of Ref. [251].

Equations I'-IV' constitute a wealth of intuitive information about light fields in scattering media neatly summarized in symbolic form, and which the reader is invited to discover. Thus I' and III' considered together show the two distinct modes of growth of R(a,b) when the medium is altered by varying the parameter a, and then the parameter b. In other words R(a,b) grows differently when layers are added to X(a,b) from below, than when layers are added from above. The precise manner of growth in each case is clearly discernable from each differential equation and can be pictured in terms of the interaction of X(a,b) with an infinitely thin layer added to X(a,b), e.g., at level a, whose reflectance and transmittance are $\rho(a)$ and $\tau(a)$, respectively. The growth of $R(a,b)$ when a is varied is far more complex than when b is varied. Unfortunately this more complex growth is necessary to contend with in the task of determining $R(a,b)$. Remarks of a similar nature can be made about the general growth patterns of $T(a,b)$ using II' and IV'. In the case of $T(a,b)$ the difference of growth rates, depending on whether a or b is varied, are less subtle than that of $R(a,b)$, and rest mainly in the order of application of the operators in the square brackets with respect to $T(a,b)$.

Functional Relations for Decomposed Light Fields

Some radiative transfer investigations are simplified if one is able to treat separately the reduced and diffuse components of the radiance field. Thus in the classical researches of Chandrasekhar, the computations were limited to computing the diffuse radiance transmitted through plane-parallel media. In addition, our discussions of the point source problem were facilitated in Sec. 6.6 by adopting for study not N but the diffuse component N^* of N. Furthermore, as noted in the *Remarks on the Interaction Method* in Sec. 3.18, the AC property of general interaction operators is easily shown to hold for those operators whose response functions describe diffuse radiant flux, i.e., radiant flux which has been scattered at least once. With such observations in mind we are motivated to study some of the salient properties of the *decomposed* R and T operators for plane-parallel media, in particular the principles of invariance (both local and global) which govern them, and the differential equations they satisfy. The extensions of the results of the present discussion to more general geometries is straightforward and the present techniques are presented so as to readily serve as the prototype for such extensions.

We shall work with the setting already established and used in carrying out the discussion from (1) to (29) above. Thus Fig. 7.1 will represent the present geometrical setting. Now, the first step in the decomposition of the R and T operators is to decompose the light field at general level y into its reduced and diffuse components. The basis for this decomposition rests in (5), (6) of Sec. 3.13.

Thus, $N_-(y)$, e.g., may be written:

$$N_-(y) = N_-^o(y) + N_-^*(y) \quad , \tag{30}$$

for every y, $a \le y \le b$, where $N_-^o(y)$ is the reduced (or residual) radiance distribution over the directions of Ξ_- at level y. $N_-^*(y)$ is the diffuse radiance distribution over the same direction set and at the same level. A similar decomposition holds for $N_+(y)$. The incident radiance distributions $N_-(a)$ and $N_+(b)$ on the slab $X(a,b)$ will, by convention, be of reduced form, i.e., we will assume $N_-^*(a) = 0$ and $N_+^*(b) = 0$. (See the Principle of Relative Scattering Order in Sec. 22 of Ref. [251]). Hence $N_-^o(a)$ and $N_+^o(b)$ serve as the incident radiance distributions on $X(a,b)$.

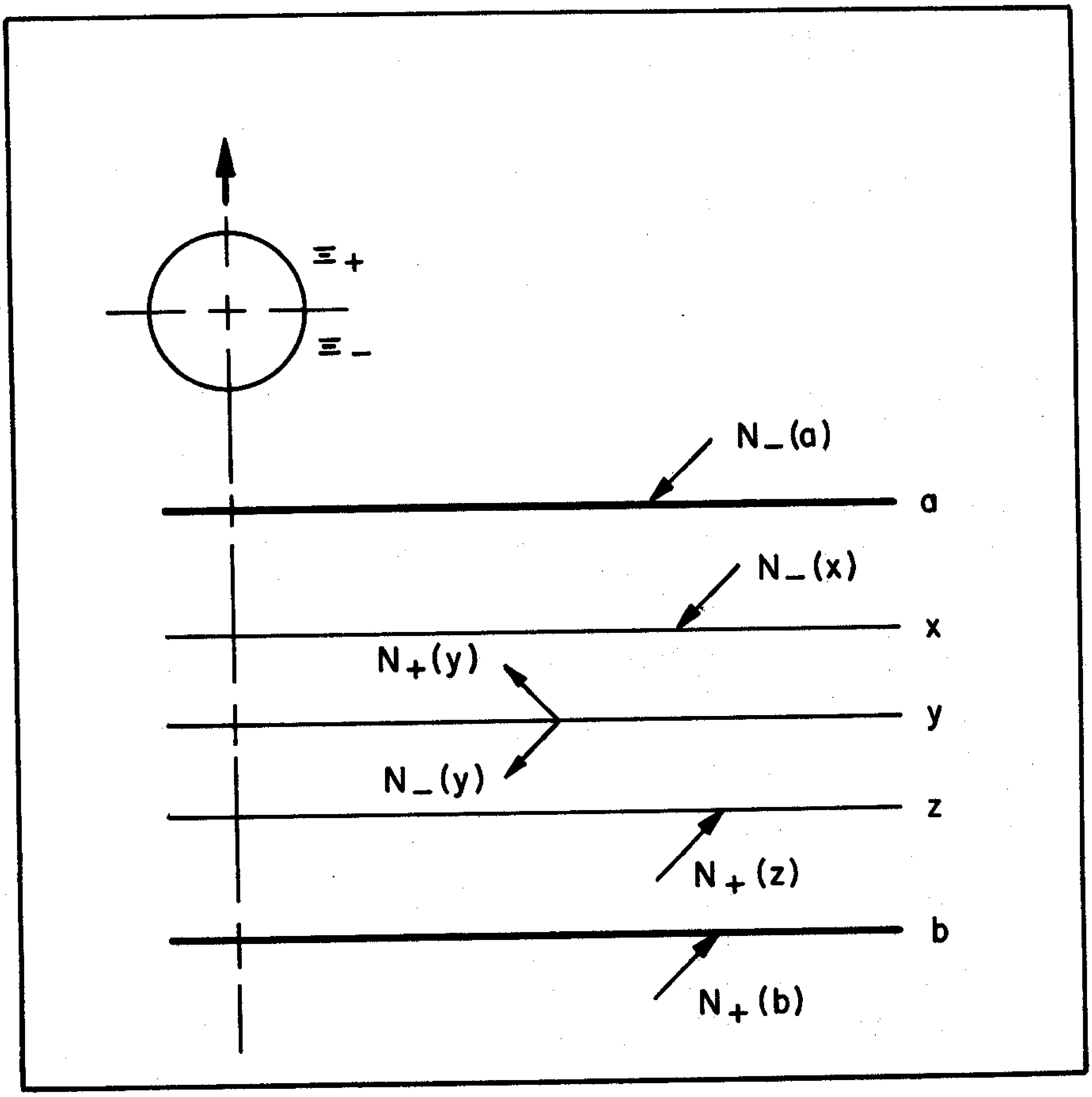

FIG. 7.1 Plane-parallel setting for the local and global forms of the Principles of Invariance.

The connection between the initial radiance function $N_-^o(a)$ over the upper boundary of X(a,b) and the residual radiance function $N_-^o(y)$ over level y within X(a,b) is readily established, using the results (4) of Sec. 3.10 and (3) of Sec. 3.11. Thus we have in general:

$$N_-^o(z) = N_-^o(x)T^o(x,z) \quad , \tag{31}$$

where we have written:

$$\text{"}T^o(x,z)\text{"} \quad \text{for} \quad \int_{\Xi_-} [\ ]T_r(p',\xi')\delta(\xi-\xi')\ d\Omega(\xi') \quad , \tag{32}$$

for $a \leq x \leq z \leq b$, and where p' is a point (i.e., an ordered triple of real numbers) in level x, and q is a point in level z such that:

$$q - p' = r\xi' \tag{33}$$

where r is determined by:

$$r = |z-x| / |\xi \cdot \mathbf{k}| \quad .$$

A companion equation to (31), written for $N_+^o(x)$, is readily stated. To see the way in which (32) is used, suppose x = a and z = y, and that the value of $N_-^o(a)$ at p' and ξ' is specifically of the form $N_-^o(p',\xi')$. Further, let $N_-^o(q,\xi)$ be the residual radiance at point q induced by $N_-^o(a)$. Then $T^o(a,y)$ acting on $N_-^o(a)$ yields the radiance:

$$N_-^o(q,\xi) = \int_{\Xi_-} N_-^o(p',\xi')T_r(p',\xi')\delta(\xi-\xi')\, d\Omega(\xi')$$

$$= N_-^o(p,\xi)T_r(p,\xi) \quad , \tag{34}$$

where $p = q - r\xi$, in which the distance r is determined by $r = |y-a|/|\xi \cdot \mathbf{k}|$.

Recalling that y is the depth parameter for X(a,b), i.e., the distance to the upper boundary of X(a,b), it follows from (31) and (2) of Sec. 3.11 that:

$$\frac{dN_-^o(y)}{dy} = -\frac{1}{|\xi \cdot \mathbf{k}|} \cdot \alpha(y)N_-^o(y) \quad . \tag{35}$$

Suppose we write:

$$\text{"}\tau^o(y)\text{"} \quad \text{for} \quad -\frac{1}{|\xi \cdot \mathbf{k}|}\,\alpha(y) \quad . \tag{36}$$

Then (35) becomes:

$$\boxed{\frac{dN_-^o(y)}{dy} = N_-^o(y)\tau^o(y)} \quad . \tag{37}$$

A similar equation may be shown to hold for $N_+^o(y)$:

$$\boxed{-\frac{dN_+^o(y)}{dy} = N_+^o(y)\tau^o(y)} \quad . \tag{38}$$

The number $\tau^o(y)$ defined in (36) (and which acts as a multiplicative operator on radiance, as in (37), (38)) is called the *local residual (or reduced) transmittance operator*. Observe the analogous roles played by $\tau^o(y)$ and $\tau(y)$ in (37), (38) and (5), (6). This observation prompts us to write:

$$\text{"}\tau^*(y)\text{"} \quad \text{for} \quad \tau(y) - \tau^o(y) \tag{39}$$

which we call the *local diffuse transmittance operator.*

In view of (39), we have:

$$\boxed{\tau(y) = \tau^o(y) + \tau^*(y)} \tag{40}$$

which is the decomposition of the local transmittance operator into its residual and diffuse parts. This should be compared with (4), so that $\tau^*(y)$ is seen to be the integral operator part of (4).

We see from (3) and (4) that the local reflectance operator $\rho(y)$ is already in diffuse form, i.e., that it already consists of just an integral of σ over Ξ_+. This fact lies at the base of the fundamental distinction between reflectance and transmittance operators whenever decomposed light fields are considered. This distinction may be carried on up to the global level where R(a,b) is necessarily already in diffuse form and where T(a,b) may be rendered into reduced and diffuse parts by writing in general:

$$\text{"}T^*(x,z)\text{"} \quad \text{for} \quad T(x,z) - T^o(x,z) \tag{41}$$

for $a \leq x \leq z \leq b$; so that:

$$\boxed{T(x,z) = T^o(x,z) + T^*(x,z)} \quad . \tag{42}$$

A similar definition holds for upward transmittances. It follows immediately from (42), (32), and from (29), (30) of Sec. 3.7 that:

$$T^*(y,y) = 0 \tag{43}$$

for every y, $a \leq y \leq b$. That is, the diffuse transmittance operator $T^*(x,z)$ reduces to the zero operator whenever x = z. This shows that we may generally picture $T^*(x,z)$ as a "soft" operator in the same sense that the reflectance operator R(x,z) for the same slab is "soft". The precise mathematical description of this "softness" of $T^*(x,z)$ and R(x,z) is that they possess the AC property with respect to depth measure. By contrast with $T^*(x,z)$, the operator T(x,z), owing to its component $T^o(x,z)$, is "hard" in the sense that:

$$T^o(y,y) = I \tag{44}$$

for every y, $a \leq y \leq b$, as may be seen by (32). That is, $T^o(x,z)$ (and hence T(x,z)) reduces to the identity operator, and certainly does not have the AC property with respect to depth measure.

The terms "soft" and "hard" as used above to describe the AC properties of operators pictorially go back to certain everyday observable phenomena of light fields in the air or the sea. For example consider a slightly hazy morning when the sky is otherwise clear. The radiance distribution of the haze appears softly variable with direction except for the bright sharp sun image discernable through the haze. If one were to describe the diffuse light field constituting the haze only (thus omitting the residual radiance of the sun) the description would use the operator $T^*(x,z)$ acting on the incident sunlight at the top of the haze layer, where x may now be the altitude of the haze layer and z the altitude of the ground. As the haze "burns off", and the haze layer becomes thinner (optically or geometrically, or both) the altitude x approaches z, and with the vanishing difference z-x so too vanishes $T^*(x,z)$. On the other hand the residual radiance from the sun transmitted through the haze layer is described by $T^o(x,z)$, which approaches the identity operator with decreasing difference z-x thereby depicting the hardening or sharpening of the sun's image as seen through the dissipating mists.

The preceding discussion has made plausible the demonstrable fact that the operator $T^*(a,b)$ for an arbitrary medium $X(a,b)$ has the AC property with respect to depth measure. This can be shown to imply, via Theorem B of Sec. 3.16, the existence of an interaction kernel function S^* for $T^*(a,b)$ such that:

$$T^*(a,b) = \int_{\Xi_-}\int_{X_a} [\ ]S^*(X;x',\xi';x,\xi)\ dA(x')\ d\Omega(\xi') . \quad (45)$$

This may be compared with (9) of Sec. 3.6. As in the earlier case "X" stands for the medium at hand--$X(a,b)$ in this case. Further, X_a is the plane boundary of $X(a,b)$ at level a. A similar integral representation for $T^*(b,a)$ can be obtained.

We now have covered all the prerequisites for the main part of the present discussion, namely for the derivation of the appropriate forms of the local and global principles of invariance for decomposed light fields, along with the differential equations for $T^*(a,b)$. We begin the main discussion with the derivation of the local forms of the principles of invariance for $N^*_\pm(y)$.

Starting with (5) in which we decompose $N_+(y)$, $N_-(y)$ and $\tau(y)$ into their reduced and diffuse parts, we perform the following calculations:

$$-\frac{d\left(N^o_+(y) + N^*_+(y)\right)}{dy} = \left(N^o_+(y) + N^*_+(y)\right)\left(\tau^o(y) + \tau^*(y)\right) + \left(N^o_-(y) + N^*_-(y)\right)\rho(y) \quad .$$

In view of (38), this may be simplified and rearranged into the form:

$$-\frac{dN^*_+(y)}{dy} = N^*_+(y)\tau(y) + N^*_-(y)\rho(y) + \left[N^o_+(y)\tau^*(y) + N^o_-(y)\rho(y)\right] \quad (46)$$

This shows how the rate of change of $N_+^*(y)$ with depth depends on the counter-flowing scattered radiances $N_\pm^*(y)$ and the counter-flowing residual radiances $N_\pm^o(y)$. In a similar way we obtain:

$$\frac{dN_-^*(y)}{dy} = N_-^*(y)\tau(y) + N_+^*(y)\rho(y) + \left[N_-^o(y)\tau^*(y) + N_+^o(y)\rho(y)\right] . \tag{47}$$

Equations (46) and (47) constitute the local forms of the principles of invariance for the diffuse light field $N^*(y)$, where we have written:

$$\text{"}N^*(y)\text{"} \quad \text{for} \quad (N_+^*(y), N_-^*(y)) \quad . \tag{48}$$

Equations (46) and (47) together are equivalent to the equation of transfer (1) for $N(y,\cdot)$. The equation of transfer for $N^*(y,\cdot)$ was studied earlier in (7) of Sec. 5.2. With only slight modifications, the preceding derivations of (46) and (47) can be made directly from (2) and (7) of Sec. 5.2.

We may cast (46), (47) into the decomposition counterpart to (9). Thus writing:

$$\text{"}N^o(y)\text{"} \quad \text{for} \quad (N_+^o(y), N_-^o(y)) \tag{49}$$

$$\text{"}\mathcal{K}^*(y)\text{"} \quad \text{for} \quad \begin{pmatrix} -\tau^*(y) & \rho(y) \\ -\rho(y) & \tau^*(y) \end{pmatrix} \tag{50}$$

we can write the system (46), (47) as:

$$\boxed{\frac{dN^*(y)}{dy} = N^*(y)\mathcal{K}(y) + N^o(y)\mathcal{K}^*(y)} \quad . \tag{51}$$

Next we look into the matter of the (global) principles of invariance for decomposed light fields. Going solely on the strength of the analogy between the pair (5),(6) and the pair I, II of Example 3 of Sec. 3.7, we should be able to immediately write down the present decomposed counterparts to I, II of Sec. 3.7, using (46), (47) as a basis. Thus we write:

I^*.

$$\boxed{N_+^*(y) = N_+^*(z)T(z,y) + N_-^*(y)R(y,z) + N_+^o(z)T^*(z,y) + N_-^o(y)R(y,z)} \tag{52}$$

and

II^*.

$$\boxed{N_-^*(y) = N_-^*(x)T(x,y) + N_+^*(y)R(y,x) + N_-^o(x)T^*(x,y) + N_+^o(y)R(y,x)} \tag{53}$$

where $a \le x \le y \le z$.

We can rigorously establish these two main principles of invariance for diffuse radiance directly from I, II of Example 3 of Sec. 3.7. Thus, to establish, say, I^* we write I of Example 3 of Sec. 3.7 as:

$$(N_+^o(y) + N_+^*(y)) = N_+^*(z)T(z,y) + N_-^*(y)R(y,z) + N_+^o(z)T^o(z,y)$$
$$+ N_+^o(z)T^*(z,y) + N_-^o(y)R(y,z) \quad .$$

Using the + counterpart to (31), namely the representation:

$$N_+^o(y) = N_+^o(z)T^o(z,y) \quad , \tag{54}$$

in the preceding expanded form of principle I, I^* follows immediately after a rearrangement of terms. The process of rewriting all the principles and functional relations of radiative transfer for decomposed light fields can now be readily carried out systematically by the interested reader. We shall leave this matter here and go on to the next observation of immediate concern.

From principles I^* and II^* above we can deduce the other two principles of invariance for the diffuse radiances $N_+^*(a)$, $N_-^*(b)$ on X(a,b). Thus, in particular, we have on setting y = a, z = b, in I^*:

$$III^*. \quad \boxed{N_+^*(a) = N_+^o(b)T^*(b,a) + N_-^o(a)R(a,b)} \quad , \tag{55}$$

and on setting x = a, y = b, in II^*:

$$IV^*. \quad \boxed{N_-^*(b) = N_-^o(a)T^*(a,b) + N_+^o(b)R(b,a)} \quad . \tag{56}$$

The final matter to be taken up in this dicussion of decomposed light fields is the derivation of the differential equations for the diffuse transmittance operator $T^*(a,b)$. The present derivations can be patterned directly after the steps (20)-(27) of the derivation of the differential equation for T(a,b). Thus, starting with principle II^* in which x = a and in which $N_-^*(a) = 0$ and $N_+(b) = 0$, we have:

$$N_-^*(y) = N_-^o(a)T^*(a,y) + N_+^*(y)R(y,a) \quad . \tag{57}$$

Taking the derivatives of each side with respect to y:

$$\frac{dN_-^*(y)}{dy} = N_-^o(a)\frac{dT^*(a,y)}{dy} + \frac{dN_+^*(y)}{dy}R(y,a) + N_+^*(y)\frac{dR(y,a)}{dy} \tag{58}$$

Here we have written:

$$"\frac{dT^*(a,y)}{dy}" \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ ]\frac{d}{dy}T^*(a,y;\xi';\xi)\,d\Omega(\xi') \tag{59}$$

where, in turn, we have written (in analogy to (32) of Sec. 3.7)

$$"T^*(x,z;\xi';\xi)" \quad \text{for} \quad |\xi\cdot\mathbf{k}|\int_{X_x}[\ ]S^*(X;y',\xi';y,\xi)\,dA(y') \tag{60}$$

and where "X" now stands for $X(x,z)$, $a \le x \le z \le b$, and X_x is the plane at depth x. Point y is in X_z. Now in (58) consider:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy}$$

which by (47) is given as:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy} = [N_-^*(b)\tau(b) + N_-^o(b)\tau^*(b)] \quad . \tag{61}$$

Using principle IV* above and (31) with y = b, we can reduce this limit to:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy} = N_-^o(a)[T^*(a,b)\tau(b) + T^o(a,b)\tau^*(b)]. \tag{62}$$

In a similar way we obtain:

Errata $$\lim_{y\to b}\frac{dN_+^*(y)}{dy} = -N_-^o(a)[T^*(a,b)\rho(y) + T^o(a,b)\rho(y)]. \tag{63}$$

Writing

$$"\frac{\partial T^*(a,b)}{\partial b}" \quad \text{for} \quad \lim_{y\to b}\frac{dT^*(a,y)}{dy} \ , \tag{64}$$

we now apply the limit operator, $\lim_{y\to b}$, to each side of (58), the result being:

$$\boxed{\begin{aligned}\frac{\partial T^*(a,b)}{\partial b} &= T^*(a,b)[\tau(b) + \rho(b)R(b,a)] \\ &\quad + T^o(a,b)[\tau^*(b) + \rho(b)R(b,a)]\end{aligned}} \tag{65}$$

which is the decomposed counterpart to (27). Observe that the structure of (65) is an inhomogeneous version of (27). That is, the gestalt of (65) is that of:

$$\frac{dT^*}{dy} = T^*A + B \tag{66}$$

while the gestalt of (27) is that of:

$$\frac{dT}{dy} = TA \tag{67}$$

where A and B are known operators.

This transition from homogeneity to nonhomogeneity of an operator equation is the general earmark of a transition from the representation of undecomposed to decomposed light fields, in which the latter are represented by N^*. This phenomenon was already seen in I^* and II^*, and earlier still in (46) and (47) (whose homogeneous counterparts are (5) and (6)); similarly with (51) and (9). A still earlier example of this transition is the transfer equation for N^* in Sec. 5.2.

The decomposed counterpart to (29) may now be written down by inspection of (65), using (29) as a guide:

Errata

$$-\frac{\partial T^*(a,b)}{\partial a} = [\tau(a) + R(a,y)\rho(a)]T^*(a,b) + [\tau^*(a) + R(a,b)\rho(a)]T^o(a,b) \tag{68}$$

This differential equation may also be derived from first principles after the manner of (65); the details are left as an exercise for the reader. In analogy to (19), equation (68) shows that $-\partial T^*(a,b)/\partial a$ for $a = b$ is given by

$$-\frac{\partial T^*(a,b)}{\partial a} = \tau^*(a) \quad . \tag{69}$$

Thus when $a = b$, $T^*(a,b) = 0$ and its 'rate of growth' is precisely the magnitude of the local diffuse transmittance operator $\tau^*(b)$. Thus the analogy between (19) and (69) is perfect.

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7.2 Differential Equations Governing the Time Dependent R and T Operators

We now extend the formulations of the preceding section to the time dependent case. The geometric setting and optical properties of the medium are unchanged except that now all functions in addition vary with time. The first step in such an extension is the derivation of the time dependent version of the local forms of the principles of invariance for a plane-parallel medium.