Operating on each side with $D_\beta(y)$:

$$N_\beta(z) = \int_A N(y) D_\alpha(y) \; [\mathrm{trn}\; D_\alpha(y)] \mathscr{M}(y,z) D_\beta(y) \; d\mu(\alpha) \quad .$$

Hence:

$$N_\beta(z) = \int_A N_\alpha(y) \mathscr{M}_{\alpha\beta}(y,z) \; d\mu(\alpha) \tag{119}$$

for every α,β in A. Operating on each side of (119) with $[\mathrm{trn}\; D_\beta(z)]$ and integrating over A:

$$N(z) = \int_A N_\beta(t) \; d\mu(\beta) =$$

$$= N(y) \int_A \int_A [\;\;] D_\alpha(y) \mathscr{M}_{\alpha\beta}(y,z) [\mathrm{trn}\; D_\beta(z)] \; d\mu(\alpha) \; d\mu(\beta) \quad .$$

Since N(y) is arbitrary, we have from this and (115):

$$\boxed{\mathscr{M}(y,z) = \int_A \int_A [\;\;] D_\alpha(y) \mathscr{M}_{\alpha\beta}(y,z) [\mathrm{trn}\; D_\beta(z)] \; d\mu(\alpha) \; d\mu(\beta)} \tag{120}$$

which is one of the possible generalizations of the type to which (92) belongs. This concludes the summary and overview of a possible general method of constructing partial groups $\Gamma_2(S)$ of operators on the subset S of the optical medium X. The problem of generalizing $\Gamma_2(a,b)$ to $\Gamma_2(S)$ will be considered once again in Sec. 7.11.

7.5 Analytic Properties of the Invariant Imbedding Operators

We now continue the work, begun in Sec. 7.1, of deriving the differential equations governing the main invariant imbedding operators. In particular we shall derive the various functional differential equations governing the operators $\mathscr{M}(x,y)$, $\mathscr{M}(x,y,z)$, and $\mathscr{M}(v,x;u,w)$. Since these operators are in turn 2×2 matrices of operators, each such differential equation is a potential plethora of differential equations for its component operators. Such a superabundance of operator differential equations would constitute an embarrassment of riches for the theory were it not for the insight gained into such operators in the preceding section. Indeed, our studies there showed that the operators of the form $\mathscr{M}(v,x;u,w)$ could be studied in terms of those of the form $\mathscr{M}(x,y,z)$, and the latter in terms of those of the form $\mathscr{M}(x,y)$. Hence the operators $\mathscr{M}(x,y)$ emerge as the undisputed victors in any contest of conceptual simplicity and inherent power of representation. In summary, then, it was shown how the members of $\Gamma_2(a,b)$ could represent, via simple algebraic formulas, all the other invariant imbedding operators of $\Gamma_3(a,b)$ and $\Gamma_4(a,b)$, plus the operators of $G_2(a,b)$, and even the classical R and T operators. Hence the multitude of operators can be reduced, formally at least, to just those in $\Gamma_2(a,b)$. This power of representation of the members of $\Gamma_2(a,b)$ will be used again in the present section to derive the requisite differential equations for all invariant imbedding operators from those for $\mathcal{M}(x,y)$. From these differential equations, in turn, the operators may be systematically constructed by various solution procedures using the inherent optical properties of the appropriate media.

Throughout this section, unless otherwise stated, we shall work with an arbitrary source-free plane-parallel medium $X(a,b)$, $a \le b$, with arbitrary incident radiance distributions $N_-(a)$ and $N_+(b)$ over the upper boundary X_a and the lower boundary X_b, respectively. As in the case of Sec. 7.4, the present results are readily generalized to wider settings, namely general one-parameter settings and general unparameterized optical media. Also, as in the case of Sec. 7.4, the exposition is primarily heuristic, with rigorous developments left for future study.

Differential Equations for $\mathcal{M}(x,y)$

Starting with the basic equation concerning the operator $\mathcal{M}(x,y)$; namely:

$$N(y) = N(x)\mathcal{M}(x,y)$$

introduced and studied in Ex. 7 of Sec. 3.7, we apply the differential operator d/dy to each side of this equation, where the differential operator occurs in (1) of Sec. 7.1. Thus:

$$\frac{dN(y)}{dy} = \frac{\partial}{\partial y}\left(N(x)\mathcal{M}(x,y)\right) = N(x)\left(\frac{\partial \mathcal{M}(x,y)}{\partial y}\right) .$$

For the reader unfamiliar with analytic (i.e., differential, integral, and general limit) operations on operators, we may note here that the rules governing these operations are the same in all essential respects as those for the everyday type of function encountered in the domain of elementary calculus. Hence for the purposes of the present discussion, the reader will require no more advanced techniques than those encountered in such a domain. Needless to add, however, the physical content of the ensuing statements are far from trivial and are worthy of further analysis and application.

Continuing now with the derivation, we use (9) of Sec. 7.1 to reduce the preceding result to:

$$\frac{dN(y)}{dy} = N(y)\mathcal{K}(y) = N(x)\frac{d\mathcal{M}(x,y)}{dy} .$$

Using the basic equation for $\mathcal{M}(x,y)$ once again, we obtain:

$$N(y)\mathcal{K}(y) = \left(N(x)\mathcal{M}(x,y)\right)\mathcal{K}(y) = N(x)\frac{d\mathcal{M}(x,y)}{dy} .$$

Under the present lighting conditions, N(x) is arbitrary so that:

$$\boxed{\frac{\partial \mathcal{M}(x,y)}{\partial y} = \mathcal{M}(x,y)\mathcal{K}(y)} \quad (1)$$

which is our first main result and which holds for arbitrary x,y in X(a,b). This differential equation for $\mathcal{M}(x,y)$ harbors the four differential equations for its four components, which are operator-valued functions. For future reference, these are:

$$-\frac{d\mathcal{M}_{++}(x,y)}{dy} = \mathcal{M}_{++}(x,y)\tau(y) + \mathcal{M}_{+-}(x,y)\rho(y) \quad (2)$$

$$-\frac{d\mathcal{M}_{-+}(x,y)}{dy} = \mathcal{M}_{-+}(x,y)\tau(y) + \mathcal{M}_{--}(x,y)\rho(y) \quad (3)$$

$$\frac{\partial\mathcal{M}_{+-}(x,y)}{\partial y} = \mathcal{M}_{++}(x,y)\rho(y) + \mathcal{M}_{+-}(x,y)\tau(y) \quad (4)$$

$$\frac{\partial\mathcal{M}_{--}(x,y)}{\partial y} = \mathcal{M}_{-+}(x,y)\rho(y) + \mathcal{M}_{--}(x,y)\tau(y) \quad . \quad (5)$$

Observe how (2),(4) and (3),(5) are autonomous and indeed are copies of (5),(6) in Sec. 7.1. Hence $(\mathcal{M}_{++},\mathcal{M}_{+-})$ pair with (N_+,N_-) as do also $(\mathcal{M}_{-+},\mathcal{M}_{--})$. The initial value of $(\mathcal{M}_{++},\mathcal{M}_{+-})$ is (I,0), while that of $(\mathcal{M}_{-+},\mathcal{M}_{--})$ is (0,I).

A companion equation to (1) (its adjoint) is obtained when the differentiation is performed with respect to x rather than y. To obtain the companion equation observe that:

$$0 = \frac{d}{dx} I = \frac{\partial}{\partial x}\left[\mathcal{M}(y,x)\mathcal{M}(x,y)\right]$$

$$= \frac{\partial\mathcal{M}(y,x)}{\partial x}\mathcal{M}(x,y) + \mathcal{M}(y,x)\frac{\partial\mathcal{M}(x,y)}{\partial x} \quad .$$

Hence:

$$\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{M}(x,y)\frac{\partial\mathcal{M}(y,x)}{\partial x}\mathcal{M}(x,y) \quad . \quad (6)$$

Applying (1) to the derivative on the right in (6) we have

$$\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{M}(x,y)[\mathcal{M}(y,x)\mathcal{K}(x)]\mathcal{M}(x,y) \quad ,$$

whence:

$$\boxed{\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{K}(x)\mathcal{M}(x,y)} \quad . \quad (7)$$

Equations (1) and (7) are reducible to n×n matrix equations using angular discretization techniques such as those to be described in Secs 7.7, 7.9 and 7.10. The initial values are of course in each case $\mathcal{M}(x,x) = I$, the identity matrix. The four functions taken in the above pairs comprising $\mathcal{M}$, are called the *fundamental solutions of the equation of transfer*. As we have already seen in 7.4, by judicious linear combinations of these solutions, we can obtain all the useful scattering properties (e.g., R, T, $\mathcal{R}$, $\mathcal{T}$) of an optical medium.

The tactic used above to find (6) is a special case of the general procedure for finding the derivative of the inverse A^{-1} of an operator, knowing the derivative of A. Thus, if A is differentiable and depends on x:

$$0 = \frac{d}{dx} I = \frac{d}{dx}(AA^{-1}) = \frac{dA}{dx}A^{-1} + A\frac{dA^{-1}}{dx} ,$$

whence:

$$\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1} \quad . \tag{8}$$

This formula is based on the standing assumption that the inverse of A exists so that the product AA^{-1} is defined, and that A and A^{-1} are in some sense differentiable. Equation (8) is a general form of the formula:

$$\frac{d(1/y)}{dx} = -\frac{1}{y^2}\frac{dy}{dx} = -y^{-1}\frac{dy}{dx}y^{-1}$$

encountered in elementary calculus for the derivative of the numerical valued function 1/y in terms of that of y. Now, however, we generally are not permitted to join together the two inverses A^{-1} in (8) since operator multiplication is generally not commutative.

Differential Equations for $\mathcal{M}(x,y,z)$

By means of the representation of $\mathcal{M}(x,y,z)$ in terms of $\mathcal{M}(y,z)$ and $\mathcal{M}(y,x)$, as given in (42) of Sec. 7.4, we can find the differential equations governing $\mathcal{M}(x,y,z)$. There are generally three such equations, one arising from differentiation of $\mathcal{M}(x,y,z)$ with respect to each of the three distinct depth variables x,y,z within [a,b]. Throughout the derivations, then, x,y,z will be distinct variables, unless specifically noted otherwise.

Thus, differentiating each side of:

$$\mathcal{M}(x,y,z) = [\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-]^{-1} \tag{9}$$

with respect to x and using (8) and (1):

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\mathcal{M}(x,y,z) \frac{\partial}{\partial x} [\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= -\mathcal{M}(x,y,z)\mathcal{M}(y,x)\mathcal{K}(x)C_-\mathcal{M}(x,y,z) \quad .$$

Now, the general relation:

$$\mathcal{M}(x,v,z) = \mathcal{M}(x,y,z)\mathcal{M}(y,v) \tag{10}$$

follows from the definition of $\mathcal{M}(y,v)$ and (44) of Sec. 7.4 along with the general relation (84) of Sec. 3.7. Compare (10) with (39) of Sec. 7.4. Using (10), we have, in particular:

$$\mathcal{M}(x,y,z)\mathcal{M}(y,x) = \mathcal{M}(x,x,z) \quad ,$$

which allows us to write:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\mathcal{M}(x,x,z)\mathcal{K}(x)C_-\mathcal{M}(x,y,z)} \quad . \tag{11}$$

We defer discussion of (11) until its two companions have been derived. Toward this end, differentiating each side of (9) with respect to y and this time using (8) along with (7):

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial y} = -\mathcal{M}(x,y,z)[-\mathcal{K}(y)\mathcal{M}(y,z)C_+ - \mathcal{K}(y)\mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= \mathcal{M}(x,y,z)\mathcal{K}(y)[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= \mathcal{M}(x,y,z)\mathcal{K}(y)[\mathcal{M}(x,y,z)]^{-1}\mathcal{M}(x,y,z) \quad .$$

Hence:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial y} = \mathcal{M}(x,y,z)\mathcal{K}(y)} \quad . \tag{12}$$

Finally, differentiating each side of (9) with respect to z:

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial z} = -\mathcal{M}(x,y,z)[\mathcal{M}(y,z)\mathcal{K}(z)C_+]\mathcal{M}(x,y,z)$$

which may be simplified, using (10), to:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial z} = -\mathcal{M}(x,z,z)\mathcal{K}(z)C_+\mathcal{M}(x,y,z)} \quad . \tag{13}$$

Now for a brief discussion of (11), (12), and (13). All three equations show how to construct $\mathcal{M}(x,y,z)$ given the relatively simpler operators $\mathcal{M}(x,z,z)$, $\mathcal{K}(z)$, C_+ (in the case of (13) or $\mathcal{M}(x,x,z)$, $\mathcal{K}(x)$, C_- (in the case of (11)). For example, in the case of (11), part (a) of Fig. 7.9 shows that by starting with the basic slab X(y,z) (shaded) and building it up to level x as shown, we can compute $\mathcal{M}(x,y,z)$ for every x, such that $x \le y \le z$. All that is needed to start the calculation is information on $\mathcal{M}(x,y,z)$ for the special case x=y. This information, in view of (44)-(47) of Sec. 3.7, is tantamount to knowledge of the standard operators R(y,z) and T(y,z). Of course, one must know in addition the local transmittance and reflectance operators for the required range of x above the level y. A similar observation holds for (13) whose geometric significance is depicted in part (c) of Fig. 7.9. Finally, Eq. (12) strikes the middle road between (11) and (13) and shows how $\mathcal{M}(x,y,z)$ can be obtained by working *inward* from either boundary of X(x,z), and initially knowing $\mathcal{M}(x,x,z)$ or $\mathcal{M}(x,z,z)$, as the case may be. The former of

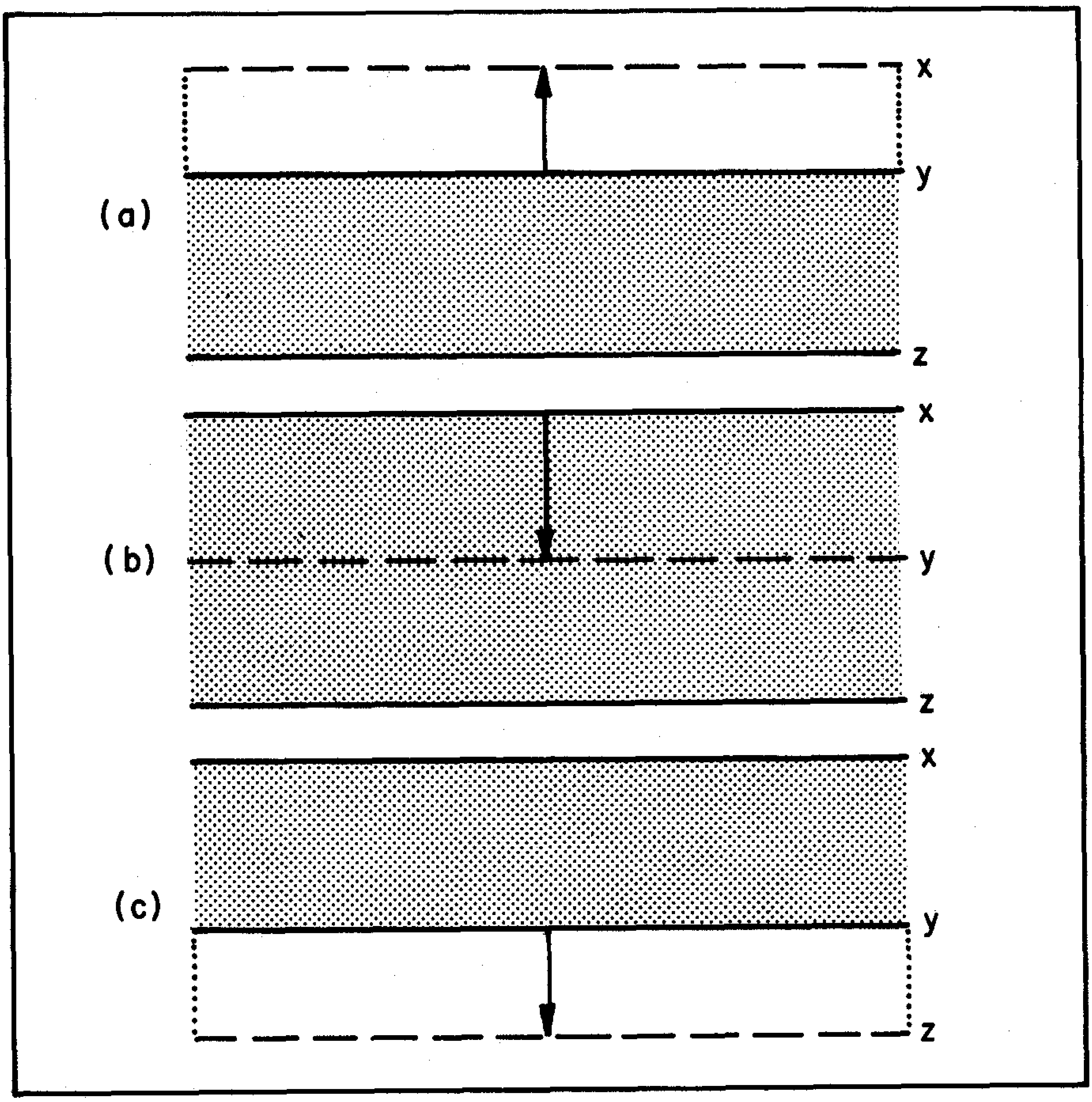

FIG. 7.9 Three ways in which to generate invariant imbedding operator $\mathcal{M}(x,y,z)$.

these cases is shown in (b) of Fig. 7.9.

It is instructive to unravel some of the information contained in these equations. We begin with (12) which yields the following four differential equations for the complete reflectance and transmittance operators:

$$-\frac{\partial \mathcal{T}(z,y,x)}{\partial y} = \mathcal{T}(z,y,x)\tau(y) + \mathcal{R}(z,y,x)\rho(y) \tag{14}$$

$$-\frac{\partial \mathcal{R}(x,y,z)}{\partial y} = \mathcal{R}(x,y,z)\tau(y) + \mathcal{T}(x,y,z)\rho(y) \tag{15}$$

$$\frac{\partial \mathcal{R}(z,y,x)}{\partial y} = \mathcal{T}(z,y,x)\rho(y) + \mathcal{R}(z,y,x)\tau(y) \tag{16}$$

$$\frac{\partial \mathcal{T}(x,y,z)}{\partial y} = \mathcal{R}(x,y,z)\rho(y) + \mathcal{T}(x,y,z)\tau(y) \quad . \tag{17}$$

Observe how (14), (16) are fundamentally similar to (5), (6) of Sec. 7.1, while (15), (17) are likewise autonomous and similar. Recall also the discussions of (1), (7) above. These earlier equations are no more fundamental than the present equations. Indeed, (14)-(17) may be used as the basis for all two-point boundary value problems by adopting the set of (two-point) fundamental solutions defined in (38)-(40) below.

Next, since the situations depicted in parts (a) and (c) of Fig. 7.9 are basically alike, we shall give only the details of unravelling Eq. (11), which goes with part (a) of the figure. The result is readily obtained by first noting that:

$$\begin{aligned}
-\mathcal{M}(x,x,z)\mathcal{K}(x)C_- &= -\begin{bmatrix}\mathcal{T}(z,x,x) & \mathcal{R}(z,x,x)\\ \mathcal{R}(x,x,z) & \mathcal{T}(x,x,z)\end{bmatrix}\begin{bmatrix}-\tau(x) & \rho(x)\\ -\rho(x) & \tau(x)\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\\
&= -\begin{bmatrix}T(z,x) & 0\\ R(x,z) & I\end{bmatrix}\begin{bmatrix}0 & \rho(x)\\ 0 & \tau(x)\end{bmatrix}\\
&= -\begin{bmatrix}0 & T(z,x)\rho(x)\\ 0 & R(x,z)\rho(x)+\tau(x)\end{bmatrix} \quad .
\end{aligned}$$

In view of this, (11) reduces to :

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\begin{bmatrix}0 & T(z,x)\rho(x)\\ 0 & R(x,z)\rho(x)+\tau(x)\end{bmatrix}\begin{bmatrix}\mathcal{T}(z,y,x) & \mathcal{R}(z,y,x)\\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z)\end{bmatrix} \quad .$$

whence:

$$-\frac{\partial \mathcal{T}(z,y,x)}{\partial x} = T(z,x)\rho(x)\mathcal{R}(x,y,z) \qquad (18)$$

$$-\frac{\partial \mathcal{R}(z,y,x)}{\partial x} = T(z,x)\rho(x)\mathcal{T}(x,y,z) \qquad (19)$$

$$-\frac{\partial \mathcal{R}(x,y,z)}{\partial x} = \big(R(x,z)\rho(x) + \tau(x)\big)\mathcal{R}(x,y,z) \qquad (20)$$

$$-\frac{\partial \mathcal{T}(x,y,z)}{\partial x} = \big(R(x,z)\rho(x) + \tau(x)\big)\mathcal{T}(x,y,z) \quad . \qquad (21)$$

The various physical interpretations of these equations are instructive and the reader may gain understanding of the dynamics of scattering problems by translating each of the preceding equations into words or appropriate mental images. For example, (18) describes how steadily flowing upward radiance (imagined incident at level z) changes at level y in X(x,z) when material is added to X(x,z) at its upper boundary X_x. Thus (refer to part (a) of Fig. 7.9) when a thin layer is added to X(x,z) at level x, the normally transmitted radiance (represented by T(z,x)) is now locally reflected in the new layer (represented by ρ(x)) and then globally reflected (as represented by $\mathcal{R}$(x,y,z)) down to layer y in X(x,z).

Further elucidation of the dynamics of scattering-absorbing media is forthcoming from the present differential equations for $\mathcal{M}$(x,y,z) by observing how the differential equations for R and T, as derived in Sec. 7.1, may be derived anew in the present setting. As an example, consider Eq. (18) of Sec. 7.1. That equation describes, in essence, how reflected radiance at level a in X(a,b) changes when an incremental layer is added to X(a,b) at level a. In terms of the present equations such a change in R(a,b) is the sum of the changes in $\mathcal{R}$(a,y,b) when a and y are simultaneously varied for the special instance when a = y, i.e., when the derivatives of $\mathcal{R}$(a,y,b) with respect to y and a are added together for the case a = y. Thus, from (15) and (20):

$$-\left(\frac{\partial \mathcal{R}(x,y,z)}{\partial x} + \frac{\partial \mathcal{R}(x,y,z)}{\partial y}\right) =$$

$$= \mathcal{R}(x,y,z)\tau(y) + \mathcal{T}(x,y,z)\rho(y) + \big(R(x,z)\rho(x)+\tau(x)\big)\mathcal{R}(x,y,z) \qquad (22)$$

Letting y approach x, the right side of this equation becomes, after rearrangements:

$$\rho(x) + \tau(x)R(x,z) + R(x,z)\tau(x) + R(x,z)\rho(x)R(x,z) \ .$$

Furthermore, the left side of (22) is related to R(x,z) by the equation:

$$\frac{\partial R(x,z)}{\partial x} = \lim_{y\to x} \left[\frac{\partial \mathcal{R}(x,y,z)}{\partial x} + \frac{\partial \mathcal{R}(x,y,z)}{\partial y}\right] \tag{23}$$

which follows from:

$$R(x,z) = \lim_{y\to x} \mathcal{R}(x,y,z) \quad .$$

These limit equalities devolve on (45) of Sec. 3.7 and the usually available continuity of $\mathcal{R}(x,y,z)$ and its derivative. Combining these results, (18) of Sec. 7.1 is obtained from (22) but now as seen in the light of a superposition of changes of the complete reflectance function $\mathcal{R}(x,y,z)$.

The remaining three equations of Sec. 7.1 may also be viewed from the new vantage point of the invariant imbedding relation. For instance, Eq. (27) of Sec. 7.1 may be obtained, in essence, from (14) and (18) via the observation that:

$$\frac{\partial T(z,x)}{\partial x} = \lim_{y\to x} \left[\frac{\partial \mathcal{T}(z,y,x)}{\partial y} + \frac{\partial \mathcal{T}(z,y,x)}{\partial x}\right] \tag{24}$$

which follows from:

$$T(z,x) = \lim_{y\to x} \mathcal{T}(z,y,x) \quad .$$

These limit equalities devolve on (44) of Sec. 3.7 and the usually available continuity of $\mathcal{T}(z,y,x)$ and its derivatives. On the other hand, Eq. (28) of Sec. 7.1 is obtained directly from (19) after passing to the limit $y\to z$ and suitable rearrangement of coordinate variables. Finally, (29) of Sec. 7.1 follows directly from (21) in a similar way. The reason for the direct derivations in the latter two cases stems from the observation that:

$$R(z,x) = \lim_{y\to z} \mathcal{R}(z,y,x)$$

and

$$T(x,z) = \lim_{y\to z} \mathcal{T}(x,y,z)$$

and that the derivatives in (19) and (21) are with respect to x.

Differential Equations for $\mathcal{M}(v,x;u,w)$

Our starting point for the present derivations may be either (50), (51) or (52) of Sec. 7.4. We choose the representation (51) of $\mathcal{M}(v,x;u,w)$ so as to build directly on the results (11)-(13) just obtained and to gain some practice in the semigroup properties of the $\mathcal{M}$-operators. In the present notation, (51) becomes:

$$\mathcal{M}(v,x;u,w) = \mathcal{M}(v,y,u)\mathcal{M}^{-1}(x,y,w) \tag{25}$$

We generally expect four distinct differential equations to govern each member of $\Gamma_*(a,b)$. Thus, assuming u,v,w,x,y,z to be pairwise distinct variables, we have first of all:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial v} = \frac{\partial \mathcal{M}(v,y,u)}{\partial v}\mathcal{M}^{-1}(x,y,w) \quad .$$

By (11):

$$\frac{\partial \mathcal{M}(v,y,u)}{\partial v} = -\mathcal{M}(v,v,u)\mathcal{K}(v)C_-\mathcal{M}(v,y,u) \quad .$$

Hence:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial v} = -\mathcal{M}(v,v,u)\mathcal{K}(v)C_-\mathcal{M}(v,x;u,w)} \quad . \tag{26}$$

This is the first of the requisite differential equations. Next, from (25):

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,y,u)\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial x} \quad .$$

But from (8) and (11):

$$\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial x} = -\mathcal{M}^{-1}(x,y,w)[-\mathcal{M}(x,x,w)\mathcal{K}(x)C_-\mathcal{M}(x,y,w)]\mathcal{M}^{-1}(x,y,w) \quad .$$

Hence the second requisite differential equation is:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,x;u,w)\mathcal{M}(x,x,w)\mathcal{K}(x)C_-} \tag{27}$$

This equation may be simplified by recalling (76) of Sec. 3.7 which states that:

$$\mathcal{M}(x,x,w) = \mathcal{M}(x,x;w,x)$$

and also the semigroup property (84) of Sec. 3.7:

$$\begin{aligned}\mathcal{M}(v,x;u,w)\mathcal{M}(x,x;w,x) &= \mathcal{M}(v,x;u,x)\\ &= \mathcal{M}(v,x,u)\end{aligned}$$

so that (27) becomes:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,x,u)\mathcal{K}(x)C_- = \frac{\partial \mathcal{M}(v,x,u)}{\partial x}C_- \quad . \tag{28}$$

Of the two preceding differential equations, (27) is the natural form of the requisite differential equation (the operator sought occurs explicitly on both sides of the equation). From a conceptual and computational point of view, (28) shows that, as far as dependence on the variable x is concerned, $\mathcal{M}(v,x;u,w)$ behaves essentially like the members of $\Gamma_2(a,b)$ or $\Gamma_3(a,b)$ (cf. (1) and (12)).

Next, from (25):

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial u} = \frac{\partial \mathcal{M}(v,y,u)}{\partial u} \mathcal{M}^{-1}(x,y,w) \quad .$$

By (13):

$$\frac{\partial \mathcal{M}(v,y,u)}{\partial u} = -\mathcal{M}(v,u,u)\mathcal{K}(u)C_+\mathcal{M}(v,y,u)$$

so that:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial u} = -\mathcal{M}(v,u,u)\mathcal{K}(u)C_+\mathcal{M}(v,x;u,w)} \quad . \tag{29}$$

Finally, from (25) once again:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} = \mathcal{M}(v,y,u) \frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial w} \quad .$$

From (8) and (13):

$$\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial w} = -\mathcal{M}^{-1}(x,y,w)\left(-\mathcal{M}(x,w,w)\mathcal{K}(w)C_+\mathcal{M}(x,y,w)\right)\mathcal{M}^{-1}(x,y,w).$$

It then follows that:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} = \mathcal{M}(v,x;u,w)\mathcal{M}(x,w,w)\mathcal{K}(w)C_+} \quad . \tag{30}$$

This equation, as (27), can be simplified slightly if we use the fact that:

$$\begin{aligned}\mathcal{M}(v,x;u,w)\mathcal{M}(x,w,w) &= \mathcal{M}(v,w;u,w)\\ &= \mathcal{M}(v,w,u) \quad .\end{aligned}$$

Hence (30) becomes:

$$\begin{aligned}\frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} &= \mathcal{M}(v,w,u)\mathcal{K}(w)C_+\\ &= \frac{\partial \mathcal{M}(v,w,u)}{\partial w} C_+ \quad .\end{aligned} \tag{31}$$

This equation and (28) show that, as far as the response variables x, w are concerned, $\mathcal{M}(v,x;u,w)$ behaves essentially like the members of $\Gamma_2(a,b)$ or $\Gamma_3(a,b)$ (see (1) and (12)). These observations could have been obtained directly using (52) of Sec. 7.4; however, the plausibility of (28) and (31) has now been reinforced by taking the preceding route.

Differential Equations for M(x,y) and Ψ(s,y)

It is interesting to derive the differential equations for the simplest of operators M(x,y) and the most complex of operators Ψ(s,y) encountered so far in our studies. Simplicity and complexity are measured here in terms of the ostensible algebraic structure of the components of M(x,y) and Ψ(s,y). As far as the simplicity and complexity of their differential equations are concerned, matters are reversed, as we shall now see. Thus for Ψ(s,y) we use (56) of Sec. 7.4 and find that:

$$\frac{\partial \Psi(s,y)}{\partial y} = [I + \Psi(s,s)]\,\frac{\partial \mathcal{M}(s,y)}{\partial y}$$

$$= [I + \Psi(s,s)]\,\mathcal{M}(s,y)\mathcal{K}(y) \quad , \quad (32)$$

whence*:

$$\boxed{\frac{\partial \Psi(s,y)}{\partial y} = \Psi(s,y)\mathcal{K}(y)} \qquad s \neq y \qquad (33)$$

This result shows that the dependence of Ψ(s,y) on y is essentially that of $\mathcal{M}(x,y)$ on y. The integration of (33) starts from the initial given operator Ψ(s,s). The derivation of the differential equation showing how Ψ(s,y) varies with s is somewhat more complex and left to the reader. The differential equations for the Ψ-operators will be considered again in Sec. 7.12 wherein they will be represented in terms of complete reflectance and transmittance operators.

Turning now to the derivation of the differential equation for M(x,y), we use as a base the representation given by (10) of Sec. 7.4. From this we see that it is necessary to find:

$$\frac{\partial}{\partial z}\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1} =$$

$$= -\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1}\left(\frac{\partial \mathcal{M}(z,x)}{\partial z}\,C_-\right)\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1}$$

in which we have used (8). Hence, with the aid of (7) and (11):

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*For all y we add $I_-\delta(s-y)$ where $I_- = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, by (56) of Sec. 7.4.

$$\frac{\partial M(x,z)}{\partial z} = -[C_+ + \mathcal{M}(z,x)C_-]^{-1}(-\mathcal{X}(z)\mathcal{M}(z,x)C_-) \times$$

$$\times \; [C_+ + \mathcal{M}(z,x)C_-]^{-1}[\mathcal{M}(z,x)C_+ + C_-]$$

$$+ \; [C_+ + \mathcal{M}(z,x)C_-]^{-1}(-\mathcal{X}(z)\mathcal{M}(z,x)C_+)$$

$$= [C_+ + \mathcal{M}(z,x)C_-]^{-1}\mathcal{X}(z)\mathcal{M}(z,x)[C_-M(x,z) - C_+] \; .$$ Errata

Hence

Errata $$\frac{\partial M(x,z)}{\partial z} = [C_+ + \mathcal{M}(z,x)C_-]^{-1}\mathcal{X}(z)[C_- - C_+M(x,z)] . \qquad (34)$$

Replacing $\mathcal{M}(z,x)$ in (34) by either of its representations, Errata (11) or (12) of Sec. 7.4, the desired differential equation is obtained. The details are left to the interested reader.

Analysis of the Differential Equation for R(y,b)

The differential equation for R(a,b), as given in (18) of Sec. 7.1, was shown in the discussion of that section to be of central importance in evaluating the reflectance and transmittance operators associated with a plane-parallel medium X(a,b). In view of this importance, it is desirable to gain as much insight as possible into the structure of the differential equation governing R(a,b). We now analyze the equation for R(a,b), in two different manners, into a relatively simple pair of linear operator equations using the invariant imbedding relation. The result will perhaps shed some light on the methods of determining radiance fields within natural optical media.

We begin with the semigroup relation (53) of Sec. 3.7:

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,y,b)\mathcal{R}(y,z,b) \quad .$$

By setting y = z in this relation, $\mathcal{R}(y,z,b)$ becomes R(y,b), so that:

$$\boxed{R(y,b) = \mathcal{T}^{-1}(a,y,b)\mathcal{R}(a,y,b)} \quad . \qquad (35)$$

This is the key representation for the reflectance operator R(y,b) in X(a,b) using the complete reflectance and transmittance operators $\mathcal{R}(a,y,b)$ and $\mathcal{T}(a,y,b)$. The inverse of the operator $\mathcal{T}(a,y,b)$ usually exists in most natural media, and so we shall proceed on the assumption of its availability, in order to see where it leads. Now, it follows from (15) and (17) that:

$$-\frac{\partial \mathcal{R}(a,y,b)}{\partial y} = \mathcal{R}(a,y,b)\tau(y) + \mathcal{T}(a,y,b)\rho(y) \tag{36}$$

$$\frac{\partial \mathcal{T}(a,y,b)}{\partial y} = \mathcal{R}(a,y,b)\rho(y) + \mathcal{T}(a,y,b)\tau(y) \ . \tag{37}$$

Therefore, on differentiating each side of (35) with respect to y:

$$\frac{\partial R(y,b)}{\partial y} =$$

$$= \frac{\partial \mathcal{T}^{-1}(a,y,b)}{\partial y} \mathcal{R}(a,y,b) + \mathcal{T}^{-1}(a,y,b) \frac{\partial \mathcal{R}(a,y,b)}{\partial y}$$

$$= -\mathcal{T}^{-1}(a,y,b)\left[\mathcal{R}(a,y,b)\rho(y) + \mathcal{T}(a,y,b)\tau(y)\right]\mathcal{T}^{-1}(a,y,b)\mathcal{R}(a,y,b)$$

$$+\mathcal{T}^{-1}(a,y,b)\left[-\mathcal{R}(a,y,b)\tau(y) - \mathcal{T}(a,y,b)\rho(y)\right] \ .$$

Using (35) again, this may be simplified to :

$$\frac{\partial R(y,b)}{\partial y} = -R(y,b)\rho(y)R(y,b) - \tau(y)R(y,b) - R(y,b)\tau(y) - \rho(y) \ .$$

On rearranging the preceding equation, we obtain (18) of Sec. 7.1. Equations (35), (36), (37) therefore constitute the required analysis of (18) of Sec. 7.1. We may summarize this finding alternatively as follows: *The system* (36), (37) *of linear differential equations for the complete* $\mathcal{R}$ *and* $\mathcal{T}$ *operators together with* (35) *uniquely determines* R(y,b) *in* X(a,b). The system (36), (37) may be represented succinctly by:

$$\boxed{\frac{d\mathcal{A}(y)}{dy} = \mathcal{A}(y)\mathcal{K}(y)} \tag{38}$$

where we have written:

$$\text{"}\mathcal{A}(y)\text{"} \quad \text{for} \quad (\mathcal{R}(a,y,b), \mathcal{T}(a,y,b)) \ .$$

Therefore the construction of $\mathcal{R}(a,y,b)$, $a < y < b$ is tantamount to solving (38) over the interval [a,b] with the initial condition:

$$\mathcal{A}(b) = (0, T(a,b)) \tag{39}$$

and then working up from level b to level y in X(a,b); or with the initial condition:

$$\mathcal{A}(a) = (R(a,b), I) \tag{40}$$

and then working down from level a to level y in X(a,b).

The physical significance of these observations is quite interesting. Suppose we are confronted with a plane-parallel optical medium X(a,b) (such as a portion of the atmosphere or the sea) and know only its overall transmittance T(a,b). *It is therefore in principle possible, via* (38) *and* (39), *to find* $\mathcal{R}(a,y,b)$, $\mathcal{T}(a,y,b)$ *for every intermediate level y within* X(a,b), *and hence* R(y,b) *for every intermediate level y within* X(a,b) *knowing the inherent optical properties of* X(a,b). A similar observation may be made using (40) and knowledge of R(a,b). Putting this in even more practical terms: one can essentially find the light field in the atmosphere at any altitude y knowing the overall transmittance T(a,b) of X(a,b) and $\mathcal{X}(y)$ throughout X(a,b); or the light field in the sea at any depth y can be obtained from R(a,b) and $\mathcal{X}(y)$ throughout X(a,b). The initial incident radiance distributions in each case are, of course, assumed given at levels a and b. Thus under suitable conditions the system (38) can yield knowledge of the radiometric situation inside a medium X(a,b), by knowing either the overall transmittance T(a,b) or overall reflectance R(a,b) of X(a,b). These observations are especially useful in the context of separable plane-parallel media, for in such media the R and T operators do not possess polarity (Sec. 7.1). Hence T(a,b) = T(b,a) and R(a,b) = R(b,a), so that the number of interaction operators for X(a,b) is cut in half.

We turn now to the second analysis of (18) of Sec. 7.1. The second manner of analyzing (18) of Sec. 7.1 is carried out by starting with the representation:

$$R(y,b) = \mathcal{M}_{--}^{-1}(b,y)\mathcal{M}_{-+}(b,y) \tag{41}$$

which is obtained from (29) of Sec. 7.4. The similarity of this representation with (35) is quite close: in each case the inverse operator is that of a transmittance-like operator, the remaining factor being a reflectance-like operator. In the present analysis we have, corresponding to (36) and (37), the following equations:

$$-\frac{\partial \mathcal{M}_{-+}(b,y)}{\partial y} = \mathcal{M}_{-+}(b,y)\tau(y) + \mathcal{M}_{--}(b,y)\rho(y) \tag{42}$$

$$\frac{\partial \mathcal{M}_{--}(b,y)}{\partial y} = \mathcal{M}_{-+}(b,y)\rho(y) + \mathcal{M}_{--}(b,y)\tau(y) \tag{43}$$

which are derived from (3) and (5). It should now be clear, without any further detailed discussion, that the system (42), (43), along with (41), determines R(y,b) for every level y in X(a,b). The parallel with the preceding analysis is completed by writing (42), (43) in matricial form and adducing the requisite initial condition, the present counterparts to (39) and (40). The only salient difference between the two analyses just given is that the concepts used in the first analysis are slightly more meaningful physically, and that the initial conditions in the second analysis are perhaps more convenient numerically. Thus, writing:

$$"\mathcal{B}(y)" \quad \text{for} \quad \{\mathcal{M}_{-+}(b,y), \mathcal{M}_{--}(b,y)\} \tag{44}$$

the system (42), (43) can be written:

$$\boxed{\frac{d\mathcal{B}(y)}{dy} = \mathcal{B}(y)\,\mathcal{X}(y)} \tag{45}$$

for $a \le y \le b$, and we have as initial condition:

$$\mathcal{B}(b) = (0, I) \quad . \tag{46}$$

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7.6 Special Solution Procedures for R(a,b) and T(a,b) in Plane-Parallel Media

In this and the remaining sections of this chapter we shall discuss some of the solution procedures for light fields in natural media suggested by the theories of the preceding sections. The discussions will also serve to exhibit the inner analytic structure of the functional equations for the R and T operators. We begin with the differential equation (18) of Section 7.1 for the reflectance operator R(a,b) of a plane-parallel medium. In order to illustrate the procedure of reducing R(a,b) to the appropriate forms on a numerical level, we assume in this section that the medium X(a,b) is homogeneous and that its volume scattering function σ is isotropic, i.e., that:

$$\sigma(z;\xi';\xi) = s/4\pi \tag{1}$$

where s is the volume total scattering function for the medium.

Starting with (18) of Sec. 7.1, reproduced here for convenience; we have:

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a) + \tau(a)R(a,b) + R(a,b)\tau(a) + R(a,b)\rho(a)R(a,b) \tag{2}$$

This is the differential equation for the reflectance operator R(a,b) for downward incident flux on X(a,b). Our immediate objective is to "shell" each term of (2) and to extract the kernel function of each of the indicated operators. It is the kernel function of R(a,b) which is to be evaluated in the present discussion, and we must somehow lay bare its presence in (2). An effective means towards this end is to postulate that the only radiance distribution on X(a,b) is incident on the upper boundary X_a of X(a,b) and is a radiance function $N_-(a)$ with Dirac-delta structure, i.e., for some positive radiance N^o and vector ξ^o in Ξ_-,

$$N_-(a)(x,\xi) = N^o\delta(\xi - \xi^o) \tag{3}$$