convenient numerically. Thus, writing:

$$"\mathcal{B}(y)" \quad \text{for} \quad \{\mathcal{M}_{-+}(b,y), \mathcal{M}_{--}(b,y)\} \tag{44}$$

the system (42), (43) can be written:

$$\boxed{\frac{d\mathcal{B}(y)}{dy} = \mathcal{B}(y)\,\mathcal{X}(y)} \tag{45}$$

for $a \le y \le b$, and we have as initial condition:

$$\mathcal{B}(b) = (0,I) \quad . \tag{46}$$

7.6 Special Solution Procedures for R(a,b) and T(a,b) in Plane-Parallel Media

In this and the remaining sections of this chapter we shall discuss some of the solution procedures for light fields in natural media suggested by the theories of the preceding sections. The discussions will also serve to exhibit the inner analytic structure of the functional equations for the R and T operators. We begin with the differential equation (18) of Section 7.1 for the reflectance operator R(a,b) of a plane-parallel medium. In order to illustrate the procedure of reducing R(a,b) to the appropriate forms on a numerical level, we assume in this section that the medium X(a,b) is homogeneous and that its volume scattering function σ is isotropic, i.e., that:

$$\sigma(z;\xi';\xi) = s/4\pi \tag{1}$$

where s is the volume total scattering function for the medium.

Starting with (18) of Sec. 7.1, reproduced here for convenience; we have:

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a)+\tau(a)R(a,b)+R(a,b)\tau(a)+R(a,b)\rho(a)R(a,b) \tag{2}$$

This is the differential equation for the reflectance operator R(a,b) for downward incident flux on X(a,b). Our immediate objective is to "shell" each term of (2) and to extract the kernel function of each of the indicated operators. It is the kernel function of R(a,b) which is to be evaluated in the present discussion, and we must somehow lay bare its presence in (2). An effective means towards this end is to postulate that the only radiance distribution on X(a,b) is incident on the upper boundary X_a of X(a,b) and is a radiance function $N_-(a)$ with Dirac-delta structure, i.e., for some positive radiance N^o and vector ξ^o in Ξ_-,

$$N_-(a)(x,\xi) = N^o\delta(\xi-\xi^o) \tag{3}$$

for every point x in X_a and direction ξ in Ξ_-. Once the response of X(a,b) to this arbitrary singular input is determined, the corresponding response to an arbitrary input is determinable by an integration over the direction set of the new input radiance distribution. The basis for this rests on the additivity and continuity properties of the function S defined in (7) of Sec. 3.6. This function, via the definitions (8)-(11) of Sec. 3.6, and (31),(32) of Sec. 3.7, yields the desired integral representations of R(a,b) and T(a,b) for X(a,b). Thus:

$$R(a,b) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ ]R(a,b;\xi';\xi)\ d\Omega(\xi') \tag{4}$$

$$T(a,b) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ ]T(a,b;\xi';\xi)\ d\Omega(\xi')\quad . \tag{5}$$

Our present goal is to describe a solution procedure for the reflectance *function* $R(a,b;\cdot;\cdot)$ on $\Xi_-\times\Xi_+$. Toward this end, we apply each of the five terms in (2) to the function $N_-(a)$ as defined in (3):

$$N_-(a)\ \frac{\partial R(a,b)}{\partial a} = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\ \frac{\partial R(a,b;\xi';\xi)}{\partial a}\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \frac{\partial R(a,b;\xi^o;\xi)}{\partial a} \tag{6}$$

$$N_-(a)\rho(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \sigma(a;\xi^o;\xi)\quad . \tag{7}$$

As for the term $\tau(a)R(a,b)$ in (2), we reduce it in two stages; first:

$$N_-(a)\tau(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\ d\Omega(\xi') - \frac{N^o\delta(\xi-\xi^o)\alpha(a)}{|\xi\cdot\mathbf{k}|}$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \left[\sigma(a;\xi^o;\xi) - \delta(\xi-\xi^o)\alpha(a)\right]\quad .$$

Second:

$$\big(N_-(a)\tau(a)\big)R(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\frac{1}{|\xi'\cdot\mathbf{k}|}\big[\sigma(a;\xi^o;\xi')-\delta(\xi'-\xi^o)\alpha(a)\big]R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\right]\sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi^o\cdot\mathbf{k}|\,|\xi\cdot\mathbf{k}|}R(a,b;\xi^o;\xi) \quad . \tag{8}$$

Next, the term $R(a,b)\tau(a)$ yields up, in turn:

$$N_-(a)R(a,b) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\delta(\xi'-\xi^o)R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}R(a,b;\xi^o;\xi) \tag{9}$$

$$\big(N_-(a)R(a,b)\big)\tau(a) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}R(a,b;\xi^o;\xi')\right]\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi\cdot\mathbf{k}|^2}R(a,b;\xi^o;\xi) \quad . \tag{10}$$

As for the term $R(a,b)\rho(a)R(a,b)$, we use (9) to obtain at once:

$$\big(N_-(a)R(a,b)\big)\rho(a) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\frac{N^o}{|\xi'\cdot\mathbf{k}|}R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

Hence:

$$\big(N_-(a)R(a,b)\rho(a)\big)R(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\left[\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\cdot\frac{1}{|\xi''\cdot\mathbf{k}|}\right]R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')R(a,b;\xi'';\xi)\,d\Omega(\xi')\right]d\Omega(\xi'') \tag{11}$$

The General Equation for $R(a,b;\xi';\xi)$

Assembling the results (6), (7), (8), (10) and (11) by means of (2), and rearranging terms, we have:

$$-\frac{\partial R(a,b;\xi^o;\xi)}{\partial a} + \alpha(a)\left(\frac{1}{\mu^o} + \frac{1}{\nu}\right) R(a,b;\xi^o;\xi) =$$

$$= \sigma(a;\xi^o;\xi) + \int_{\Xi_-} \sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\, \frac{d\Omega(\xi')}{\mu'}$$

$$+ \int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\, \frac{d\Omega(\xi')}{\nu'}$$

$$+ \int_{\Xi_-} \left[\int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')R(a,b;\xi'';\xi)\, \frac{d\Omega(\xi')}{\nu'}\right] \frac{d\Omega(\xi'')}{\mu''} \tag{12}$$

where we have written:

$$\text{"}\mu\text{"} \quad \text{for} \quad |\xi\cdot\mathbf{k}| \tag{13}$$

whenever ξ is in Ξ_- and:

$$\text{"}\nu\text{"} \quad \text{for} \quad |\xi\cdot\mathbf{k}| \tag{14}$$

whenever ξ is in Ξ_+. The minus sign before the derivative term in (12) rests on the physically based convention of measuring distance positively in the downward (or inward) direction in $X(a,b)$ from the boundary X_a. Equation (12) is the general integrodifferential equation for the reflectance function. Despite its apparent formidability, the equation is, in the last analysis, relatively tractable, since $R(a,b;\xi';\xi)$ is constructable from (12) starting with the initial condition:

$$R(b,b;\xi';\xi) = 0 \tag{15}$$

for every (ξ',ξ) in $\Xi_-\times\Xi_+$. In other words, (12) and (15) define the R function in terms of an initial value (or one-point boundary value) problem, a type of problem eminently suitable for the grist of modern electronic computer mills.

The Isotropic Scattering Case for R

A further simplification in the reduction of (2) is now possible, using the homogeneity and the isotropic scattering property (1) of $X(a,b)$. Writing:

$$\text{"}\rho\text{"} \quad \text{for} \quad s/\alpha$$

as usual, (12) reduces to:

$$-\frac{1}{\alpha}\frac{\partial R(a,b;\xi^o;\xi)}{\partial a} + \left(\frac{1}{\mu^o} + \frac{1}{\nu}\right) R(a,b;\xi^o;\xi) =$$

$$= \frac{\rho}{4\pi}\left[1 + \int_{\Xi_-} R(a,b;\xi';\xi)\frac{d\Omega(\xi')}{\mu'} + \int_{\Xi_+} R(a,b;\xi^o;\xi')\frac{d\Omega(\xi')}{\nu'}\right.$$

$$\left. + \int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')R(a,b;\xi'';\xi)\frac{d\Omega(\xi')}{\nu'}\cdot\frac{d\Omega(\xi'')}{\mu''}\right] \quad (16)$$

A further reduction is possible in (16) by noting the following two facts. First, from an examination of (16) or the equation of transfer for the uniform scattering case (i.e., under the assumption (1) on σ) it follows that the radiance distribution $N(z,\cdot)$ at depth z in $X(a,b)$ is azimuth-independent, that is

$$N(z,\xi) = N(z,\xi')$$

whenever

$$\xi\cdot\mathbf{k} = \xi'\cdot\mathbf{k} \quad .$$

Indeed, the path function for such a medium as the present one is of the form:

$$N_*(z,\xi) = \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\, d\Omega(\xi')$$

$$= \frac{s}{4\pi}\int_{\Xi} N(z,\xi')\, d\Omega(\xi')$$

$$= \frac{sh(z)}{4\pi} \quad .$$

Hence N_* depends on depth only so that the path radiance of a path of sight with initial point of depth z_o, direction ξ, and length r is:

$$N_r^*(z,\xi) = \int_0^r N_*(z',\xi)T_{r-r'}(z_o,\xi)\, dr'$$

$$= \frac{s}{4\pi}\int_0^r h(z')e^{-\alpha(r-r')}\, dr'$$

where

$$z' = z_o - r'\xi\cdot\mathbf{k}$$

This shows quite clearly that $N_r^*(z,\xi)$ is azimuth independent. For if the path is changed only in azimuth, (so that $\xi\cdot\mathbf{k}$ is unchanged) the result N_r^* of the preceding calculation is basically unchanged. This azimuthal independence of

N_r^* is then inherited by $N_r(z,\xi)$ for all ranges r, depths z, and all directions ξ in Ξ_+ (and, interestingly, for all ξ' in Ξ_-, except one, namely ξ^o, because of the singular residual radiance $N^o(z,\xi^o)$ at each depth z). Hence in particular $N_r(a,\xi)$ with $r = (b-a)/|\xi\cdot\mathbf{k}|$, namely the reflected radiance from X(a,b), is azimuth independent. In this way the values $R(a,b;\xi';\xi)$ themselves are seen to be azimuth independent of each direction ξ', and ξ. This independence serves to cut down the number of variables needed to describe $R(a,b;\xi';\xi)$. Indeed, we need henceforth only write:

$$\text{"}R(a,b;\mu',\nu)\text{"} \quad \text{for} \quad R(a,b;\xi';\xi) \tag{17}$$

in order to go on with the solution procedure, where μ' and ν are defined in (13), (14). (See Fig. 7.10.) Hence we reduce the number of directional variables from four (two real numbers each for ξ',ξ) to two, namely μ' and ν. This, then, is the first simplification (16) may undergo.

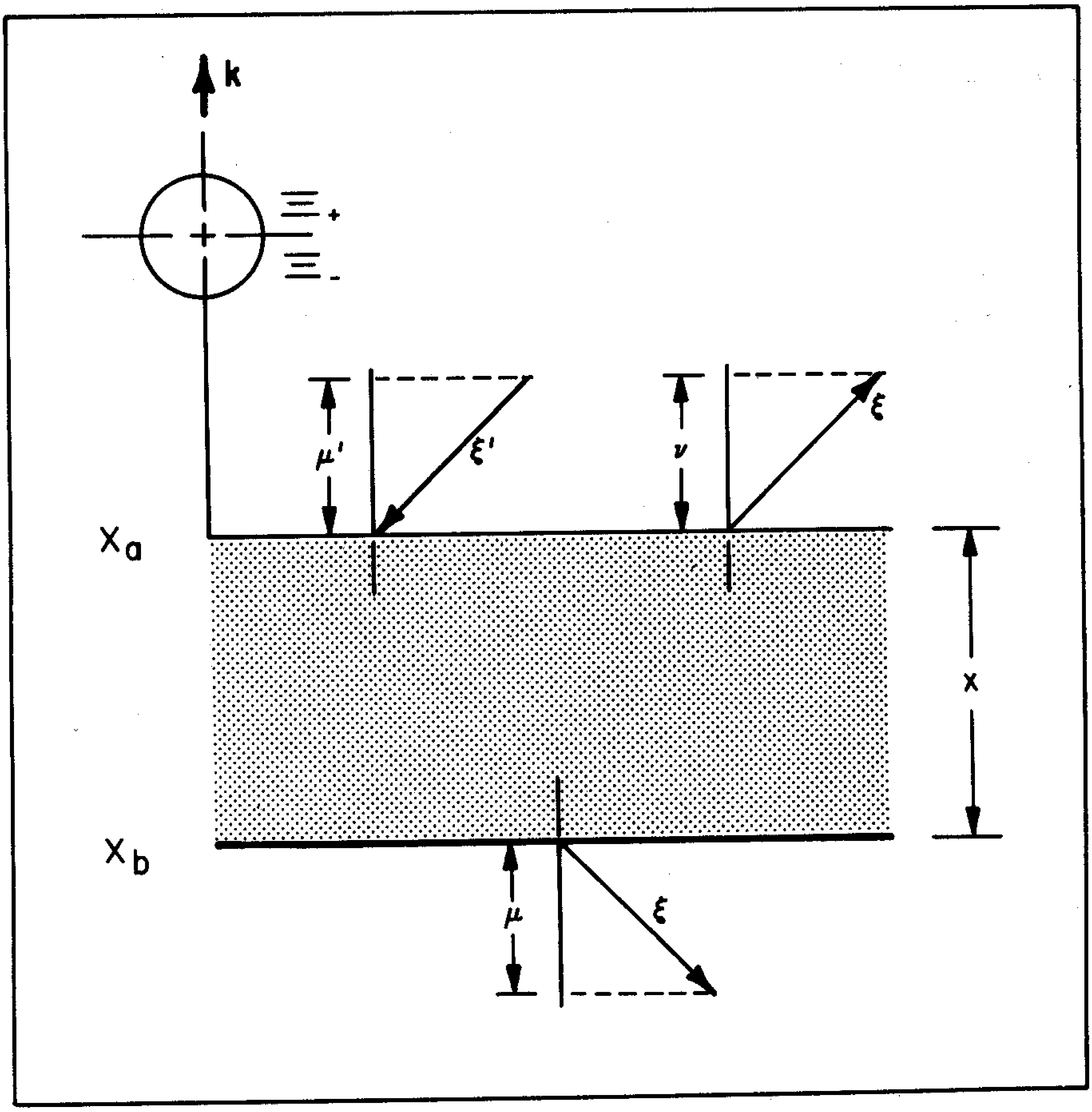

FIG. 7.10 Direction conventions for reflectance and transmittance operators on a slab X(a,b).

The remaining simplification of (16) is to note that the sum of the four terms on the right of (16) may be neatly factored into the product of two terms. Combining these two reductions, and noting that, in view of (17), we may write:

$$\int_{\Xi_-} R(a,b;\xi';\xi)\ \frac{d\Omega(\xi')}{\mu'} = \int_0^{2\pi}\int_0^1 R(a,b;\mu',\nu)\ \frac{d\mu' d\phi}{\mu'}$$

$$= 2\pi \int_0^1 R(a,b;\mu',\nu)\ \frac{d\mu'}{\mu'} \qquad (18)$$

along with:

$$\int_{\Xi_+} R(a,b;\xi,\xi')\ \frac{d\Omega(\xi')}{\nu'} = 2\pi \int_0^1 R(a,b;\mu',\nu)\ \frac{d\nu}{\nu} \qquad (19)$$

we have, at last:

$$-\frac{1}{\alpha}\frac{\partial R(a,b;\mu',\nu)}{\partial a} + \left(\frac{1}{\mu'} + \frac{1}{\nu}\right) R(a,b;\mu',\nu) =$$

$$= \frac{\rho}{4\pi}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\ \frac{d\mu'}{\mu'}\right]\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\ \frac{d\nu}{\nu}\right] \qquad (20)$$

This equation may be 'rationalized' if desired, by writing:

"$r(a;\mu',\nu)$" for $4\pi R(a,b;\mu',\nu)$,

thereby suppressing also the inactive fixed variable b and resulting in the equation:

$$-\frac{1}{\alpha}\frac{\partial r(a;\mu',\nu)}{\partial a} + \left(\frac{1}{\mu'} + \frac{1}{\nu}\right) r(a;\mu',\nu) =$$

$$= \rho\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\ \frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\ \frac{d\nu}{\nu}\right] \qquad (21)$$

The variable a in (21) may be changed to one, say x which is the distance, in terms of attenuation lengths, measured positively upward from the lower boundary X_b of $X(a,b)$. See Fig. 7.10. Then the derivative term becomes positive, the initial condition becomes:

$$r(0;\mu',\nu) = 0$$

for $0 < \mu' \leq 1$, $0 < \nu \leq 1$, and (21) is thereby ready for numerical solution. The numerical solution is facilitated by using reciprocity, namely that $r(x;\mu',\nu) = r(x;\nu,\mu')$. This is readily established using the functional equations for $R(a,b;\xi',\xi)$ and the isotropy of the medium. Reciprocity will be discussed further in Sec. 7.12. Observe that the homogeneity of $X(a,b)$ is used in an essential way in reaching (21). Hence (21) may be used as it stands for separable media, i.e., media in which ρ is independent of depth. A slight generalization is possible by letting ρ vary with depth.

A Sample Numerical Solution for $r(x;\mu',\nu)$

A numerical solution of (21) was recently constructed using an electronic computer [15]. Figures 7.11, 7.12, 7.13, and 7.14 summarize some typical results of the computation. Reference [15] should be consulted for full details. However, the following basic information of radiative transfer interest may be noted here: for homogeneous media with scattering-absorption ratios $\rho \leq 0.5$ (called "λ" in the cited reference) a thickness of three optical depths is essentially equivalent to infinite thickness as far as reflectance is concerned, the agreement being to two or three decimal places. Thus $r(3,\mu',\nu)$ differs insignificantly from $r(\infty,\mu',\nu)$ for ρ values encountered in natural optical media. In general, as ρ increases (i.e., as s/α increases) toward 1, the infinite medium reflectance $r(\infty,\mu,\nu)$ is approached more slowly by $r(x;\mu',\nu)$. For example when $\rho = 0.9$, six optical depths are

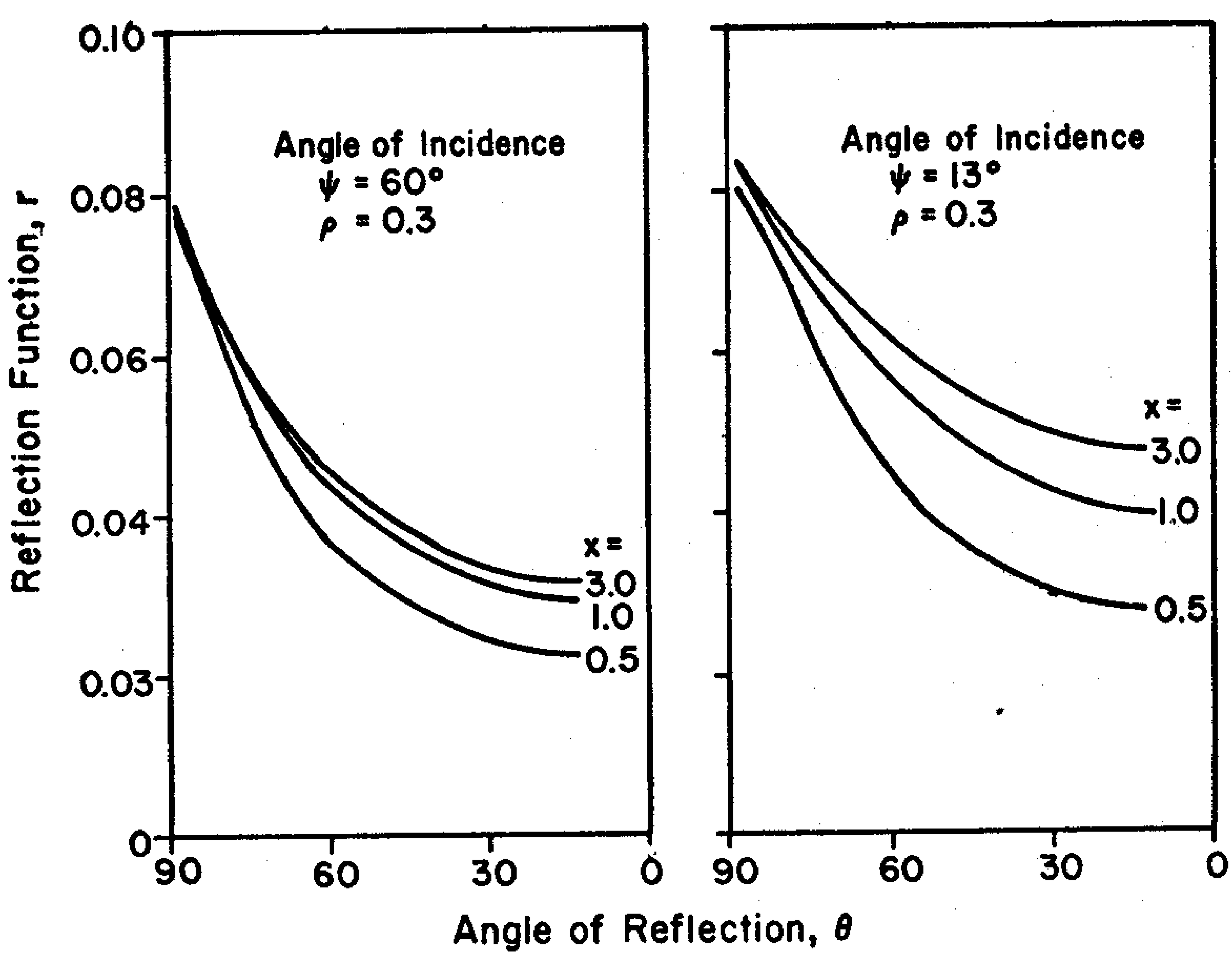

FIG. 7.11 Some typical curves with scattering-attenuation ratio $\rho = 0.3$. (From [15] by permission.)

needed to reasonably simulate infinite optical depth; and when $\rho = 1.0$, more than twenty attenuation lengths are needed to obtain agreement with $r(\infty,\mu',\nu)$ where three lengths sufficed above.

We observe in passing that the integrodifferential equation (21) yields immediately the equation governing $r(\infty,\mu',\nu)$. For, the derivative of $r(\infty,\mu',\nu)$ with respect to a is zero (cf. (30) of Sec. 7.3 and the comments below it), so that we obtain the following nonlinear integral equation for $r(\infty,\mu',\nu)$, (written, for brevity, as "$r(\mu',\nu)$"):

$$\left[\frac{1}{\mu'} + \frac{1}{\nu}\right] r(\mu',\nu) = \rho\left[1 + \frac{1}{2}\int_0^1 r(\mu',\nu)\,\frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(\mu',\nu)\,\frac{d\nu}{\nu}\right]$$

This equation played an important part in the early phases of modern radiative transfer theory (cf. [1], [2], [43]).

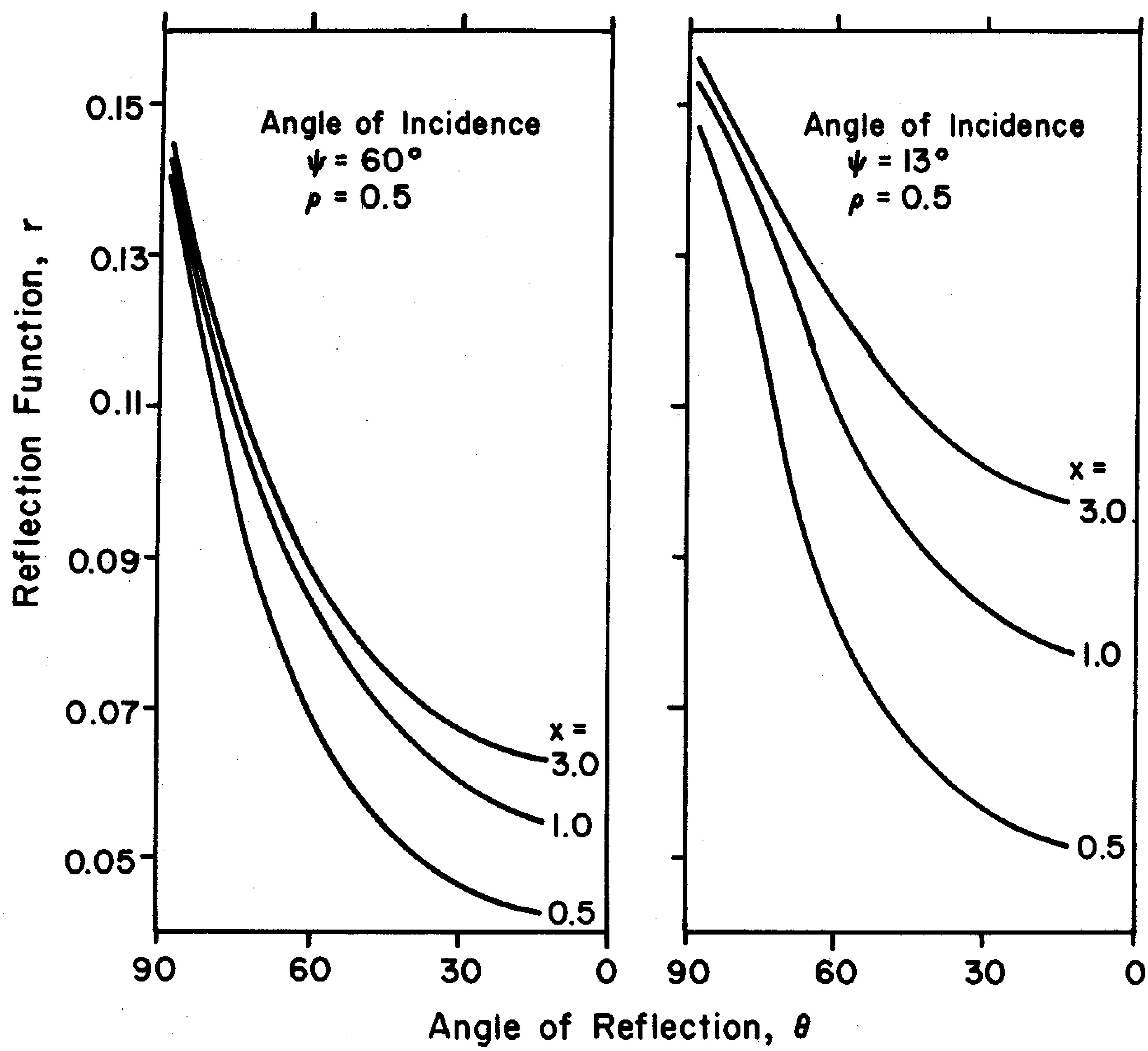

FIG. 7.12 Some typical curves with scattering-attenuation ratio $\rho = 0.5$. (From [15] by permission.)

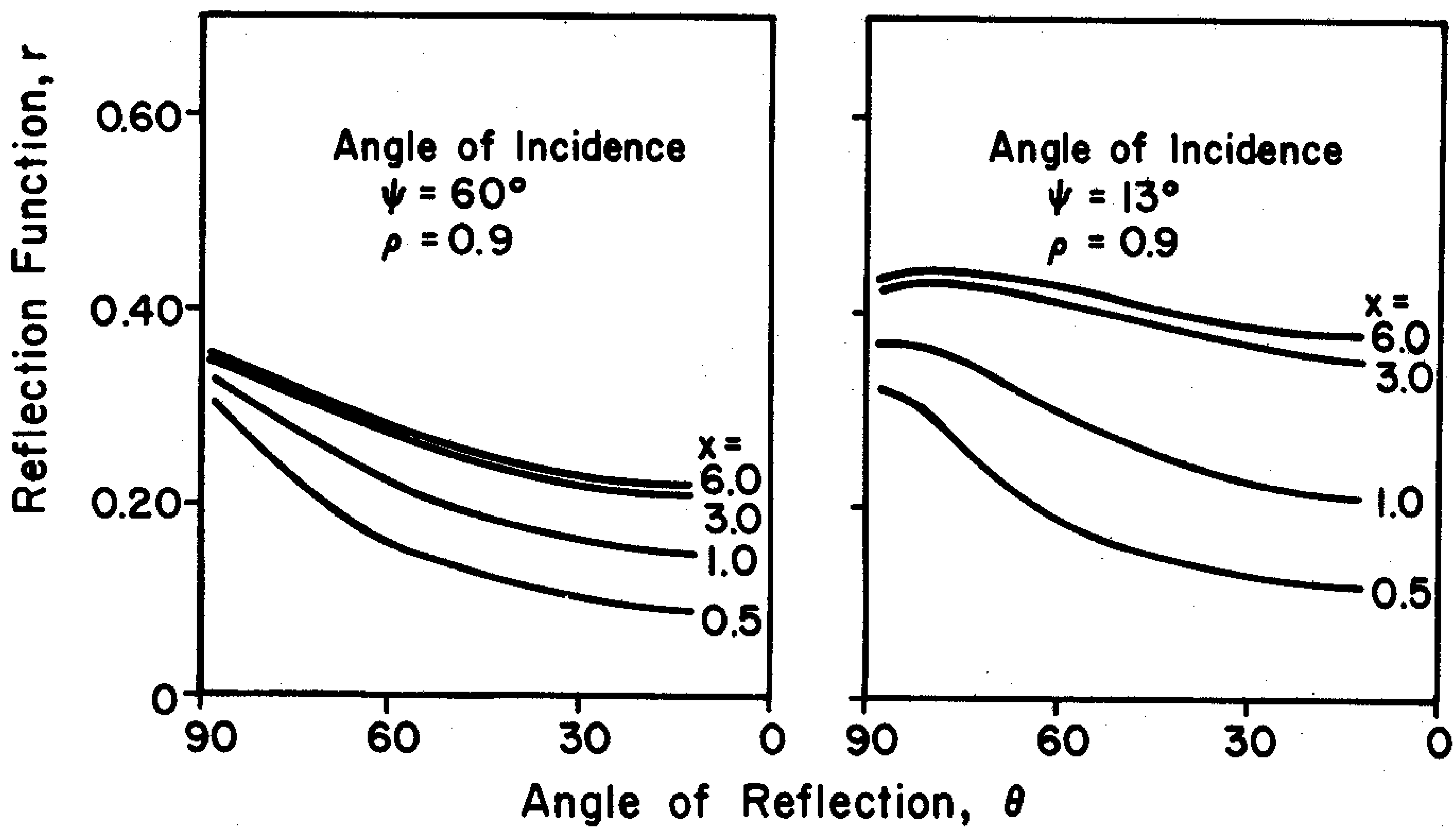

FIG. 7.13 Some typical curves with scattering-attenuation ratio ρ = 0.9. (From [15] by permission.)

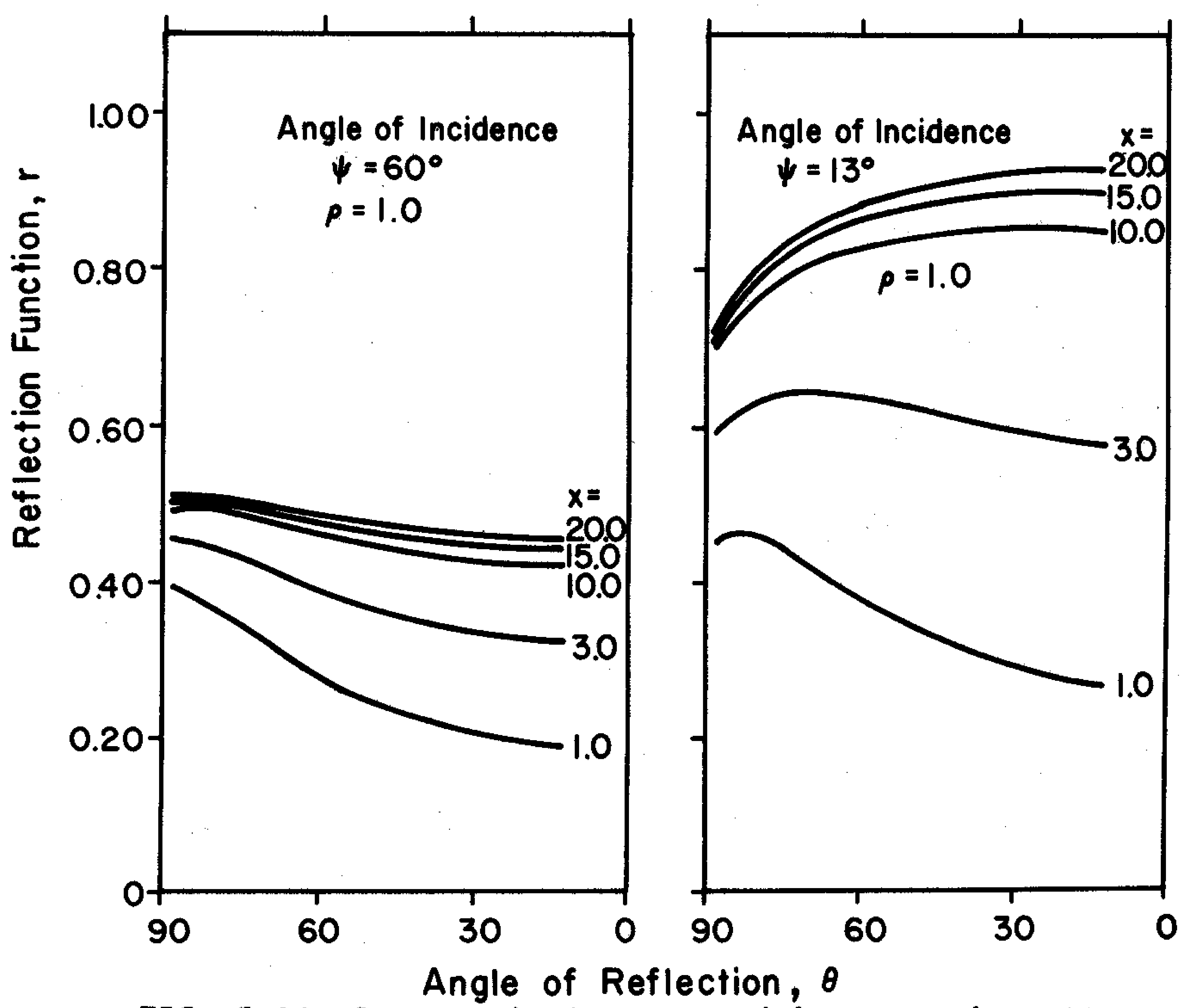

FIG. 7.14 Some typical curves with scattering-attenuation ratio ρ = 1.0. (From [15] by permission.)

The General Equation for $T^*(a,b;\xi';\xi)$

The tabulations of $R(x;\mu,\nu)$ in [16] may be built upon to obtain the companion transmittance function $T(x;\mu',\mu)$ by using equations (27) or (29) of Sec. 7.1, suitably reduced. Since the R and T operators for the present homogeneous space $X(a,b)$ do not possess polarity, i.e., since we have $R(a,b) = R(b,a)$ and $T(a,b) = T(b,a)$, $X(a,b)$ has associated with it only two operators, so that finding $T(x;\mu',\mu)$ will round out the basic information needed to determine the light field within $X(a,b)$ given the external incident radiances.

The reduction details of (29) of Sec. 7.1 to function form generally proceed as do those of the operator equation for $R(a,b)$. Since the resultant equation is of some importance, we now pause to sketch the details of the reduction. First we observe that the ultimate use of the reduced equation will be in a numerical procedure rather than a theoretical discussion; therefore it would be desirable to use (68) of 7.1 instead of (29) of Sec. 7.1, for the reason that the angular dependence of $T^*(a,b;\xi';\xi)$ is continuous while that of $T(a,b;\xi';\xi)$ is discontinuous. The basis for this fact is given in detail in the discussion on "Functional Relations for Decomposed Light Fields" in Sec. 7.1.

Starting with the derivative term in (68) of Sec. 7.1 we have:

$$N_-(a)\,\frac{\partial T^*(a,b)}{\partial a} = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\,\frac{\partial T^*(a,b;\xi';\xi)}{\partial a}\,d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\,\frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} \tag{22}$$

Then, analogously to the second stage of finding (8) of Sec. 7.6:

$$\big(N_-(a)\tau(a)\big)T^*(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\frac{1}{|\xi'\cdot\mathbf{k}|}\Big[\sigma(a;\xi^o;\xi')-\delta(\xi'-\xi^o)\alpha(a)\Big]T^*(a,b;\xi';\xi)\,d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\frac{1}{|\xi'\cdot\mathbf{k}|}\cdot\sigma(a;\xi^o;\xi')T^*(a,b;\xi';\xi)\,d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi^o\cdot\mathbf{k}|\,|\xi\cdot\mathbf{k}|}\,T^*(a,b;\xi^o;\xi) \quad . \tag{23}$$

Using the expression for N(a)R(a,b)ρ(a) found just prior to (11) of Sec. 7.6, we have:

$$\left(N_-(a)R(a,b)\rho(a)\right)T^*(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\frac{1}{|\xi''\cdot\mathbf{k}|}\right]R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')T^*(a,b;\xi'';\xi)\,d\Omega(\xi')\,d\Omega(\xi'') \tag{24}$$

Some entirely new terms are next forthcoming from (68) of Sec. 7.1:

$$N_-(a)\tau^*(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\,d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\sigma(a;\xi^o;\xi) \quad . \tag{25}$$

Next, using (32) of Sec. 7.1 on the result just obtained:

$$\left(N_-(a)\tau^*(a)\right)T^o(a,b) = \int_{\Xi_-}\left[\frac{N^o}{|\xi'\cdot\mathbf{k}|}\sigma(a;\xi^o;\xi')\right]T_r(\not{p}',\xi')\delta(\xi-\xi')\,d\Omega(\xi')$$

$$= \frac{N^o\,e^{-\alpha r}}{|\xi\cdot\mathbf{k}|}\sigma(a;\xi^o;\xi) \tag{26}$$

where:

$$r = |b-a|/|\xi\cdot\mathbf{k}| \quad . \tag{27}$$

Finally, applying $T^o(a,b)$ now to $N_-(a)R(a,b)\rho(a)$:

$$\left(N_-(a)R(a,b)\rho(a)\right)T^o(a,b) =$$

$$= \int_{\Xi_-}\left[\frac{N^o}{|\xi'\cdot\mathbf{k}|}\int_{\Xi_+}\frac{1}{|\xi''\cdot\mathbf{k}|}R(a,b;\xi^o;\xi'')\sigma(a;\xi'';\xi')\,d\Omega(\xi')\right]T_r(\not{p}',\xi')\delta(\xi-\xi')\,d\Omega(\xi')$$

$$= \frac{N^o\,e^{-\alpha r}}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\frac{1}{|\xi'\cdot\mathbf{k}|}R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\,d\Omega(\xi') \tag{28}$$

where r is given in (27).

The preceding results are now ready for assembly in (68) of Sec. 7.1:

$$- \frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} + \frac{\alpha(a)}{\mu^o} T^*(a,b;\xi^o;\xi) =$$

$$= \int_{\Xi_-} \sigma(a;\xi^o;\xi')T^*(a,b;\xi';\xi) \frac{d\Omega(\xi')}{\mu'} +$$

$$+ \int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')T^*(a,b;\xi'';\xi) \frac{d\Omega(\xi')}{\nu'} \frac{d\Omega(\xi'')}{\mu''} +$$

$$+ e^{-\alpha r}\left[\sigma(a;\xi^o;\xi) + \int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi) \frac{d\Omega(\xi')}{\nu'}\right] \quad (29)$$

This is the requisite integrodifferential equation for $T^*(a,b;\xi';\xi)$ associated with the initial condition: (cf. (43) of Sec. 7.1):

$$T^*(b,b;\xi';\xi) = 0$$

for every ξ' in Ξ_- and ξ in Ξ_+.

The Isotropic Scattering Case for T^*

Under the conditions of homogeneity and isotropic scattering, (29) may be further reduced. Thus, analogously t the reduction of (12) to (16), (29) now goes over into:

$$- \frac{1}{\alpha} \frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} + \frac{1}{\mu^o} T^*(a,b;\xi^o;\xi) =$$

$$= \frac{\rho}{4\pi} \int_{\Xi_-} T^*(a,b;\xi';\xi) \frac{d\Omega(\xi')}{\mu'} +$$

$$+ \frac{\rho}{4\pi} \int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')T^*(a,b;\xi'';\xi) \frac{d\Omega(\xi')}{\mu'} \frac{d\Omega(\xi'')}{\nu''} +$$

$$+ \frac{\rho}{4\pi} e^{-\alpha r}\left[1 + \int_{\Xi_+} R(a,b;\xi^o;\xi') \frac{d\Omega(\xi')}{\nu'}\right] .$$

The preceding iterated integrals now uncouple, and, as before, the induced azimuthal symmetry encourages us to write:

$$\text{"}T^*(a,b;\mu',\mu)\text{"} \quad \text{for} \quad T^*(a,b;\xi';\xi) \quad , \tag{30}$$

so that we may use (19), and its present counterpart:

$$\int_{\Xi_-} T^*(a,b;\xi';\xi)\,\frac{d\Omega(\xi')}{\mu'} = 2\pi\int_0^1 T^*(a,b;\mu',\mu)\,\frac{d\mu'}{\mu'}$$

to come down to:

$$-\frac{1}{\alpha}\,\frac{\partial T^*(a,b;\mu',\mu)}{\partial a} + \frac{1}{\mu'}\,T^*(a,b;\mu',\mu) =$$

$$= \frac{\rho}{4\pi}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu}\right]\left[2\pi\int_0^1 T^*(a,b;\mu',\mu)\,\frac{d\mu'}{\mu'}\right] +$$

$$+ \frac{\rho}{4\pi}\,e^{-\alpha r}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu}\right] \quad . \tag{31}$$

This may be "rationalized" by writing:

$$\text{"}t^*(a;\mu',\mu)\text{"} \quad \text{for} \quad 4\pi T^*(a,b;\mu',\mu)$$

so that we have, after a final rearrangement:

$$-\frac{1}{\alpha}\,\frac{\partial t^*(a;\mu',\mu)}{\partial a} + \frac{1}{\mu'}\,t^*(a;\mu',\mu) =$$

$$= \rho\left[e^{-\alpha r} + \frac{1}{2}\int_0^1 t^*(a;\mu',\mu)\,\frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\nu}{\nu}\right] \tag{32}$$

This is the required integrodifferential equation for the diffuse transmittance function. In this equation, the function values:

$$\frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\nu}{\nu} \quad ,$$

which depend generally on a and μ', are known from the solution procedure for (21) and enter the solution procedure of (32) as given data, along with the initial condition:

$$t^*(0;\mu',\mu) = 0$$

for t^*, $0 < \mu' \le 1$, $0 < \mu \le 1$. Matters can be arranged so that the depth variable x of $t^*(x;\mu',\mu)$ is in attenuation lengths measured upward from the lower boundary X_b of X(a,b), thereby eliminating the minus sign in (32), and absorbing "α" into the derivative notation. That is, for numerical purposes, we can always change variables in (32) according to the equation $x = -y\alpha$, $a \le y \le b$. (See Figure 7.10.) The solution problem of (32) is now a straightforward initial value problem which may be reduced, by standard Gaussian quadrature procedures applied to the indicated integrals, to a finite system of simultaneous first order differential equations.

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7.7 General Solution Procedures for R(a,b) and T(a,b) in Plane-Parallel Media

We return to the general integrodifferential equations for $R(a,b;\xi';\xi)$ and $T^*(a,b;\xi';\xi)$ as given in (12) and (29) of Sec. 7.6, and develop a general numerical procedure for their solution without the benefit of homogeneity and isotropic scattering within the medium X(a,b). The approach we shall follow is quite direct, one which requires a minimum of numerical preliminaries, thereby leaving such matters for choice in the individual programming procedure for the numerical solution. For example, with only minor changes, the following analysis may be repeated using Gaussian quadrature procedures. For the purposes of the present exposition, the determination problem for the operators R(a,b) and T(a,b) is considered solved when their correct functional equations have been found and suitably reduced to an initial value problem for some set of approximating (or occasionally exact) differential equations. Perhaps the greatest value of the following discussion is to allow students of the subject to come to grips with the inner workings of the integral operators R(a,b), and T(a,b). Once this is done, perhaps some efficient solution procedures will eventually come to mind.

We begin with a partition of Ξ_- and Ξ_+ into m and n sets of directions A_i and B_i, respectively (see Fig. 7.15); that is, we assume:

$$\Xi_- = \bigcup_{i=1}^{m} A_i \; ; \quad A_i \cap A_j = \phi \tag{1}$$

$$\Xi_+ = \bigcup_{i=1}^{n} B_i \; ; \quad B_i \cap B_j = \phi \quad . \tag{2}$$

These partitions of Ξ_- and Ξ_+, if sufficiently fine, let the integrals over them be reduced to simple numerical sums, as follows.

Consider, for example, the integral term occurring in (12) of Sec. 7.6: