

$$t^*(0; \mu', \mu) = 0$$

for t^* , $0 < \mu' \leq 1$, $0 < \mu \leq 1$. Matters can be arranged so that the depth variable x of $t^*(x; \mu', \mu)$ is in attenuation lengths measured upward from the lower boundary X_b of $X(a, b)$, thereby eliminating the minus sign in (32), and absorbing " α " into the derivative notation. That is, for numerical purposes, we can always change variables in (32) according to the equation $x = -y\alpha$, $a \leq y \leq b$. (See Figure 7.10.) The solution problem of (32) is now a straightforward initial value problem which may be reduced, by standard Gaussian quadrature procedures applied to the indicated integrals, to a finite system of simultaneous first order differential equations.

7.7 General Solution Procedures for $R(a, b)$ and $T(a, b)$ in Plane-Parallel Media

We return to the general integrodifferential equations for $R(a, b; \xi'; \xi)$ and $T^*(a, b; \xi'; \xi)$ as given in (12) and (29) of Sec. 7.6, and develop a general numerical procedure for their solution without the benefit of homogeneity and isotropic scattering within the medium $X(a, b)$. The approach we shall follow is quite direct, one which requires a minimum of numerical preliminaries, thereby leaving such matters for choice in the individual programming procedure for the numerical solution. For example, with only minor changes, the following analysis may be repeated using Gaussian quadrature procedures. For the purposes of the present exposition, the determination problem for the operators $R(a, b)$ and $T(a, b)$ is considered solved when their correct functional equations have been found and suitably reduced to an initial value problem for some set of approximating (or occasionally exact) differential equations. Perhaps the greatest value of the following discussion is to allow students of the subject to come to grips with the inner workings of the integral operators $R(a, b)$, and $T(a, b)$. Once this is done, perhaps some efficient solution procedures will eventually come to mind.

We begin with a partition of E_- and E_+ into m and n sets of directions A_i and B_i , respectively (see Fig. 7.15); that is, we assume:

$$E_- = \bigcup_{i=1}^m A_i ; \quad A_i \cap A_j = \phi \quad (1)$$

$$E_+ = \bigcup_{i=1}^n B_i ; \quad B_i \cap B_j = \phi \quad (2)$$

These partitions of E_- and E_+ , if sufficiently fine, let the integrals over them be reduced to simple numerical sums, as follows.

Consider, for example, the integral term occurring in (12) of Sec. 7.6:

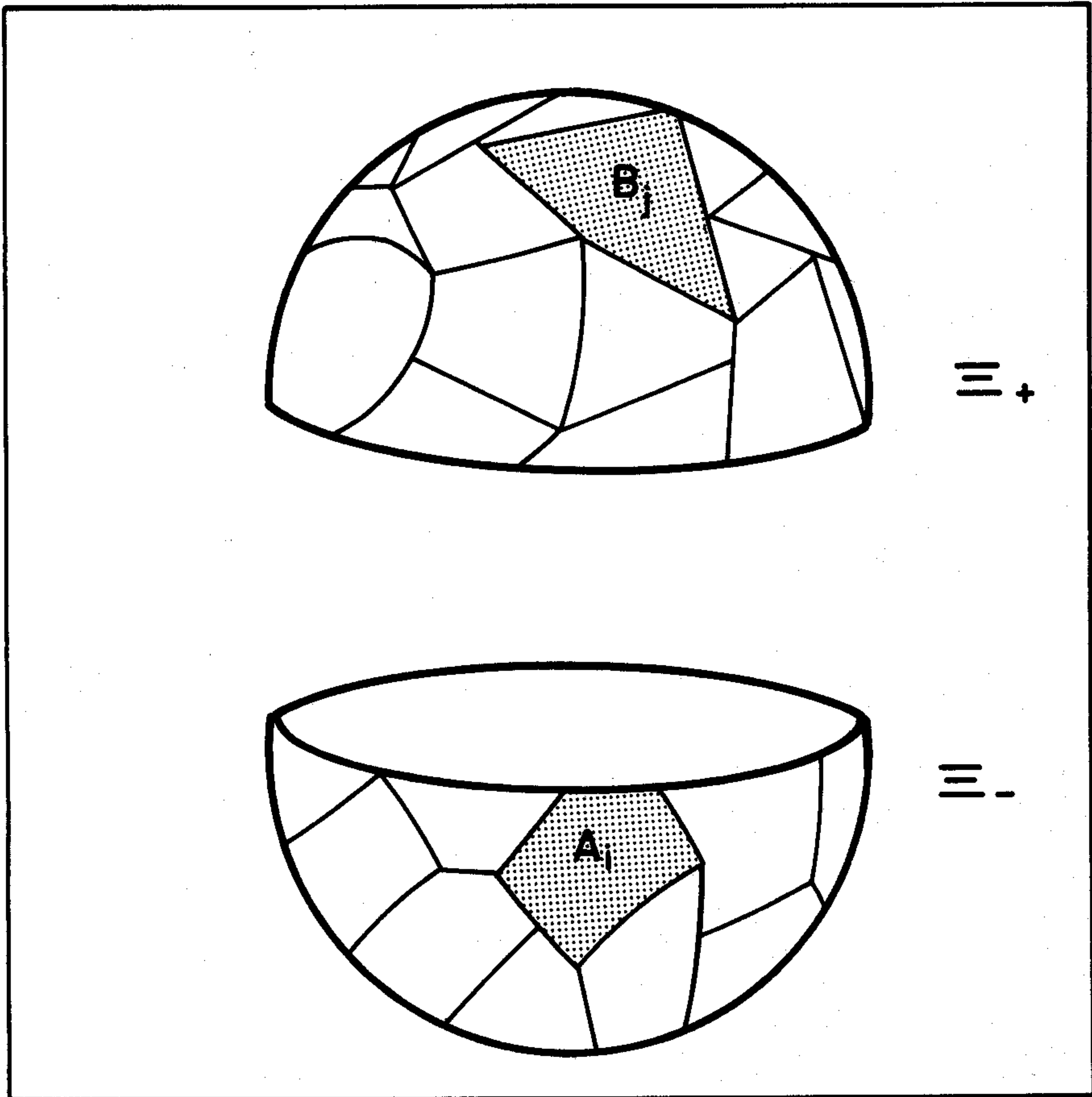


FIG. 7.15 General partitions of each half of the sphere of unit directions.

$$\int_{E_-} \sigma(a; \xi^0; \xi') R(a, b; \xi'; \xi) \frac{d\Omega(\xi')}{\mu'}$$

which with the partition of E_- in force, may be written equivalently as the sum:

$$\sum_{i=1}^m \int_{A_i} \sigma(a; \xi^0; \xi') R(a, b; \xi'; \xi) \frac{d\Omega(\xi')}{\mu'}$$

Now, if the partition of E_- is sufficiently fine and σ and R , as functions of ξ' are continuous (a condition always available in geophysical settings) the integral over A_i may be represented arbitrarily closely by a term of the form:

$$\sigma(a; \xi^0; \xi_i^!) R(a, b; \xi_i^!; \xi) \frac{\Omega(A_i)}{\mu_i^!}$$

where $\xi_i^!$ is a direction in A_i and $\mu_i^!$ is $|\xi_i^! \cdot \mathbf{k}|$.

Now ξ^0 and ξ as they occur in the reduced term, happen also (for some j and k) to be in partition members A_j and B_k respectively, so that we can completely replace the functions σ and R , for the present purposes, by sets of numerical quantities, as follows. We write:

$$"R(a, b; i, j)" \quad \text{for} \quad R(a, b; \xi_i; \xi_j) \quad (3)$$

where ξ_i and ξ_j are selected fixed directions in A_i and B_j , respectively. Hence the function $R(a, b; \cdot; \cdot)$ is replaced by the mn numbers $R(a, b; i, j)$. These direction selections are fixed for the remainder of this discussion. Further, we write:

$$"r_-(a; i, j)" \quad \text{for} \quad \sigma(a; \xi_i; \xi_j) \quad (4)$$

whenever ξ_i is in A_i and ξ_j is in B_j :

$$"r_+(a; i, j)" \quad \text{for} \quad \sigma(a; \xi_i; \xi_j) \quad (5)$$

whenever ξ_i is in B_i and ξ_j is in A_j :

$$"t_-(a; i, j)" \quad \text{for} \quad \sigma(a; \xi_i; \xi_j) \quad (6)$$

whenever ξ_i is in A_i and ξ_j is in A_j :

$$"t_+(a; i, j)" \quad \text{for} \quad \sigma(a; \xi_i; \xi_j) \quad (7)$$

whenever ξ_i is in B_i and ξ_j is in B_j .

The values in each of the four cases just defined are readily determinable from the given volume scattering function σ . Observe that we are not assuming that the medium is isotropic, so that no use will be made of reciprocity principles (of which $r(a; \mu', \nu) = r(a; \nu, \mu')$ was an instance in Sec. 7.6). With these four definitions we may disassemble (12) of Sec. 7.6 into the following system of differential equations which can approximate (12) of 7.6 arbitrarily closely:

$$\begin{aligned} & - \frac{\partial R(a, b; i, j)}{\partial a} + \alpha(a) \left[\frac{1}{\mu_i} + \frac{1}{\nu_j} \right] R(a, b; i, j) = \\ & = r_-(a; i, j) + \sum_{k=1}^m \tau_-(a; i, k) R(a, b; k, j) \frac{\Omega(A_k)}{\mu_k^!} + \\ & + \sum_{k=1}^n R(a, b; i, k) \tau_+(a; k, j) \frac{\Omega(B_k)}{\nu_k} \\ & + \sum_{\ell=1}^m \sum_{k=1}^n R(a, b; i, k) r_+(a; k, \ell) R(a, b; \ell, j) \frac{\Omega(B_k)}{\nu_k^!} \frac{\Omega(A_\ell)}{\mu_\ell} \quad (8) \end{aligned}$$

The unknowns in this system are the mn functions $R(a,b;i,j)$, $i = 1, \dots, m$; $j = 1, \dots, n$. Holding b fixed, a is allowed to vary from b up to any arbitrary distance above b . Hence we have an initial value problem, which is got underway using the initial condition:

$$R(b,b;i,j) = 0$$

for every choice of the mn directional pairs. The $(m+n)^2$ dimensionless numbers such as:

$$\rho_-(a;i,j) \cdot \frac{1}{\alpha(a)}, \quad \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n \end{array}$$

$$\tau_-(a;i,k) \frac{\Omega(A_k)}{\mu'_k} \cdot \frac{1}{\alpha(a)}, \quad \begin{array}{l} i = 1, \dots, m \\ k = 1, \dots, m \end{array}$$

$$\tau_+(a;k,j) \frac{\Omega(B_k)}{\nu_k} \cdot \frac{1}{\alpha(a)}, \quad \begin{array}{l} k = 1, \dots, n \\ j = 1, \dots, n \end{array}$$

and

$$\rho_+(a;k,l) \frac{\Omega(B_k)}{\nu'_k} \frac{\Omega(A_l)}{\mu_l} \frac{1}{\alpha(a)}, \quad \begin{array}{l} k = 1, \dots, n \\ l = 1, \dots, m \end{array}$$

can be assembled neatly into four matrices, denoted, say, by: " $r_-(a)$ ", " $t_-(a)$ ", " $t_+(a)$ ", " $r_+(a)$ ", respectively. (For an alternative procedure, see (17)-(21) below) Then writing " $R(a,b)$ " for the $m \times n$ matrix formed of the numbers $R(a,b;i,j)$, (8) can be written succinctly as:

$$\begin{aligned} - \frac{1}{\alpha(a)} \frac{\partial R(a,b)}{\partial a} + [DR(a,b) + R(a,b)E] &= \\ &= r_-(a) + t_-(a)R(a,b) + R(a,b)t_+(a) + R(a,b)r_+(a)R(a,b) \\ &= [r_-(a) + R(a,b)t_+(a)] + [t_-(a) + R(a,b)r_+(a)]R(a,b) \end{aligned}$$

(9)

where we have written:

$$"D" \quad \text{for} \quad \begin{bmatrix} 1/\mu_1 & & & 0 \\ & 1/\mu_2 & & \\ & & \ddots & \\ 0 & & & 1/\mu_m \end{bmatrix} \quad (10)$$

$$"E" \quad \text{for} \quad \begin{bmatrix} 1/\nu_1 & & & 0 \\ & 1/\nu_2 & & \\ & & \ddots & \\ 0 & & & 1/\nu_n \end{bmatrix} \quad (11)$$

A check of the linear dimensions of the matrices shows that they are all properly commensurate. Thus, $r_-(a)$ is $m \times n$, $t_-(a)$ is $m \times m$, $t_+(a)$ is $n \times n$ and $r_+(a)$ is $n \times m$. The choice of the partitions of Ξ_- and Ξ_+ governs the dimensions of these matrices. In some settings it is quite possible and, indeed, desirable to partition Ξ_- and Ξ_+ in essentially the same manner so that $m = n$ and the μ_i and ν_i are equal for each $i = 1, \dots, n$. Hence we would have $D = E$ and all matrices would be $n \times n$, and there would be n^2 equations in n^2 unknowns implicit in (9). As in the earlier reductions in Sec. 7.6, a transition to dimensionless depth parameters would be numerically convenient. Hence the change of variables given there should be adopted for numerical work. Equation (9) can be used to find both $R(a, b; \xi'; \xi)$ and $R(b, a; \xi''; \xi')$ in the case of non-separable media, simply by integrating from level X_b to X_a in the first case and from X_a to X_b in the second case (cf. Fig. 7.10 in which the integration from X_b to X_a is depicted).

It remains to reduce Equation (29) of Sec. 7.6 to its approximating matricial counterpart. We retain the general partition of Ξ used for $R(a, b)$ and write:

$$"T^*(a, b; i, j)" \quad \text{for} \quad T^*(a, b; \xi_i, \xi_j)$$

whenever ξ_i is in A_i and ξ_j is in A_j . In this way we generate the $m \times m$ matrix $T^*(a, b)$. The system of differential equations approximate to (29) of Sec. 7.6 may then be written:

$$\begin{aligned} - \frac{\partial T^*(a, b; i, j)}{\partial a} + \frac{\alpha(a)}{\mu_i} T^*(a, b; i, j) = & \\ = \sum_{k=1}^m \tau_-(a; i, k) T^*(a, b; k, j) \frac{\Omega(A_k)}{\mu_k} + & \\ + \sum_{\lambda=1}^m \sum_{k=1}^n R(a, b; i, k) \rho_+(a; k, \lambda) T^*(a, b; \lambda, j) \frac{\Omega(B_k)}{\nu_k} \frac{\Omega(A_\lambda)}{\mu_\lambda} + & \\ + T_{r_j}(a, b) \left[\tau_-(a; i, j) + \sum_{k=1}^n R(a, b; i, k) \rho_+(a; k, j) \frac{\Omega(B_k)}{\nu_k} \right] & \quad (12) \end{aligned}$$

where " $T_{r_j}(a, b)$ " denotes the beam transmittance $T_{r_j}(a, \xi_j)$ and r_j is now the distance r , as defined in (27) of Sec. 7.6 associated with ξ_j . Using the battery of matrices defined above, this can be cast into the compact matricial form:

$$\begin{aligned}
-\frac{1}{\alpha(a)} \frac{\partial T^*(a,b)}{\partial a} + DT^*(a,b) &= \\
&= t_-(a)T^*(a,b) + R(a,b)r_+(a)T^*(a,b) + \\
&+ [t_-(a) + R(a,b)r_+(a)]T^0(a,b)
\end{aligned} \tag{13}$$

where we have written:

$$\text{"T}^0(a,b)\text{" for } \begin{bmatrix} T_{r_1}(a,b) & & & 0 \\ & T_{r_2}(a,b) & & \\ & & \ddots & \\ 0 & & & T_{r_n}(a,b) \end{bmatrix} \tag{14}$$

The differential equation (13) may be written, compactly as:

$$\boxed{-\frac{1}{\alpha(a)} \frac{\partial T^*(a,b)}{\partial a} + DT^*(a,b) = [t_-(a) + R(a,b)r_+(a)]T(a,b)} \tag{15}$$

where we have written:

$$\text{"T}(a,b)\text{" for } T^0(a,b) + T^*(a,b) \tag{16}$$

The resemblance of the right side of (15) with that of (29) of Sec. 7.1 is striking. Similarly with (9) above and (18) of Sec. 7.1. The solution procedures for (9) and (15) can run along parallel to each other, for as (9) is solved for $R(a,b)$, these values could be fed into (15) a fraction of a second later to help construct $T^*(a,b)$. Indeed, a study of (9) and (15) shows that whole groups of terms of matrices are shared by both equations and that their simultaneous computation would help produce efficient computation programs.

A concluding word about the choice of definitions of the various matrices, such as $r_-(a)$ and $R(a,b)$ made above, is in order. These definitions are not unique and may be replaced by variants which, in the press of numerical work, may be found more amenable to the computational procedures than those exhibited above. For example, instead of $R(a,b;i,j)$, we could use as unknowns the terms:

$$R(a,b;i,j)/v_j \tag{17}$$

and then $R(a,b)$ would be made up anew of such entries. Once this is done, it automatically dictates the following recombinations of terms which are guided by an examination of (8) and the list of terms following it. Thus, instead of $\rho_-(a;i,j)/\alpha(a)$ we would have:

$$\rho_-(a; i, j) = \frac{1}{v_j} \frac{1}{\alpha(a)} \quad (18)$$

Further, we would adopt:

$$\tau_-(a; i, k) = \frac{\Omega(A_k)}{\mu_k'} \frac{1}{\alpha(a)} \quad (19)$$

$$\tau_+(a; k, j) = \frac{\Omega(B_k)}{v_k} \frac{1}{\alpha(a)} \quad (20)$$

as before. But now we would use:

$$\rho_+(a; k, l) = \frac{\Omega(B_k)}{v_k'} \frac{\Omega(A_l)}{\alpha(a)} \quad (21)$$

The resulting equations would have the same gestalt as (9) and (15) so that the same general numerical procedure would be applicable to either system.

7.8 The Method of Modules for Deep Homogeneous Media

Now that we have some explicit computational procedures for finding the R and T operators for a plane-parallel medium (as given, for example, in Secs. 7.6 and 7.7), we turn to the task of finding the actual light fields within the medium. Finding $R(a, b)$ and $T(a, b)$ for a plane-parallel medium $X(a, b)$ allows one to determine the reflected and transmitted radiance at the boundaries of $X(a, b)$, but they do not directly supply the radiance distributions at internal depths y , $a < y < b$, in $X(a, b)$. The concept that allows the systematic determination of these internal radiance distributions, given the standard reflectance and transmittance operators $R(x, z)$ and $T(x, z)$, $a \leq x \leq z \leq b$, within $X(a, b)$, is the invariant imbedding relation (36) of Sec. 3.7.

The general method of obtaining $N_+(y)$ and $N_-(y)$ for depths y , $a < y < b$, in $X(a, b)$ is delineated by the equations of Example 4 in Sec. 3.7 in which the invariant imbedding relation is derived. Our present purpose is to apply those general equations to a commonly encountered situation in hydrologic optics: the problem of the penetration of light into the sea, lakes, and other natural hydrosols. Now, some interesting features about such natural hydrosols is, first of all, that they are in many important instances homogeneous (or separable) and infinitely deep optically. This implies, among other facts, that the reflectances $R(a, y)$ reach an asymptotic value within a few attenuation lengths down from the surface X_a . That is, for all practical purposes, $R(a, y) = R(a, z)$ for y and z below some depth x . This phenomenon was touched upon in the discussions of the numerical solutions for $r(x; \mu', \nu)$ in Sec. 7.6. Another fact that may be