$$\rho_-(a;i,j)\ \frac{1}{\nu_j}\ \frac{1}{\alpha(a)} \quad . \tag{18}$$

Further, we would adopt:

$$\tau_-(a;i,k)\ \frac{\Omega(A_k)}{\mu_k'}\ \frac{1}{\alpha(a)} \tag{19}$$

$$\tau_+(a;k,j)\ \frac{\Omega(B_k)}{\nu_k}\ \frac{1}{\alpha(a)} \tag{20}$$

as before. But now we would use:

$$\rho_+(a;k,l)\ \frac{\Omega(B_k)}{\nu_k'}\ \frac{\Omega(A_l)}{\alpha(a)} \quad . \tag{21}$$

The resulting equations would have the same gestalt as (9) and (15) so that the same general numerical procedure would be applicable to either system.

7.8 The Method of Modules for Deep Homogeneous Media

Now that we have some explicit computational procedures for finding the R and T operators for a plane-parallel medium (as given, for example, in Secs. 7.6 and 7.7), we turn to the task of finding the actual light fields within the medium. Finding $R(a,b)$ and $T(a,b)$ for a plane-parallel medium $X(a,b)$ allows one to determine the reflected and transmitted radiance at the boundaries of $X(a,b)$, but they do not directly supply the radiance distributions at internal depths y, $a < y < b$, in $X(a,b)$. The concept that allows the systematic determination of these internal radiance distributions, given the standard reflectance and transmittance operators $R(x,z)$ and $T(x,z)$, $a \le x \le z \le b$, within $X(a,b)$, is the invariant imbedding relation (36) of Sec. 3.7.

The general method of obtaining $N_+(y)$ and $N_-(y)$ for depths y, $a < y < b$, in $X(a,b)$ is delineated by the equations of Example 4 in Sec. 3.7 in which the invariant imbedding relation is derived. Our present purpose is to apply those general equations to a commonly encountered situation in hydrologic optics: the problem of the penetration of light into the sea, lakes, and other natural hydrosols. Now, some interesting features about such natural hydrosols is, first of all, that they are in many important instances homogeneous (or separable) and infinitely deep optically. This implies, among other facts, that the reflectances $R(a,y)$ reach an asymptotic value within a few attenuation lengths down from the surface X_a. That is, for all practical purposes, $R(a,y) = R(a,z)$ for y and z below some depth x. This phenomenon was touched upon in the discussions of the numerical solutions for $r(x;\mu',\nu)$ in Sec. 7.6. Another fact that may be of use in considering such deep natural hydrosols is that $R(y,\infty) = R(z,\infty)$ for all depths y and z below the surface X_a. This is the type of insight which arose in the study of (3) of Sec. 7.3; it now arises in a planetary oceanographic context rather than a stellar atmospheric context. Finally, the homogeneity (or more generally the separability) of X(a,b) bars polarity from infecting the R and T operators, so that $R(x,z) = R(z,x)$ and $T(x,z) = T(z,x)$ for all depths x,z in X(a,b). Hence these operators depend only on the geometrical thickness (or optical thickness if X(a,b) is separable) of X(a,b). We shall use these three major features of natural hydrosols in the discussions, now to begin.

The Invariant Imbedding Relation for Deep Hydrosols

Let us examine the invariant imbedding relation (36) of Sec. 3.7:

$$\big(N_+(y),N_-(y)\big) = \big(N_+(z),N_-(x)\big)\mathcal{M}(x,y,z) \tag{1}$$

with the preceding physical observations in mind. To represent the fact that the medium X(a,b) is a natural hydrosol and infinitely deep either optically or geometrically, we set $a = 0$ and $b = \infty$ in X(a,b). This fact of infinite depth makes itself felt in (1) when we set $z = b = \infty$ and $x = a = 0$, in such a way that (1) becomes:

$$\big(N_+(y),N_-(y)\big) = \big(0,N_-(0)\big)\mathcal{M}(0,y,\infty) \quad . \tag{2}$$

Note that our observations lead us to set $N_+(b) = N_+(\infty) = 0$; a physically obvious condition to impose at present. Equation (2) is the invariant imbedding relation for deep hydrosols (and for semi-infinite media in general) which are irradiated only at their upper boundaries X_a. The operator equations yielded by (2) are:

$$N_+(y) = N_-(0)\mathcal{R}(0,y,\infty) \tag{3}$$

$$N_-(y) = N_-(0)\mathcal{T}(0,y,\infty) \tag{4}$$

These equations, as simple as they are, can be made simpler by invoking the semigroup property for the complete reflectance operator $\mathcal{R}(a,y,\infty)$ as given in (53) of Sec. 3.7. When that property is adapted to the present case, we have:

$$\mathcal{R}(0,y,\infty) = \mathcal{T}(0,y,\infty)R(y,\infty) \quad . \tag{5}$$

This was obtained by setting $y = z$ and $b = \infty$ in (53) of Sec. 3.7. Using this in (3), we go on to obtain:

$$N_+(y) = N_-(0)\mathcal{T}(0,y,\infty)R(y,\infty) \quad . \tag{6}$$

Now our introductory discussion elicited the fact that $R(y,\infty)$ is independent of y in deep homogeneous media. In view of this, let us write:

$$\text{"}R_\infty\text{"} \quad \text{for} \quad R(y,\infty) \tag{7}$$

for every $y \geq 0$. R_∞ is readily calculated using the techniques of Secs. 7.6 or 7.7.

The problem of describing the upwelling ($N_+(y)$) and downwelling ($N_-(y)$) radiance distributions in $X(a,\infty)$ has now been reduced to the determination of R_∞ for $X(0,\infty)$ and the complete transmittance operator $\mathcal{T}(0,y,\infty)$ for $X(a,\infty)$. On examining the representation of $\mathcal{T}(a,y,b)$, as given in (42) of Sec. 3.7 for the present case, we see that:

$$\mathcal{T}(0,y,\infty) = T(0,y)[I - R_\infty R(y,0)]^{-1} \quad . \tag{8}$$

In view of our observations about the effects of homogeneity of $X(0,\infty)$ on the imbedding relation, we see that $\mathcal{T}(0,y,\infty)$ depends only on the difference $|y-0| = y$ of the depths 0 and y. More generally, we may state that for any three depths x,y,z:

$$\mathcal{T}(x,y,\infty) = T(x,y)[I - R_\infty R(y,x)]^{-1} \tag{9}$$

and:

$$\mathcal{T}(y,z,\infty) = T(y,z)[I - R_\infty R(z,y)]^{-1} \tag{10}$$

in which $\mathcal{T}(x,y,\infty)$ and $\mathcal{T}(y,z,\infty)$ depend, respectively, only on $|y-x|$, and $|z-y|$. By multiplying these complete transmittance operators together we obtain, by virtue of the semigroup property (52) of Sec. 3.7:

$$\mathcal{T}(x,z,\infty) = \mathcal{T}(x,y,\infty)\,\mathcal{T}(y,z,\infty) \tag{11}$$

where $\mathcal{T}(x,z,\infty)$ again depends only on $|z-x|$. These observations suggest that we write:

$$\text{"}\mathcal{T}(s)\text{"} \quad \text{for} \quad \mathcal{T}(x,z,\infty)$$

whenever:

$$s = |z-x| \quad .$$

Hence (11) may be written more succinctly as:

$$\boxed{\mathcal{T}(r + s) = \mathcal{T}(r)\,\mathcal{T}(s)} \tag{12}$$

and equations (3) and (4) may be reduced to:

$$\boxed{\begin{aligned} N_+(y) &= N_-(0)\,\mathcal{T}(y) R_\infty \\ N_-(y) &= N_-(0)\,\mathcal{T}(y) \end{aligned}} \quad . \tag{13}$$

These are the requisite invariant imbedding equations for the light field at depth y in an infinitely deep homogeneous hydrosol irradiated at its upper boundary by an arbitrary given radiance distribution $N_-(0)$. The equations (13) may be interpreted in either the integral operator form (the general, exact interpretation) or in matrix form (the approximate interpretation) where the matrices are built up from those in Sec. 7.7. When the matrix interpretation of (13) is intended in the discussions below, bold face type will be used.

It is Equation (12) which brings the name "semigroup" into the present discussion. For by considering the collection $\{\mathcal{T}(r)\}$ of all complete transmittance operators $\mathcal{T}(r)$, $r \geq 0$, we see that this collection is closed under composition $\mathcal{T}(r)\mathcal{T}(s)$ of every two operators $\mathcal{T}(r)$ and $\mathcal{T}(s)$ (the composition, by (12), is $\mathcal{T}(r+s)$). Further, the associativity law holds:

$$\mathcal{T}(r)[\mathcal{T}(s)\mathcal{T}(t)] = [\mathcal{T}(r)\mathcal{T}(s)]\mathcal{T}(t)$$

and finally, $\mathcal{T}(0) = I$, the identity property holds. The collection $\{\mathcal{T}(r)\}$, so endowed, is called a *semigroup*, with unit, and is an instance of a more general concept of the same name in advanced functional analysis.

The Module Equations

A practical numerical procedure for determining the light field at depths y in $X(0,\infty)$ is suggested by the system (13) and (8). Suppose we agree to partition the natural hydrosol into layers of equal thickness d (in either meters or attenuation lengths). For example, a practical choice of d may be between 1/2 to 1 attenuation length. Once d is fixed, we compute the operators $T(0,d)$ or the matrices $\mathbf{T}(0,d)$, $\mathbf{R}(0,d)$, and $\mathbf{R}_\infty$, according to the procedures in, say, Sec. 7.7. Then, by (8), we find $\mathcal{T}(d)$. It follows from (12) and (13) that we can go on to obtain $N_+(jd)$ and $N_-(jd)$ for any integer $j \geq 0$ by means of the equations:

$$\boxed{\begin{aligned} N_+(jd) &= N_-(0)\mathcal{T}^j(d)R_\infty \\ N_-(jd) &= N_-(0)\mathcal{T}^j(d) \end{aligned}} \qquad (14)$$

Hence the problem of finding $N_\pm(y)$ in the sea or in lakes or other deep natural optical media has been reduced to the problem of raising a fixed matrix or integral operator $\mathcal{T}(d)$ to an integral power, a relatively simple operation in this day of electronic computers. A slab in $X(0,\infty)$ of thickness d is called a *module* of $X(0,\infty)$, and once this thickness is fixed, the determination of $N_\pm(jd)$ is a mere mechanical detail of computation from the *module equations* (14). Linear interpolation procedures should be sufficient to determine $N_\pm(y)$ for $jd \leq y < (j+1)(d)$.

Empirical Bases for the Use of the Module Equations

The module equations (14), which have been deduced from the invariant imbedding relation, are basically theoretical equations whose lineage can be traced all the way back to the interaction principle of Chapter 3. Be this as it may, the system (14) nevertheless suggests the intriguing possibility of computing the light fields in natural optical media by knowing just two bits of empirical information about the media, namely $\mathbf{R}_\infty$ and $\mathcal{T}(d)$. To explain the idea behind the measurements of these quantities consider the following ideal experiment. We calm the surface of the sea and remove the atmosphere of the earth, and position the sun in the sky so that its parallel rays irradiate the sea, in turn, in each of the directions ξ_i associated with the m parts A_i of Ξ_-, as defined in (1) of Sec. 7.7. For each incident direction ξ_i, we then measure the radiance reflected from the sea in each of the response directions ξ_j associated with the n parts B_j of Ξ_+, as defined in (2) of Sec. 7.7. Thus, in effect, we can determine the mn numbers $R(0,\infty;i,j)$ as defined in (3) of Sec. 7.7. Analogously to $R(a,b;\xi';\xi)$, $\mathcal{T}(d)$ can be given a matricial form, that of a m×m matrix to be exact. By going down d units of depth in the sea and measuring $N(d,\xi_j)$ when the sun is irradiating the surface in the direction ξ_i, we, in effect, find the entry $\mathcal{T}_{ij}(d)$ of $\mathbf{\mathcal{T}}(d)$. (The sun's radiance must be normalized to unity for each irradiation.)

The procedure just sketched for determining the matrices $\mathbf{\mathcal{T}}(d)$ and $\mathbf{R}_\infty$ is, of course, not to be taken seriously --at least not literally. It does, however, contain the germ of a possibly workable procedure for finding $\mathbf{\mathcal{T}}(d)$. Our observations, in the form of (9) and (10), show that $\mathbf{\mathcal{T}}(d)$ is in principle determinable if we measure a sufficient number of $N_-(x)$ and $N_-(x+d)$ values at some convenient depths $x \geq 0$ below the surface. For then:

$$N_-(x+d) = N_-(x)\,\mathcal{T}(d) \tag{15}$$

and converting the measured radiances $N_-(x+d)$ and $N_-(x)$ into m-component vectors (on the basis of the partition of Ξ_- into the parts A_i) and $\mathcal{T}(d)$ into an m×m matrix of unknown parts, we have a set of m equations in m^2 unknowns $\mathcal{T}_{ij}(d)$. What is needed, then, is a set of m measured vectors $\mathbf{N}_-(x)$ and their m measured correspondents $\mathbf{N}_-(x+d)$, and obtained in such a way that the set of vectors $\mathbf{N}_-(x)$ is linearly independent.

To see this in more detail, let us denote the jth column of the matrix $\mathbf{\mathcal{T}}(d)$ by "$\mathbf{\mathcal{T}}_j(d)$", and let us denote the ith measured radiance vector $N_-(x)$ by "$\mathbf{N}_-^i(x)$" for every depth x in $X(0,\infty)$. Further, let "$N_j^i(x)$" denote the jth component of $\mathbf{N}_-^i(x)$. Then the expanded matrix form of (15) leads to the following relation: for every i, j=1,...,m:

$$N_j^i(x+d) = \mathbf{N}_-^i(x) \cdot \mathbf{\mathcal{T}}_j(d) \tag{16}$$

where the dot denotes the dot product for vectors.

Suppose we write:

$$\text{"}\mathbf{N}_-(x)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathbf{N}_-^1(x) \\ \mathbf{N}_-^2(x) \\ \vdots \\ \mathbf{N}_-^m(x) \end{bmatrix} \quad . \tag{17}$$

Then assembling the m^2 equations of (16), we have:

$$\mathbf{N}_-(x+d) = \mathbf{N}_-(x)\,\mathbf{\mathcal{T}}(d) \quad . \tag{18}$$

The linear independence of the vectors $\mathbf{N}_-^i(x)$, i=1,...,m, implies the existence of the inverse $\mathbf{N}^{-1}(x)$ of the m×m matrix $\mathbf{N}(x)$. Equation (18) then yields:

$$\mathcal{T}(d) = N_-^{-1}(x)N_-(x+d) \tag{19}$$

which holds for every depth x and module X(x,x+d) of X(0,∞). In this way we find an empirical basis for the complete transmittance matrix for a module in a natural hydrosol.

It remains to determine the matrix R_∞ empirically. However, this determination is exactly analogous to that of $\mathcal{T}(d)$ and if "$N_-(x)$" and "$N_+(x)$" denote the incident and reflected m×m matrices of radiances measured at any depth x such that $N_-(x)$ is invertible, then:

$$R_\infty = N_-^{-1}(x)N_+(x) \tag{20}$$

In (20) we have assumed that Ξ_+ and Ξ_- are both partitioned similarly into m pieces. Once measurements within a natural hydrosol have been made so that $\mathcal{T}(d)$ and R_∞ are obtained, then the module equations yield the light field at any integral depth jd in the medium knowing $N_-(0)$. These observations show that the radiative transfer problem of the penetration of light into the sea can be solved on either an empirical level or a theoretical level knowing the basic reflectance operator R_∞ for X(0,∞) and complete transmittance operator $\mathcal{T}(d)$ for a module X(x, x+d) of X(0,∞).

The operators R_∞ and $\mathcal{T}(d)$ as used above are inherent optical properties of the hydrosol in the sense that they are independent of the light fields within the medium and that they depend only on the intrinsic physical makeup of the medium (cf. closing remarks of Sec. 3.12, and also Chap. 11 for definitions and discussions of inherent optical properties).

One of the significant features of the module equations is that they may be formulated, solved, and applied completely on the global level within the medium X(0,∞), and need make no appeal either directly or indirectly to the local properties of the medium such as the volume attenuation and scattering functions α and σ of the medium. Further discussion of the problem of determining the global optical properties of a medium using measured radiometric data is made in Sec. 13.10.

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7.9 The Method of Semigroups for Deep Homogeneous Media

The results of the preceding section, in the form of the module method of solution of radiative transfer problems in the sea and the air, were so simple and direct that we are encouraged to explore the method in more detail, with an eye toward obtaining a general method applicable to all media. Thus our purpose in this section is to begin with the basis for the module equations, namely the system (13) of Sec. 7.8, and study the effect on the module equations when the module thickness is allowed to go to zero but with the depth z (=jd) held fixed. The resultant equations will reveal a general pattern which suggests the requisite generalization, namely the *method of semigroups*.