

$$\mathcal{T}(d) = N_-^{-1}(x)N_-(x+d) \quad (19)$$

which holds for every depth x and module $X(x, x+d)$ of $X(0, \infty)$. In this way we find an empirical basis for the complete transmittance matrix for a module in a natural hydrosol.

It remains to determine the matrix R_∞ empirically. However, this determination is exactly analogous to that of $\mathcal{T}(d)$ and if " $N_-(x)$ " and " $N_+(x)$ " denote the incident and reflected $m \times m$ matrices of radiances measured at any depth x such that $N_-(x)$ is invertible, then:

$$R_\infty = N_-^{-1}(x)N_+(x) \quad (20)$$

In (20) we have assumed that Ξ_+ and Ξ_- are both partitioned similarly into m pieces. Once measurements within a natural hydrosol have been made so that $\mathcal{T}(d)$ and R_∞ are obtained, then the module equations yield the light field at any integral depth jd in the medium knowing $N_-(0)$. These observations show that the radiative transfer problem of the penetration of light into the sea can be solved on either an empirical level or a theoretical level knowing the basic reflectance operator R_∞ for $X(0, \infty)$ and complete transmittance operator $\mathcal{T}(d)$ for a module $X(x, x+d)$ of $X(0, \infty)$.

The operators R_∞ and $\mathcal{T}(d)$ as used above are inherent optical properties of the hydrosol in the sense that they are independent of the light fields within the medium and that they depend only on the intrinsic physical makeup of the medium (cf. closing remarks of Sec. 3.12, and also Chap. 11 for definitions and discussions of inherent optical properties).

One of the significant features of the module equations is that they may be formulated, solved, and applied completely on the global level within the medium $X(0, \infty)$, and need make no appeal either directly or indirectly to the local properties of the medium such as the volume attenuation and scattering functions α and σ of the medium. Further discussion of the problem of determining the global optical properties of a medium using measured radiometric data is made in Sec. 13.10.

7.9 The Method of Semigroups for Deep Homogeneous Media

The results of the preceding section, in the form of the module method of solution of radiative transfer problems in the sea and the air, were so simple and direct that we are encouraged to explore the method in more detail, with an eye toward obtaining a general method applicable to all media. Thus our purpose in this section is to begin with the basis for the module equations, namely the system (13) of Sec. 7.8, and study the effect on the module equations when the module thickness is allowed to go to zero but with the depth z ($=jd$) held fixed. The resultant equations will reveal a general pattern which suggests the requisite generalization, namely the *method of semigroups*.

The Semigroup Equations for $\mathcal{T}(z)$

Consider (14) of Sec. 7.8, which describes the downward radiance distribution at depth jd , $j \geq 0$, and where d is the thickness of the module $X(0,d)$ for the homogeneous infinitely deep medium $X(0,\infty)$. If we halve the depth of the module, then we must double the powers to be used to find $N.(jd)$, i.e., we are observing that:

$$\mathcal{T}^{2j}(d/2) = \mathcal{T}^j(d) \quad , \quad (1)$$

which follows from (12) of Sec. 7.8. More generally, for every positive integer n :

$$\mathcal{T}^{nj}(d/n) = \mathcal{T}^j(d) \quad (2)$$

which follows by induction on n , starting with (1). Setting $j = 1$ in (2) we see that the resulting equation, namely

$$\mathcal{T}^n(d/n) = \mathcal{T}(d) \quad ,$$

demonstrates quite graphically that the module transmittance is the product of an arbitrarily large number of transmittances of 'submodules' of thickness d/n . Now, (12) of Sec. 7.8 shows that $\mathcal{T}(0) = I$, the identity operator on the set \mathcal{N} of all downward radiance distributions (\mathcal{N} is defined in the invariant imbedding statement Sec. 3.9). The continuity of $\mathcal{T}(s)$, which holds for the most part in all natural optical media, then implies that the transmittance operators for the submodules $X(0,d/n)$ approach the identity operator I . That is:

$$\lim_{s \rightarrow 0} \mathcal{T}(s) = I \quad .$$

All this is quite clear when one reflects on the definition of $\mathcal{T}(s)$, being an instance of a complete transmittance operator. But now, in the light of the present approach, wherein the analogy of $\mathcal{T}(s)$ and beam transmittance T_s just waiting to appear explicitly upon the scene, in this light we are moved to consider next the limit:

$$\lim_{s \rightarrow 0} \frac{I - \mathcal{T}(s)}{s}$$

which is motivated by the defining equation for volume attenuation function α (Sec. 3.11):

$$\alpha = \lim_{s \rightarrow 0} \frac{1 - T_r}{r} \quad .$$

The limit involving $\mathcal{T}(s)$ will, of course be an operator of some kind rather than a number, as is α ; however, the analogy now a-building seems so suggestive that we are next moved to write:

$$\text{"A"} \quad \text{for} \quad \lim_{s \rightarrow 0} \frac{I - \mathcal{T}(s)}{s} \quad (3)$$

Hence, for any depth differences s , we have directly from (3):

$$A + \epsilon(s) = \frac{I - \mathcal{J}(s)}{s} \quad (4)$$

where " $\epsilon(\cdot)$ " denotes an operator (actually defined implicitly by (4)) which goes to zero as its argument goes to zero. Solving (4) for $\mathcal{J}(s)$:

$$\mathcal{J}(s) = (I - sA) + o(s) \quad (5)$$

where " $o(\cdot)$ " denotes $\epsilon(\cdot)s$. The closer s is to zero, the closer $o(s)$ is to the zero operator. This equation is analogous to

$$T_s = (1 - sa) + o(s)$$

for beam transmittance (where " $o(\cdot)$ " in the latter equation is of course distinct from that in (5)).

The momentum of these definitions, and discoveries of analogy carry us on to consider the present analogous structure to the differential equation for beam transmittance:

$$\frac{dT_r}{dr} = -\alpha T_r$$

(cf. (2) of Sec. 3.11). Thus, we are led to form the difference quotient:

$$\frac{\mathcal{J}(r+s) - \mathcal{J}(r)}{s}$$

and obtain its limit as s goes to zero. In preparation for this, we write:

$$\begin{aligned} \frac{\mathcal{J}(r+s) - \mathcal{J}(r)}{s} &= \frac{\mathcal{J}(s) - I}{s} \mathcal{J}(r) \\ &= -(A + \epsilon(s)) \mathcal{J}(r) \end{aligned}$$

which follows on use of (12) of Sec. 7.8 and (4) above. Therefore we have:

$$\frac{d\mathcal{J}(r)}{dr} = -A\mathcal{J}(r) \quad (6)$$

wherein we have written

$$\frac{d\mathcal{T}(r)}{dr} \quad \text{for} \quad \lim_{s \rightarrow 0} \frac{\mathcal{T}(r+s) - \mathcal{T}(r)}{s}$$

We pause now to restate the purpose of the present discussion. We wish to find the continuous counterparts to the module equations (14) of Sec. 7.8. The first step, just completed, makes clear the structure of $\mathcal{T}(s)$ when s is allowed to approach zero. This structure is shown in (5) and (6). The next step is to find the continuous counterpart to $\mathcal{T}^j(d)$ as d goes to zero but such that jd is some fixed depth z . In view of the analogy between \mathcal{T}_r and T_r which has guided the developments so far, it is clear that this next step should be equivalent to finding the operator version of:

$$T_r = \exp \{-ar\} \quad .$$

This observation requires us to find the operator analog of $\exp \{-ar\}$. At this point we recall that the Maclaurin series development of $\exp \{-ar\}$ shows promise of being extendable to the operator context, especially since we have the basic derivative formula (6) to work from. Therefore by means of (6), taking all the integral derivatives of $\mathcal{T}(r)$ in succession, we obtain:

$$\frac{d^2 \mathcal{T}(r)}{dr^2} = - \frac{d}{dr} (A \mathcal{T}(r)) = -A \frac{d\mathcal{T}(r)}{dr} = A^2 \mathcal{T}(r)$$

$$\frac{d^3 \mathcal{T}(r)}{dr^3} = \frac{d}{dr} (A^2 \mathcal{T}(r)) = A^2 \frac{d\mathcal{T}(r)}{dr} = -A^3 \mathcal{T}(r)$$

and in general:

$$\frac{d^j \mathcal{T}(r)}{dr^j} = (-1)^j A^j \mathcal{T}(r) \quad . \quad (7)$$

Using the identity property of $\mathcal{T}(r)$, namely that $\mathcal{T}(0) = I$, (7) yields:

$$\left. \frac{d^j \mathcal{T}(r)}{dr^j} \right|_{r=0} = (-1)^j A^j \quad .$$

Following through on the Maclaurin series analogy we then write:

$$\text{"exp \{-Ar\}"} \quad \text{for} \quad \sum_{j=0}^{\infty} \frac{(-1)^j (Ar)^j}{j!} \quad . \quad (8)$$

This definition makes sense from a strictly operational point of view. For we can perform, at least in principle, the iterations of the operator A to find A^j for every integer j . Furthermore we can multiply A^j by the real number $(-1)^j r^j / j!$

in each case; and we can add together any finite number n of such combinations to end up with

$$\sum_{j=0}^n \frac{(-1)^j (Ar)^j}{j!}$$

as a well-defined operator. Granted all this, it appears that, at least on a numerical or empirical level, the exponential characterization of $\mathcal{T}(r)$ is settled. The mathematical reader, however, will wish to dwell on the convergence problem entailed in the definition (8). Such considerations are quite simple and are readily characterized in terms of the radiometric norm (Ex. 15 of Sec. 2.11) which supplies the necessary machinery in algebraic radiative transfer theory to handle problems of convergence of operator sequences. Such a digression is not pertinent in the present discussion, and we can safely pass it by without serious effect on the remainder of our study. Interested readers wishing to study such matters in more detail are referred to Ref. [110], a book devoted almost exclusively to the extension of the ideas, inherent in e^{-ar} , to their most general settings.

With the definition (8) and on the basis of the analogy of (8) with the scalar (numerical) context, we see that:

$$\mathcal{T}(r) = \exp \{-Ar\} \quad (9)$$

With this representation, $\mathcal{T}(r)$ exhibits directly and succinctly all its important properties ((12) of Sec. 7.8, and (3), (5), (6) above). We now may write (13) of Sec. 7.8 as:

$$\begin{aligned} N_+(y) &= N_+(0) \exp \{-Ay\} R_\infty \\ N_-(y) &= N_-(0) \exp \{-Ay\} \end{aligned} \quad (10)$$

Equations (9), (10) are the requisite semigroup equations for $\mathcal{T}(r)$ as they are applied to the determination of $N_\pm(y)$ in an infinitely deep homogeneous plane-parallel medium. The operator A is called the *infinitesimal generator* of the semigroup formed by the transmittance operators $\mathcal{T}(r)$. (The *semigroup* structure stems primarily from the property (12) of Sec. 7.8.) Readers acquainted with the theory of stochastic processes (in continuous time, say) will observe via (6) or (9) that a radiative transfer process in a deep homogeneous optical medium may be viewed as a Markov process which evolves continuously with depth in that medium.

The Infinitesimal Generator A

One final point remains in the preparation of the system (10) for actual numerical application, or in further theoretical work, and that is in determining the explicit

dependence of A on the inherent optical properties of the medium. Thus we have the task of finding how A depends on the volume attenuation and scattering functions α and σ .

The key to the required answer rests in the equation (37) of Sec. 7.5. For, setting $a = 0$ and $b = \infty$, in that equation, we have:

$$\frac{\partial \mathcal{T}(0, y, a)}{\partial y} = \mathcal{R}(0, y, \infty)\rho(y) + \mathcal{T}(0, y, \infty)\tau(y)$$

or, in the contracted notation presently in use, this is:

$$\frac{d\mathcal{T}(y)}{dy} = \mathcal{R}(0, y, \infty)\rho + \mathcal{T}(y)\tau$$

By (5) of Sec. 7.8 this can be reduced to:

$$\frac{d\mathcal{T}(y)}{dy} = \mathcal{T}(y)[R_{\infty}\rho + \tau] \quad (11)$$

Comparison of (11) with (6) leads us to the required representation:

$$A = -(\tau + R_{\infty}\rho) \quad (12)$$

Equation (9) can now be written as:

$$\mathcal{T}(r) = \exp(\tau + R_{\infty}\rho)r \quad (13)$$

Thus, the infinitesimal generator A of the semigroup $\{\mathcal{T}(r)\}$ of complete transmittance operators is characterizable as the sum of two operators: τ the local transmittance operator, and $R_{\infty}\rho$, the product of the (global) reflectance operator R_{∞} of $X(0, \infty)$ and the local reflectance operator ρ . Observe that, by the homogeneity of $X(0, \infty)$, all three operators comprising A are independent of depth in $X(0, \infty)$, so that A has the same property. Further, observe that:

$$\mathcal{T}(r)A = A\mathcal{T}(r) \quad ,$$

by virtue of the discussion leading to (6), and in particular the semigroup relation (12) of Sec. 7.8. The approximate matrixial form of A is readily forthcoming from those of τ , ρ and R_{∞} as given by the discussions in Sec. 7.7 and 7.8. In particular, we use ρ_+ and τ_- in place of ρ and τ .

The role of the infinitesimal generator A as compared to that of the volume attenuation function α , as these roles are viewed from the theory of radiative transfer as a whole, is characterizable succinctly as follows: A is to N as α is to N^0 . That is, A is the logarithmic derivative of downwelling observable radiance distributions in a plane-parallel medium, while α is the logarithmic derivative of directly transmitted (i.e., residual) radiance along a path. Putting it still

another way, A is the 'volume attenuation function' for the *natural* (undecomposed) light field in deep homogeneous media such as the seas, lakes, and optically dense atmospheric media. This may be seen by returning to (13) of Sec. 7.8 and taking the derivative of $N_-(y)$ with respect to y . Thus:

$$\begin{aligned} \frac{dN_-(y)}{dy} &= N_-(0) \frac{d\mathcal{T}(y)}{dy} \\ &= N_-(0) (-A \mathcal{T}(y)) = N_-(0) \mathcal{T}(y) (-A) \\ &= -N_-(y)A \end{aligned}$$

When we apply the derivative operator d/dy to the upwelling radiance distributions as given in (13) of Sec. 7.8, we have:

$$\begin{aligned} \frac{dN_+(y)}{dy} &= N_-(0) \frac{d\mathcal{T}(y)}{dy} R_\infty = N_-(0) (-A \mathcal{T}(y)) R_\infty \\ &= -N_-(y)AR_\infty \end{aligned}$$

This representation of the depth rate of change of $N_+(y)$, while not as direct as that for $N_-(y)$, still shows that the logarithmic depth rate of change of $N_+(y)$ is essentially A . Since commutativity of A and R_∞ need not generally hold, we cannot generally place " R_∞ " next to " $N_-(y)$ " in the preceding equation to get $N_+(y)$ as a result. This asymmetry in the local behaviour of $N_+(y)$ and $N_-(y)$ is a slight and inessential notational irregularity in the otherwise conceptually pleasing and powerful formulations of the method of modules and the method of semigroups. A search for a more symmetric treatment of the depth rates of change of $N_+(y)$ and $N_-(y)$ leads to the method of groups to be considered in the following section.

7.10 The Method of Groups for Deep Homogeneous Media

Once the flush of discovery of the semigroup equations (10) of Sec. 7.9 has passed and the critical eye runs over their asymmetric forms, one is moved to search for a new set of equations which incorporates both the conceptual and computational power of that set with a more pleasing symmetry of form. In this section we embark on such a search and are rewarded with a set of equations which fulfills all these requirements and more. The additional dividend is a novel perspective of Chandrasekhar's classical method of solution of the transfer equation in plane-parallel homogeneous media [43] from the heights of group theory and the modern theory of differential equations. As a result, we can view Chandrasekhar's classical method as but one of a large family of possible solution procedures unified from the viewpoint of invariant imbedding theory. This insight then unites with that encountered in Secs. 6.1-6.4, in which novel views of the spherical