## 7.11 Method of Groups for General Optical Media

The various methods of solution of the equation of transfer, such as the method of modules (Sec. 7.8), the method of semigroups (Sec. 7.9), and the method of groups in the preceding section hold within them a common core which, if extracted, can guide the construction of a method of solution of the radiance field in arbitrary optical media. This section is devoted to the isolation of the common conceptual kernel of those methods and to a brief exposition of the general method of solution it suggests.

### Analysis of the Group Method: Initial Data

We begin with a recapitulation of the ground-forms for the two basic methods. The semigroup method rests on the semigroup relation (12) of Sec. 7.8 for the complete transmittance operator $\mathcal{T}(r)$ (i.e., $\mathcal{T}(x,z,\infty)$ where $r = |z-x|$). The fundamental equations for the light field in this method are given by the system (10) of Sec. 7.9 or the system (14) of Sec. 7.8, depending on whether the continuous variable y or the discrete variable $y = jd$ is used. The method of groups rests on (1) of Sec. 7.10 for the invariant imbedding operator $\mathcal{M}(r)$ (i.e., $\mathcal{M}(x,z)$ where $r = z-x$), which holds for all real numbers r and s. The equations (9) or (25) of Sec. 7.10 may be used to find the light field at any depth y in $X(0,\infty)$.

What are the basic data needed in the computational applications of each method? The data needed are: (a) $\alpha,\sigma$ throughout $X(0,\infty)$ and either (b): $N(0)$, the complete radiance distribution at level 0; or (c) $N_-(0)$ and $R_\infty$, i.e., the downward incident radiance $N_-(0)$ at level 0, and the reflectance operator $R_\infty$ for $X(0,\infty)$. Thus, the inherent optical properties $\alpha$ and $\sigma$ are indispensable in finding $N(y)$ using either method. However, we clearly have an option on the initial radiance data. Alternative (b) requires the full radiance distribution at level 0. Alternative (c) requires only the downward incident radiance on level 0, but along with the reflectance operator for $X(0,\infty)$. Alternative (b) is possible when preliminary empirical estimates of $N(0)$ are available. As a result of having both $N_-(0)$ and $N_+(0)$ available, we then obviate the need of $R_\infty$. However, in theoretical studies only $N_-(0)$ is generally available for use. The remaining part of $N(0)$, namely, $N_+(0)$, is simply some more unknown data to be sought along with $N(y)$, $y > 0$. Clearly, for deep homogeneous media $X(0,\infty)$, having to find $N_+(0)$ is tantamount to finding $R_\infty$ for $X(0,\infty)$. We thus come to the first conclusion in our analysis of the group and semigroup methods: *Each method requires as given data either alternatives (a) and (b); or (a) and (c).* The first alternative is the *empirical alternative*; the second, the *theoretical alternative*. In discussion the extension of the method of groups to more general media, the theoretical alternative demands more attention than the empirical alternative. Hence when the extension is made below, it will be made with an eye to the adoption of the theoretical alternative, thereby resulting in a more powerful method of solution in the sense that it does not depend on basically superfluous preliminary empirical measurements.

## Analysis of the Group Method: Limitations of the Equation of Transfer

Having settled the matter of what kind of initial data shall be required in the general method, we seek the theoretical equations which may be the basis of the new method. Now, both equations (12) of Sec. 7.8 and (1) of Sec. 7.10 use one-dimensional parameters, namely the depths r and s in $X(0,\infty)$. Physically, the interpretation of these equations is that, given the operators $\mathcal{J}(r)$ and $\mathcal{J}(s)$ (or $\mathcal{M}(r)$ and $\mathcal{M}(s)$) for two contiguous segments of a vertical path in $X(0,\infty)$, one knows how to find the operator $\mathcal{J}(r+s)$ (or $\mathcal{M}(r+s)$) associated with the union of the two path segments. What is the analogous case in general media? To fix ideas, suppose we still have $X(0,\infty)$, but that $X(0,\infty)$ is no longer homogeneous, nor even stratified: $X(0,\infty)$ is a natural chaos of variations in $\alpha,\sigma$ and initial incident radiances over the upper boundary $X_0$. It is now clear that the light field can vary markedly over planes $X_y$ at depth y in $X(0,y)$. Hence it is no longer sufficient to simply give the depth in $X(0,\infty)$ in the description of the light field in $X(0,\infty)$; a full specification of the point in question must be given.

In this more general setting what then is the counterpart to the simple vertical path used in the stratified plane-parallel case? Figure 7.16 depicts a possible candidate in the form of a general path $\mathcal{P}$ with initial point $x_0$ at the boundary $X_0$ of $X(0,\infty)$ and terminal point x in $X(0,\infty)$. Here "$x_0$" and "x" denote ordered triples of real numbers giving the coordinates of $x_0$ and x with respect to some terrestrial frame of reference. It should be noted that $x_0$ need not be on $X_0$ for what follows. We have simply placed it there to fix ideas. In the homogeneous stratified case, $\mathcal{P}$ can be vertical and, given N(0) at $x_0$, we can find N(y) at any distance y along $\mathcal{P}$ using (9) of Sec. 7.10 (in the method of groups) or using (10) of Sec. 7.9 (using the method of semigroups). Alternatively, we can integrate directly along $\mathcal{P}$ using (26) of Sec. 7.10 or (38) of Sec. 7.5 to find N(y). This then suggests that we merely need to specify $\mathcal{P}$ as in Fig. 7.16 and, with the initial radiance N(0) given at $x_0$ integrate methodically along $\mathcal{P}$. But what of the curvilinear structure of $\mathcal{P}$? This appears to present no obstacles, at least in principle. For, let "**t**" denote the unit vector to $\mathcal{P}$ at x, and let $\Xi$ be given a fixed partition as in Fig. 7.15 (see also (1), (2) of Sec. 7.7). Thus no matter where x is in $X(0,\infty)$, $\Xi$ has the given fixed partition. Then for $\xi$ in $\Xi$ the equation of transfer at x may be written:

$$\xi\cdot\mathbf{t}\,\frac{dN(x,\xi)}{dr} = -\alpha(x)N(x,\xi) + N_*(x,\xi) \qquad (1)$$

where r is distance measured along $\mathcal{P}$ at x in the direction of the tangent **t**. If the partition of $\Xi$ is now introduced, an approximating system to (1) can be formed using the techniques explained, e.g., in Sec. 7.7 or in (25) of Sec. 7.10. As a result, at each point x of $\mathcal{P}$ a system of ordinary differential equations just like (26) of Sec. 7.10 describes the light

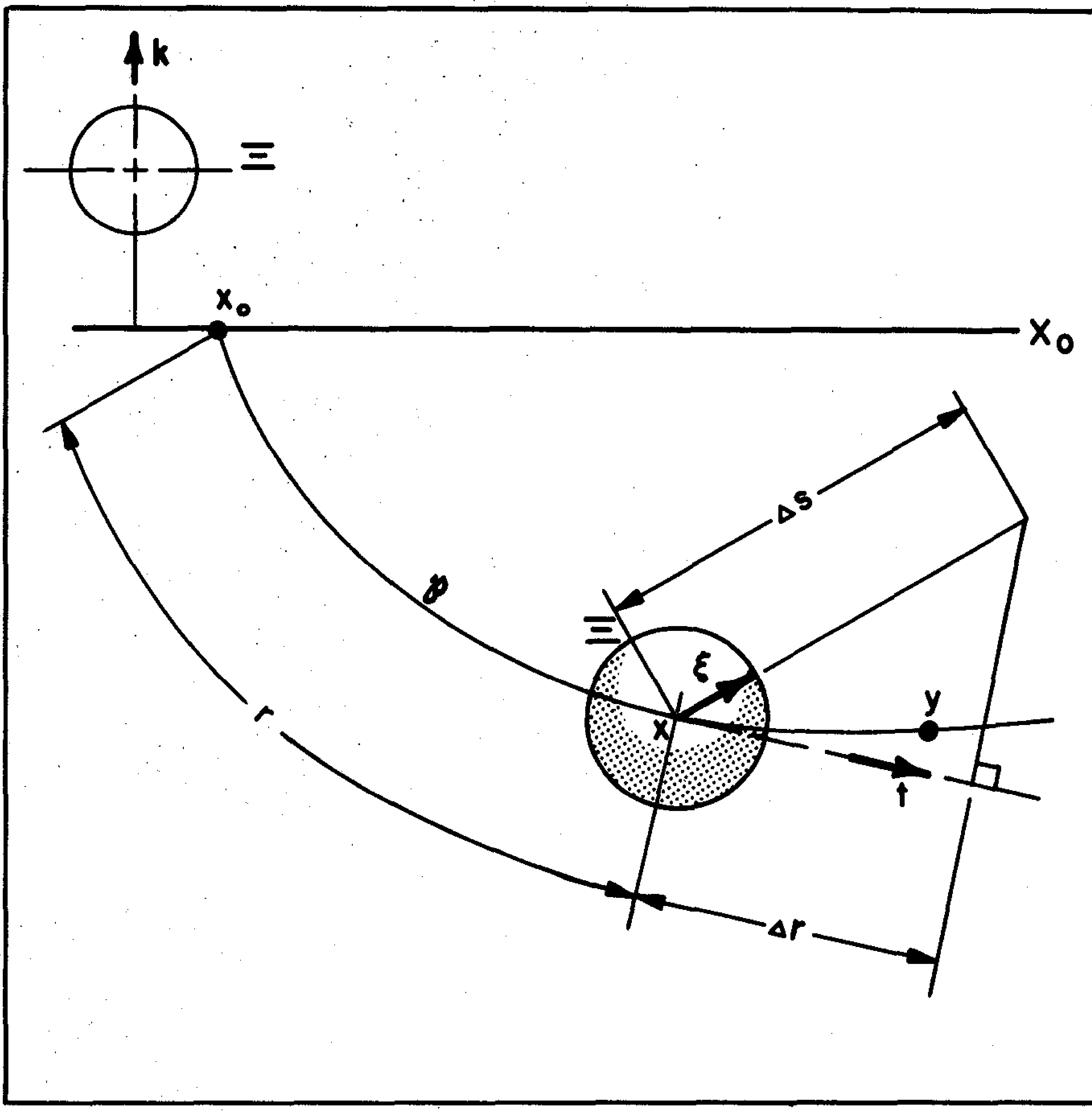


FIG. 7.16 Analysis of the group method; limitations of the equation of transfer.

field at x. Thus, if we write:

"$N_i(r)$" for $N(x,\xi_i)$

"$N_{*i}(r)$" for $N_*(x,\xi_i)$

"$\alpha(r)$" for $\alpha(x)$

where $\xi_i$ is a fixed representative in the partition of $\Xi$ and x is at a distance r from $X_0$, then (1) becomes:

$$\xi_i \cdot t \frac{dN_i(r)}{dr} = -\alpha(r)N_i(r) + N_{*i}(r) , \qquad (2)$$

$i = 1,\ldots, p$ where $p = m+n$, and we no longer explicitly distinguish between the members $A_i$, $B_j$ of the partition. It appears that by knowing all p components $N_i(r)$ at x, we can compute $dN_i(r)/dr$ for each i from the right side of (2), and then use this derivative value to estimate each of the p values $N_i(r+\Delta r)$ for some reasonable incremental distance $\Delta r$ along the path in the direction **t**. In this way we can perhaps computationally inch our way along $\mathcal{P}$ and find $N(x,\xi_i)$, at least in principle, at any point x in $X(0,\infty)$ and for any of the p directions $\xi_i$!

Encouraged by the seemingly successful generalization of the homogeneous stratified case to the nonhomogeneous case as outlined above, we go on to see whether the preceding computational scheme can be phrased succinctly in group-theoretic terms. Granted the system (2) can be integrated along a given path $\mathcal{P}$ starting with the initial radiance distribution $N(x_0)$, we can then find $N(x_1)$ at $x_1$, a point a distance r along $\mathcal{P}$ from $x_0$. Then for the same reasons we may go on to find $N(x_2)$ at point $x_2$ of $\mathcal{P}$. Suppose we summarize the construction activity over the segment between $x_0$ and $x_1$ by means of an operator $\mathcal{N}(x_0,x_1)$, and similarly let $\mathcal{N}(x_1,x_2)$ map $N(x_1)$ into $N(x_2)$. Then we could say,

$$\mathcal{N}(x_0,x_2) = \mathcal{N}(x_0,x_1)\,\mathcal{N}(x_1,x_2) \tag{3}$$

in analogy to the group relation which holds for the operator $\mathcal{M}(x,y)$ of $\Gamma_2(0,\infty)$. We thus appear to have arrived at the requisite group-theoretic relations in the form of (3) for the general case.

Before going any further and before we develop specific numerical schemes on (3) as a base, it would be well to test the validity of that scheme on some easily visualized case for which we know the answers. Toward this end, we suppose $X(0,\infty)$ is homogeneous once again. Now, however, we assume the incident radiance distributions on the upper boundary $X_0$ of $X(0,\infty)$ to vary with location on $X_0$. To fix ideas, let $N_-(0)$ be vertical collimated radiance and let it undulate sinusoidally in magnitude along the direction from left to right with period $r_0$, as in Fig. 7.17, and be constant along directions normal to the Figure. Thus the light field in $X(0,\infty)$ is quite clearly *not* stratified, even though the inherent optical properties of $X(0,\infty)$ are about as innocuous as can be without being trivial. Since the argumentation leading to (3) was for an arbitrary path $\mathcal{P}$ in $X(0,\infty)$, let us now choose $\mathcal{P}$ to be a horizontal infinitely long path going from left to right just below $X_0$, as in Fig. 7.17. With this arrangement fixed we now turn to the system (2) and observe that the associated matrix operator $\mathcal{K}$ is, by the homogeneity of $X(0,\infty)$, independent of distance r along $\mathcal{P}$. It follows that, as far as the intrinsic structure of (2) is concerned, we have reverted to the full group-theoretic context of Sec. 7.10. In particular, the asymptotic radiance theorem states that $N(r)e^{kr}$ should have a limit as $r\to\infty$, i.e., there should be a fixed radiance distribution toward which the p-component vector $N(r)$ goes. Now, under the conditions just defined this conclusion is patently false! Quite obviously the radiance distributions along $\mathcal{P}$ vary sinusoidally in dependence with distance r along $\mathcal{P}$. Observe next that while we do not know exactly what $N(r)$ is at r from $x_0$, we do know that $N(r) = N(r+r_0)$, i.e., that the

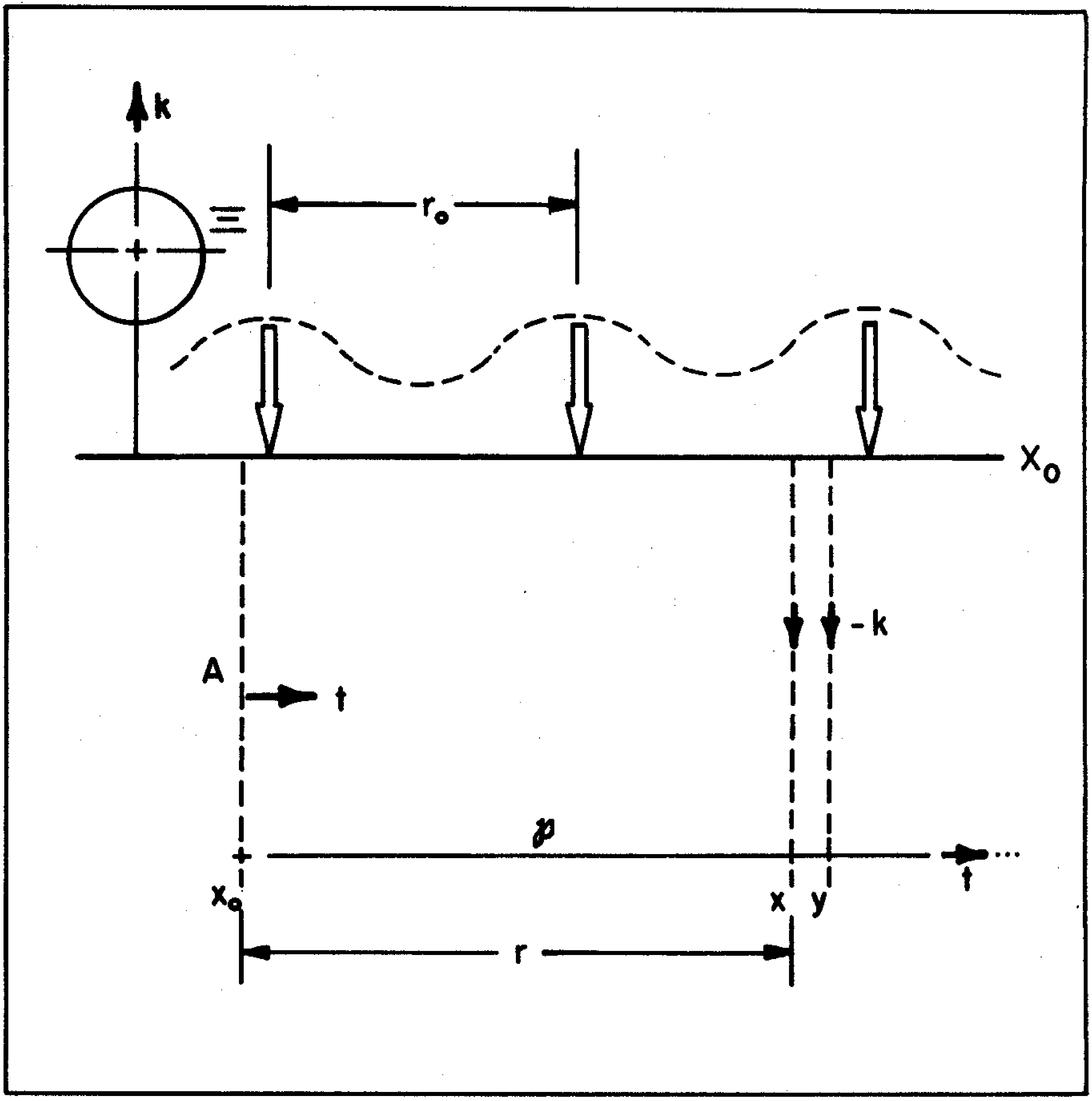


FIG. 7.17 The equation of transfer cannot directly relate radiances along parallel paths.

light field varies periodically with the same period $r_0$ as the incident radiance field on $X_0$. We have thus come to a contradiction, and the next task is to understand just where the reasoning leading to the general integration scheme along $\mathscr{P}$ was fallacious.

To detect the fallacy at hand, we quickly can dismiss equation (2) itself as the epicenter of difficulty; similarly, the asymptotic radiance theorem based on the characteristic representation (28) of Sec. 7.10 cannot be the trouble center. We therefore descend on the remaining possibility; namely, the equation of transfer itself, and this, understandably, is done with a measure of trepidation. The trouble appears to stem from the use of the equation of transfer in the setting depicted in Fig. 7.16. We therefore review with care the meaning of the terms in (1). The only strange aspect of (1) is the derivative term. But this has been correctly translated from the general directional derivative term $\xi \cdot \nabla N(x,\xi)$ that customarily appears in the equation of transfer. Indeed, let s be distance measured along direction $\xi$, as in Fig. 7.16. Then:

$$\xi \cdot \nabla N(x,\xi) = \frac{dN(x,\xi)}{ds} = \xi \cdot \mathbf{t}\, \frac{dN(x,\xi)}{dr}$$

by virtue of the simple geometric fact that:

$$\xi \cdot \mathbf{t} \Delta s = \Delta r$$

where $\Delta s$ and $\Delta r$ are two arbitrary distances along directions $\xi$ and $\mathbf{t}$ but related as shown in Fig. 7.16. Therefore, whatever the source of difficulty, it is not one born of simple algebraic errors. It remains, therefore, to consider conceptual errors of application of the equation of transfer.

In going from (2) to the integration scheme over path $\mathcal{P}$ some error of interpretation of (1) was committed. The error must center on the intended meaning, i.e., the intended physical interpretation of the derivative term of (1). It was hoped that knowledge of the value of $\xi \cdot \mathbf{t} dN(x,\xi)/dr$ would permit an estimate of $N(y,\xi)$ at the neighboring point $y = x + \mathbf{t}\Delta r$ on $\mathcal{P}$. When phrased in this way (rather than in the abbreviated notation of (2)) the difficulty starts to resolve: *the derivative term* $\xi \cdot \nabla N(x,\xi)$ *of the equation of transfer is intended to give the rate of change of* $N(x,\xi)$ *at* x *in the direction* $\xi$ *and in no other direction. Hence an attempt at extrapolating the value* $N(x,\xi)$ *at* x *in some direction* $\xi'$*, using the equation of transfer, is permissible only when* $\xi' = \xi$. Therefore the integration scheme of (2) along $\mathcal{P}$ holds only when $\mathcal{P}$ is a straight line with direction $\xi$.

The preceding italicized observation stands out in bold relief when, now forewarned, we consider the following quite simple test situation. Let $X(0,\infty)$ be a purely absorbing medium. Thus $\sigma = 0$ throughout $X(0,\infty)$. Let $X(0,\infty)$ be irradiated by vertical collimated light as in Fig. 7.16, but now the spatial dependence over the upper boundary need not even be periodic, but simply some arbitrary given form. The equation of transfer can readily predict $N(x,-\mathbf{k})$, the downward radiance at any point x along any vertical path $\mathcal{P}$. However, given $N(x,-\mathbf{k})$, we cannot use the equation to predict $N(y,-\mathbf{k})$ where y is a point the same depth from $X_0$ and just next to x. From this we infer that in any optical medium *the equation of transfer is generally powerless to describe or interrelate directly the radiance flow at two neighboring points* x *and* y *which are directed along parallel paths containing* x *and* y.

In the simple case of a purely absorbing medium, it is clear that to know $N(x,\cdot)$ at each point of plane $X_y$ in $X(0,\infty)$, it is necessary to know $N(0,\cdot)$ at each point of plane $X_0$ for the directions $\xi$ in $\Xi_-$. It seems reasonable that this is indeed the case also for media with arbitrary scattering mechanisms extant within them. It shall turn out that this is so. Thus, in the counterexample of Fig. 7.17, the initial data at point $x_0$ is generally inadequate to predict radiance parallel to $\mathcal{P}$ at points above and below $\mathcal{P}$. What is needed is initial data over a *whole* vertical plane A (seen dashed, end on) throughout $X(0,\infty)$ which then permits a methodical computational march away from the data plane in the direction of its normal $\mathbf{t}$, a march which eventually can in principle sweep through all of $X(0,\infty)$.

We have deliberately travelled the route just taken, i.e., from (1) to the preceding fallacy, and then to the resolution of the fallacy just above, principally to uncover the resultant insight into the nature of the equation of transfer enunciated above. Thus, while it is quite obvious from following any of the classical derivations of the equation of transfer, just what the equation *can do* in a given optical medium, it does not seem to have been emphasized what the equation of transfer *cannot do* by way of direct interrelation of the light fields at two neighboring points in the medium.

## Analysis of the Group Method: Summarized

These observations on the limitations of the equation of transfer complete our analysis of the semigroup and group methods by showing the minimum number of necessary steps that must be taken in generalizing the methods to arbitrary optical media. *In particular, we have learned that the general operators* $\mathcal{M}(x,z)$ *of the simple homogeneous case, which worked well in predicting* $N(z)$ *knowing* $N(x)$ *along a vertical path on a stratified plane-parallel* $X(0,\infty)$, *must now be replaced by operators* $\mathcal{M}(x,z)$ *which relate the radiance distributions over all of plane* $X_x$ *to the radiance distributions over all of plane* $X_z$. In short, we must develop the generalization of the group $\Gamma_2(0,\infty)$ for $X(0,\infty)$ in the context of a one-parameter representation of the space $X(0,\infty)$, i.e., a representation of $X(0,\infty)$ which conceives of $X(0,\infty)$ as a full three-dimensional body comprised of a one-parameter set of parallel planes. In this way we return to the general concept of a one-parameter optical medium as given in Example 2 of Sec. 3.9. Furthermore, the discussions following (93) of Sec 7.4 may now be restudied with profit. In the light of the preceding analysis, that discussion now takes on a deeper meaning which can be developed in concrete terms as follows.

## The General Method of Groups

The geometric setting for the general method of groups is a rectangular parallelepiped $X(a,b,c)$ of dimensions $a,b,c$, and which is oriented and defined with the help of Fig. 7.18. The unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ of the usual right-hand cartesian coordinate frame are shown. The standard hydrologic optics coordinate system measures depth $z$ as positive, increasing in the direction $-\mathbf{k}$. Analogously, the $x$ and $y$ measurements are positive, increasing in the directions $-\mathbf{i},-\mathbf{j}$, respectively. This measuring convention is simply a logical extension of the useful plane-parallel convention of measuring $z$ positive along $-\mathbf{k}$. If desired, the $\mathbf{i}$ and $\mathbf{j}$ unit vectors may be reversed to obtain a right-hand coordinate system of more familiar appearance. Also the $\mathbf{k}$ unit vector may be reversed with $\mathbf{i}$ and $\mathbf{j}$

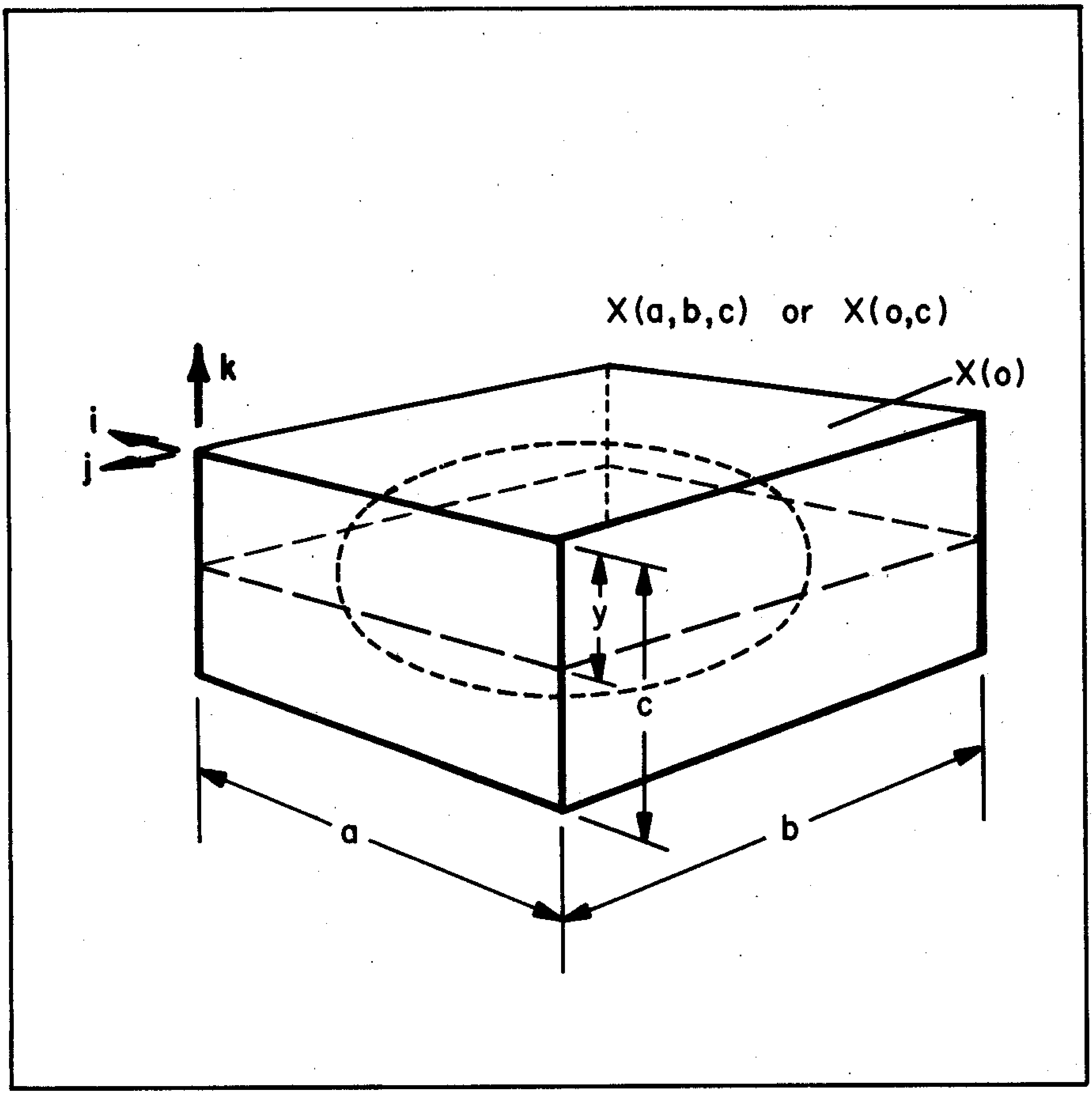


FIG. 7.18 The parallelepiped within which an arbitrary radiative transfer process can evolve and be studied.

suitably adjusted. We shall not adopt this latter reversal, as it will necessitate a massive revision of all direction conventions developed so far, and will cause difficulties in treating the unified planetary radiative transfer problems in which the atmosphere above the top plane boundary X(0) of X(a,b,c) is allowed to interact with X(a,b,c). We call X(a,b,c) a *monobloc*: it is the general version of a plane-parallel medium. The latter type of medium is the special case, of X(a,b,c) for which $a = b = \infty$. We assume that an incident radiance distribution is defined over X(0) and that there are no further sources on or within X(a,b,c). We assume also that $\alpha$ and $\sigma$ are specified throughout X(a,b,c). A one-parameter representation of the monobloc is fixed by writing:

$$X(a,b,c) = \bigcup_{0 \leq z \leq c} X(z) \tag{4}$$

where X(z) is the plane section of X(a,b,c) normal to the direction **k** and at depth z below the plane X(0). Hence each X(z) is a plane of fixed dimensions a by b. Having fixed the parametrization direction of X(a,b,c) as being parallel to **k**, we can suppress the dimensions a and b of X(a,b,c) and write simply:

$$\text{"}X(0,c)\text{"} \quad \text{for} \quad X(a,b,c) \quad . \tag{5}$$

More generally, an arbitrary subslab of X(0,c) between levels x and z is denoted by "X(x,z)" in complete analogy to the plane-parallel context discussed earlier in this chapter.

We note in passing that the choice of the direction of parametrization need not be along **k**. It can, for example, be along **i** or **j**, i.e., we could slice up X(a,b,c) by planes normal to **i** (in which case X(a,b,c) is denoted by "X(0,a)" in analogy to (5)) or normal to **j** (so that X(a,b,c) is denoted by "X(0,b)"). For the general theory developed below we could even slice up X(a,b,c) by parallel planes cocked at some outlandish angle, and being normal to an arbitrary direction $\xi_o$. Finally, the parametrization could even be accomplished with non-plane surfaces. However, as we shall presently see, the apparently special monobloc X(a,b,c), the orthodox-looking parametrization (4), and the special lighting conditions are of sufficient generality to subsume *all* cases encountered in practice.

The basic equation of the group method is the operator form of the equation of transfer:

$$\frac{dN(y)}{dy} = N(y)\mathcal{K}(y) \tag{6}$$

as developed in Sec. 7.1. Under the present lighting conditions we have, from principle of invariance I of Example 2 in Sec. 3.9:

$$N_+(y) = N_-(y)R(y,c) \quad . \tag{7}$$

This was obtained by setting z = c in principle I and using the fact that $N_+(c) = 0$. Finally, for convenience, we repeat (18) of Sec. 7.1 here (now adapted to X(y,c)):

$$-\frac{\partial R(y,c)}{dy} = \rho(y) + \tau(y)R(y,c) + R(y,c)\tau(y) + R(y,c)\rho(y)R(y,c) . \tag{8}$$

It is clear that the derivation of (8), originally performed in a stratified plane-parallel setting, holds also for the present monobloc setting. Indeed, as shown in Equation I' of Sec. 25 of Ref. [251], the gestalt of (8) persists in arbitrary optical media in euclidean three space. Equations (6), (7) and (8) are the basic equations of the general method of groups, and are used in numerical procedures as follows.

*Stage One.* Discretize the directional variables of equations (6), (7), (8) by partitioning Ξ after the manner of Secs. 7.9, 7.10. The end results are matricial versions of the three basic equations. The matrix version of each term will be written below in boldface type. Thus the function N(y) becomes the vector **N**(y) and the functional components of N(y), namely $N_+(y)$ and $N_-(y)$ become vectors $\mathbf{N}_+(y)$, $\mathbf{N}_-(y)$ as illustrated earlier in this chapter.

*Stage Two.* Solve (8) for all reflectance matrices **R**(y,c), $0 \le y \le c$ given the initial condition $\mathbf{R}(c,c) = 0$ (the zero matrix). Integration proceeds from R(c,c) through R(y,c) to R(0,c). Thus, we in effect build up X(0,c) layer by layer and compute **R**(y,c) at each intermediate stage X(y,c) of construction.

*Stage Three.* Solve (6) for all radiance vectors $\mathbf{N}_-(y)$, $0 \le y \le c$ given the initial radiance $\mathbf{N}_-(0)$. Toward this end, use (6) in expanded form:

$$\frac{d\mathbf{N}_-(y)}{dy} = \mathbf{N}_-(y)\tau(y) + \mathbf{N}_+(y)\rho(y)$$

in which (7) has been substituted in the equation for $N_-(y)$:

$$\frac{d\mathbf{N}_-(y)}{dy} = \mathbf{N}_-(y)[\tau(y) + \mathbf{R}(y,c)\rho(y)] \quad . \qquad (9)$$

Equation (9) is solved, starting at level y = 0, and is used to work down through X(0,c) to level c. At each level y, $0 \le y \le c$, R(y,c) is used, as indicated, and is taken from the result of *Stage Two*. At each level y, $\mathbf{N}_+(y)$ is obtained from the matricial version of (7). Equation (9) governs $\mathbf{N}_-(y)$; the latter is an m-component vector (cf. (23), (24) of Sec. 7.10). Equations (7) and (9) are therefore used to burrow methodically from one layer in X(0,c) down to the next, gathering up new values of $\mathbf{N}_+(y)$ along the way. Observe how knowledge of $\mathbf{N}_-(y)$ over *all of* level y permits the derivatives of the components of $\mathbf{N}_-(y)$ to be computed, from which estimates of the components $\mathbf{N}_-(y+\Delta y)$ are obtained. Equation (7) then yields $\mathbf{N}_+(y+\Delta y)$.

## Observations on the Method of Groups

A comparison of the preceding three stages of computation, especially the last two, shows that we are generalizing the method of semigroups as summarized in the system (13) of Sec. 7.8. In the present case $R_\infty$ is replaced by R(y,c) and the infinitesimal generator A (as in (12) of Sec. 7.9) now uses depth-variable operators ρ, τ, and R.

A further examination of the three stages outlined above shows that in any radiative transfer problem, of all the global properties of an extended medium, only its standard reflectance is really indispensable along with the complete transmittance operator $\mathcal{T}(0,y,c)$:

$$N_-(y) = N_-(0)\,\mathcal{T}(0,y,c)$$

which implies (9) upon differentiation of each side with respect to y. (Use (17) of Sec. 7.5 and (52), (53) of Sec. 3.7.) These two operators and their governing laws are exhibited in general in (52) and (53) of Sec. 3.7. These were used in Sec. 7.8 to develop the semigroup method in the homogeneous plane-parallel setting. The group-theoretic structure residing just below the surface activity of *Stages One, Two,* and *Three* is latent in (14) of Sec. 7.10 and may be summarized as follows. Write:

$$"\mathcal{A}(x,z)" \quad \text{for} \quad (\mathcal{R}(x,z,c), \mathcal{T}(x,z,c)) \tag{10}$$

where $\mathcal{R}(x,z,c)$ and $\mathcal{T}(x,z,c)$ are the complete reflectance and transmittance operators associated with $X(0,c)$ and $x,z$ are arbitrary levels in $X(0,c)$ such that $x \leq z$. Observe that $\mathcal{A}(x,z)$ operates on $N_-(x)$ to yield $(N_+(z), N_-(z))$. For any two such operator pairs as $\mathcal{A}(x,y)$ and $\mathcal{A}(y,z)$, write:

$$"\mathcal{A}(x,y)*\mathcal{A}(y,z)" \quad \text{for} \quad (\mathcal{T}(x,y,c)\mathcal{R}(y,z,c), \mathcal{T}(x,y,c)\mathcal{T}(y,z,c)) \tag{11}$$

By (52) and (53) of Sec. 3.7 we see immediately that:

$$\boxed{\mathcal{A}(x,y)*\mathcal{A}(y,z) = \mathcal{A}(x,z)} \quad . \tag{12}$$

Furthermore, the binary operation * defined in (11) is associative and $\mathcal{A}(x,x)$ for every x clearly serves as the identity element in the sense that:

$$\mathcal{A}(x,x)*\mathcal{A}(x,y) = \mathcal{A}(x,y) \quad . \tag{13}$$

By extending the meaning of $\mathcal{T}(x,y,c)$ and $\mathcal{R}(x,y,c)$ to the case where x and y are not restricted to the relation $x \leq y$, the set $\{\mathcal{A}(x,y): \ 0 \leq x, \ y \leq c\}$ becomes a partial group. This extension can be made by following the suggestions given around (44) of Sec. 7.4. The partial group $\{\mathcal{A}(x,y): \ 0 \leq x, \ y \leq c\}$, which we denote by "$A_2(0,c)$", is clearly isomorphic to $\Gamma_2(0,c)$ introduced in Example 7 of Sec. 3.7.

It may be well to also make some observations of a practical nature concerning the integration of equations (8) and (9). Consider (9) first. The ith component of the radiance vector function $\mathbf{N}_-(z)$ at level z is of the form $N(x,y,z,\xi_i)$ where $\xi_i$ is in the ith partition element $A_i$ of $\Xi_-$. Here x,y,z now are the three coordinates of a point in the monobloc (see Fig. 7.18). Let us write "$N_i(x,y,z)$" or "$N_i(p)$" for this ith component of $\mathbf{N}_-(z)$, where "p" stands for "(x,y,z)". Equation (9) gives the rate of change of $N_i(p)$ at point p from which we may estimate $N_i(p+\xi_i\Delta r)$ where $\Delta r$ is an increment of path length along the direction $\xi_i$. Our detailed analysis of the equation of transfer earlier in this section shows that this is the only type of extrapolation that the equation permits. However, now that $N_i(p)$ is known for every $i = 1,\ldots,m$, and for every p over the plane $X(z)$, this limited mode of extrapolation is clearly adequate to propagate the

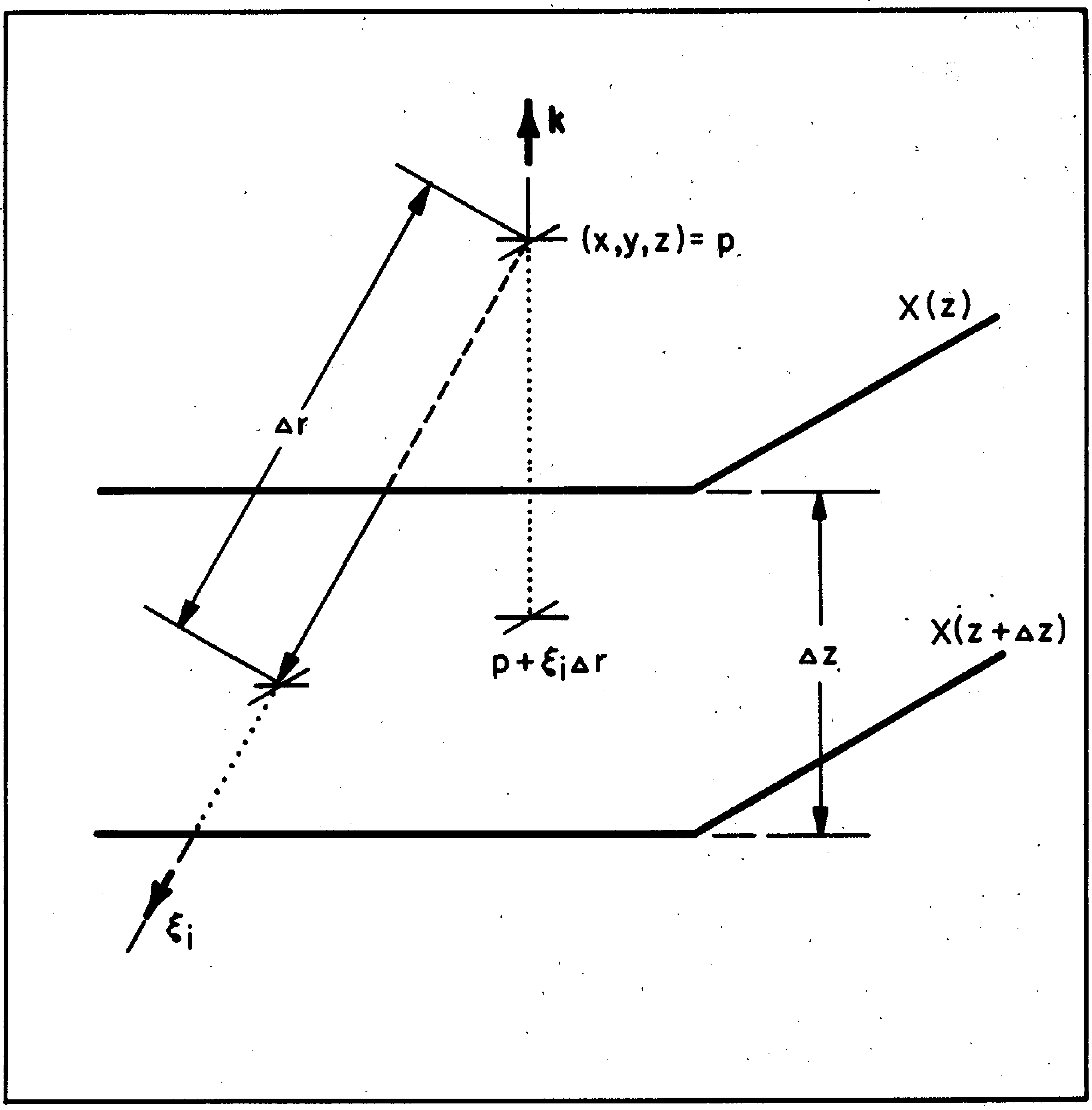


FIG. 7.19 How to propagate the radiance computations from one parameter surface to the next.

radiance field from plane X(z) to plane X(z+Δz) for suitably chosen Δz. Fig. 7.19 helps show how the function $\mathbf{N}_-(z+\Delta z)$ over X(z+Δz) is obtained from $\mathbf{N}_-(z)$ known over X(z). In particular we have for each p (= (x,y,z)) in X(z) and $\xi_i$ in $\Xi_-$:

$$N_i(p+\xi_i\Delta r) = N_i(p) + \frac{dN_i(p)}{dz}\Delta z \tag{14}$$

where:

$$\Delta z = -\xi_i\cdot\mathbf{k}\Delta r \quad . \tag{15}$$

Once $\mathbf{N}_-(z+\Delta z)$ has been obtained, we use (7) to find $\mathbf{N}_+(z+\Delta z)$ by means of the formula:

$$\mathbf{N}_+(z+\Delta z) = \mathbf{N}_-(z+\Delta z)\mathbf{R}(z+\Delta z,c) \quad . \tag{16}$$

Now that (14) and (15) have been displayed, and the mode of propagation of $\mathbf{N}_-(z)$ to $\mathbf{N}_-(z+\Delta z)$ has been made clear, it may be well to observe that there is no royal road to the solution of the radiative transfer problem in a general monobloc such as $X(a,b,c)$. The sheer number of dimensions of $X(a,b,c)$ and $\Xi_-$ must always combine to dampen the enthusiasm of the most intrepid computer. At least now that the invariant imbedding techniques have shown the fundamental structure of the present type of transfer problem (as outlined in *Stages One* to *Three* above), we can rest in the knowledge that the theory has progressed as far as it can go on the phenomenological level, and that what remains is the development of more adequate numerical procedures to use on (8) and (9) for the general monobloc $X(a,b,c)$. Of course this is not meant to discount the use of other procedures such as those based on the classical techniques of Chapter 6, or on the natural mode of solution (Chapter 5), or the canonical mode (Chapter 4), or their equivalents. As far as Eq. (8) is concerned, one such attempt has been made using invariant imbedding techniques in Ref. [251] wherein the solution of (8) is carried out on a monobloc using the approach of discrete-space theory developed in that reference (see, in particular, Chapter X).

## The Method of Groups and the Inner Structure of Natural Light Fields

We now round out our discussion of the method of groups and also bring to a close some matters raised in Example 7 of Sec. 3.7 by outlining a proof of the general group-theoretic structure of light fields in natural optical media.

Let $X$ be an arbitrary connected source-free subset of euclidean three-space. Let $\alpha,\sigma$ be given throughout $X$ and let $X$ be irradiated arbitrarily on its boundary. A parametrization of $X$ is introduced so that:

$$X = \bigcup_{a \le z \le b} X(z) \quad . \tag{17}$$

This decomposition of $X$ into a family of two-dimensional surfaces $X(z)$ is illustrated in (a) of Fig. 7.20. In this way $X$ becomes a one-parameter optical medium.

To each subslab $X(x,z)$ of $X$, shaded in (a) of Fig. 7.20, we can assign reflectance and transmittance operators after the manner explained in Examples 2, 4, and 5 of Sec. 3.9 so that the invariant imbedding relation holds for $X$. Hence equations (6), (7), (8) can be suitably extended to the setting in $X$ so that the general counterpart to (12) holds, and a partial group $A_2(a,b)$ can be assigned to $X$. In particular a computation procedure for $\mathbf{N}_-(y)$ can be initiated and sustained that will propagate $N_-(z)$ across each parameter surface $X(z)$ within $X$ in a manner completely analogous to that based on (14)-(16).

The parametrization (17), being quite general, leads to an instructive mode of description of the inner structure of the light field. As an interesting special case of (17),

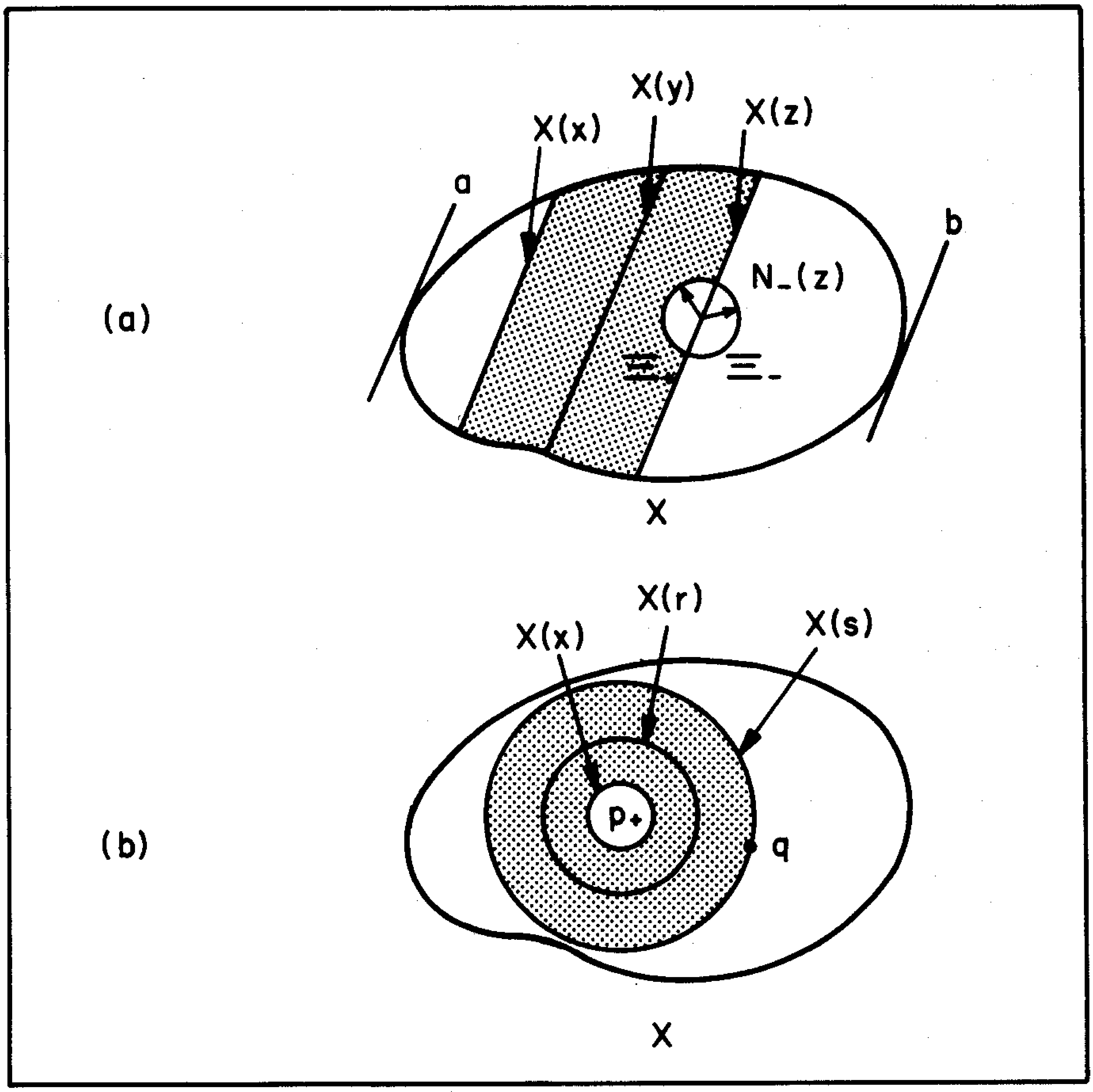


FIG. 7.20 There are many ways in which an optical medium can be made over into a one-parameter medium.

consider the parametrization of X by spherical shaped surfaces X(r) within X of radius r about an internal point p of X. We define X(r) as the intersection of a sphere of radius r with X. See (b) of Fig. 7.20. Suppose the light field is given on arbitrarily small spherical surface X(r). Then using the general one-parameter versions of (12), the radiance field can be computed at any point q in X, where q lies on X(s) for some radius s. Conversely, knowledge of the light field on some sphere about q as center could lead in principle to the determination of the light field at p after re-parametrization of X about q. This then is the most general description of the inner structure of natural light fields in an arbitrary optical medium X as defined above.

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