The loss of symmetry (where we use the term in the sense of Definition 6) is a phenomenon that arises because of the adoption of radiance as the basic radiometric concept rather than irradiance or alternatively, radiant density. Had we used the latter concept, then symmetry would hold (in the scalar irradiance context) for inhomogeneous isotropic plane-parallel media. Symmetry would be lost in such a context only when isotropy was lost. By adopting radiance over irradiance we reap the benefits of a more detailed description of the light field at the expense of the classical symmetries possessed by irradiance. Furthermore, the reflectance and transmittance functions R and T in the scalar irradiance context, being scalars, commute; i.e., symbolically, $RT = TR$. By adopting radiance, R and T become integral operators or matrices, and these objects are notoriously noncommutative, thus blocking still further the passage of certain symmetries of the scalar formulations to the field of operator formulations.

Conclusion

In conclusion, then, the elevation of the local notions of homogeneity, separability, isotropy, and reversibility to the global settings in plane-parallel media is quite possible. However, only the local concept of reversibility is generally inherited by the space on the global level (in the form of reciprocity). But this inheritance is precarious and can conceivably vanish on graduation to arbitrarily shaped anisotropic media in which the radiometric concept used is radiance rather than irradiance or scalar irradiance. Thus all the classical symmetries are in principle left behind in the search for general invariant properties of scattering-absorbing media. The general principles of invariance, the invariant imbedding relations and their various semigroup properties are important examples of general properties of optical media which are invariant under the transition from local to global formulations within those media. This has been shown in detail in Chapter VI of Ref. [251], for general discrete spaces.

Further study of the problem of the extension of local symmetries to the global level are best handled by means of the standard $\mathscr{V}$-operator $\mathscr{V}(X;a,b)$. A detailed study of such extensions has yet to be made. It would be of interest to formulate the appropriate counterparts to homogeneity, and isotropy for general media using $\mathscr{V}(X;a,b)$, and then to find theorems, if possible, which are the appropriate generalization of the Polarity and Reciprocity theorems.

7.13 Functional Relations for Media with Internal Sources

In this section we return to the problem of internal sources in optical media introduced in Example 3 of Sec. 3.9 and reconsidered in Sec. 6.7. From a theoretical point of view the problem was completely solved in Sec. 3.9 and, in view of the methods of determination of the R and T operator discussed in Sec. 7.7, we may say that the practical numerical means of solving the internal-source problem are also well in hand. However, there remain several most interesting questions on the conceptual level, questions that arise when one examines the functional relations (35) and (36) of Sec. 6.7 with an eye toward the intuitive meaning of the equations and of their connection with the invariant imbedding relations that may be written down for the same medium. Specifically, we are confronted with two equations (35), (36) of Sec. 6.7, derived directly from the equation of transfer, and which are ostensibly statements of a certain type of invariance for scalar irradiance h(x). Their physical meanings as given by Elliott [88], are, however, occluded by the fact that their main terms f_0 and f_c are Fourier transforms of the scalar irradiance function $h(x)$ (cf. (23) of Sec. 6.7) rather than h(x) itself. Therefore, one of the principal goals in this section is the development of a systematic method of derivation of the counterparts to (35), (36) of Sec. 6.7 for the case of radiance in a general one-parameter optical medium X(a,b) with an arbitrary set of sources on various levels within X(a,b), using only the concepts inherent in the invariant imbedding relation for the medium. We therebye shall establish intuitively meaningful generalizations of the Elliott equations and also extend their domain of validity. An additional dividend is accrued throughout in the form of further insight into the interconnections among the Ψ-operators and the invariant imbedding operators. These connections arise as a matter of course during the derivations. Throughout this section, let "X(a,b)" denote a one-parameter optical medium with artibrary α,σ. In particular X(a,b) will not be assumed isotropic, so that there are generally four local operators $\rho_\pm(t)$, $\tau_\pm(t)$ (cf. Sec. 7.1). Throughout this section sources shall be confined, for simplicity and without any serious loss of generality, to single depths s within the slab X(a,b), $a \le s \le b$. For sources at several discrete levels, superposition of the results below will yield the desired field expression. By passing to the limit of numbers of discrete sources, the theoretical way to continuously distributed sources is opened. These generalizations are left to the reader. For helpful hints in this direction see (36) of Sec. 3.9 and its discussion. Also see the paragraph on Two-D Models for Internal Sources in Sec. 8.5, and consult Example 10 of Sec. 8.7.

Preliminary Relations

One important dividend of the present efforts is a collection of auxiliary functional relations between the $\mathcal{R}, \mathcal{T}$ operators and the ψ-operator of Sec. 3.9. These equations place the interrelations of the ψ-operator into a deeper perspective than is available from (31)-(34) of Sec. 3.9. Of particular interest at present are the connections between $\mathcal{R}(a,s,b)$, $\mathcal{T}(a,s,b)$, and $\psi(s,s)$, where $a \le s \le b$. It follows from (20)-(23) of Sec. 3.9 and (40)-(43) of Sec. 3.7 that:

$$\mathcal{R}(a,s,b) = T(a,s)\, \Psi_{-+}(s,s:a,b) \tag{1}$$

$$\mathcal{R}(b,s,a) = T(b,s)\, \Psi_{+-}(s,s:a,b) \tag{2}$$

$$\mathcal{T}(a,s,b) = T(a,s)\left(I + \Psi_{--}(s,s:a,b)\right) \tag{3}$$

$$\mathcal{T}(b,s,a) = T(b,s)\left(I + \Psi_{++}(s,s:a,b)\right) \tag{4}$$

where we have written; for $a \leq s \leq b$, $a \leq y \leq b$:

$$\text{"}\Psi(s,y:a,b)\text{"} \quad \text{for} \quad \Psi(s,y) \tag{5}$$

to point up specifically the fact that $\Psi(a,y)$ belongs to $X(a,b)$. (See the remarks following the applications of the interaction method in Example 3 of Sec. 3.9.)

Equations (1)-(4) show clearly the interrelation between the complete reflectance operators and the local Ψ-operator for $X(a,b)$. These relations will be helpful in constructing a dual class of $\mathcal{R}$ or $\mathcal{T}$ operators needed subsequently.

Another set of functional relations, needed in the derivations below, is the following, which again is based on (20)-(23) of Sec. 3.9:

$$\Psi_{++}(s,s:a,b) = \Psi_{+-}(s,s:a,b)R(s,b) \tag{6}$$

$$= R(s,a)\,\Psi_{-+}(s,s:a,b) \tag{7}$$

$$I + \Psi_{++}(s,s:a,b) = [I - R(s,a)R(s,b)]^{-1} \tag{8}$$

$$\Psi_{--}(s,s:a,b) = \Psi_{-+}(s,s:a,b)R(s,a) \tag{9}$$

$$= R(s,b)\,\Psi_{+-}(s,s;a,b) \tag{10}$$

$$I + \Psi_{--}(s,s:a,b) = [I - R(s,b)R(s,a)]^{-1} \tag{11}$$

$$\Psi_{+-}(s,s:a,b) = R(s,a)[I + \Psi_{--}(s,s:a,b)] \tag{12}$$

$$= [I + \Psi_{++}(s,s:a,b)]R(s,a) \tag{13}$$

$$\Psi_{-+}(s,s:a,b) = R(s,b)[I + \Psi_{++}(s,s:a,b)] \tag{14}$$

$$= [I + \Psi_{--}(s,s:a,b)]R(s,b) \tag{15}$$

Integral Representations of the Local Ψ-Operators

We are now ready for the derivations of the representations of the local Ψ-operators in terms of the simpler $\mathcal{R}$ and $\mathcal{T}$ operators of the invariant imbedding relation. We fix attention at first on the setting within $X(a,b)$ depicted in Fig. 7.23.

An internal source is at level s in $X(a,b)$. The source may be a point, or some arbitrary discrete or continuous set of points on level s, and of arbitrary directional structure at each point of the set. We consider first the upward

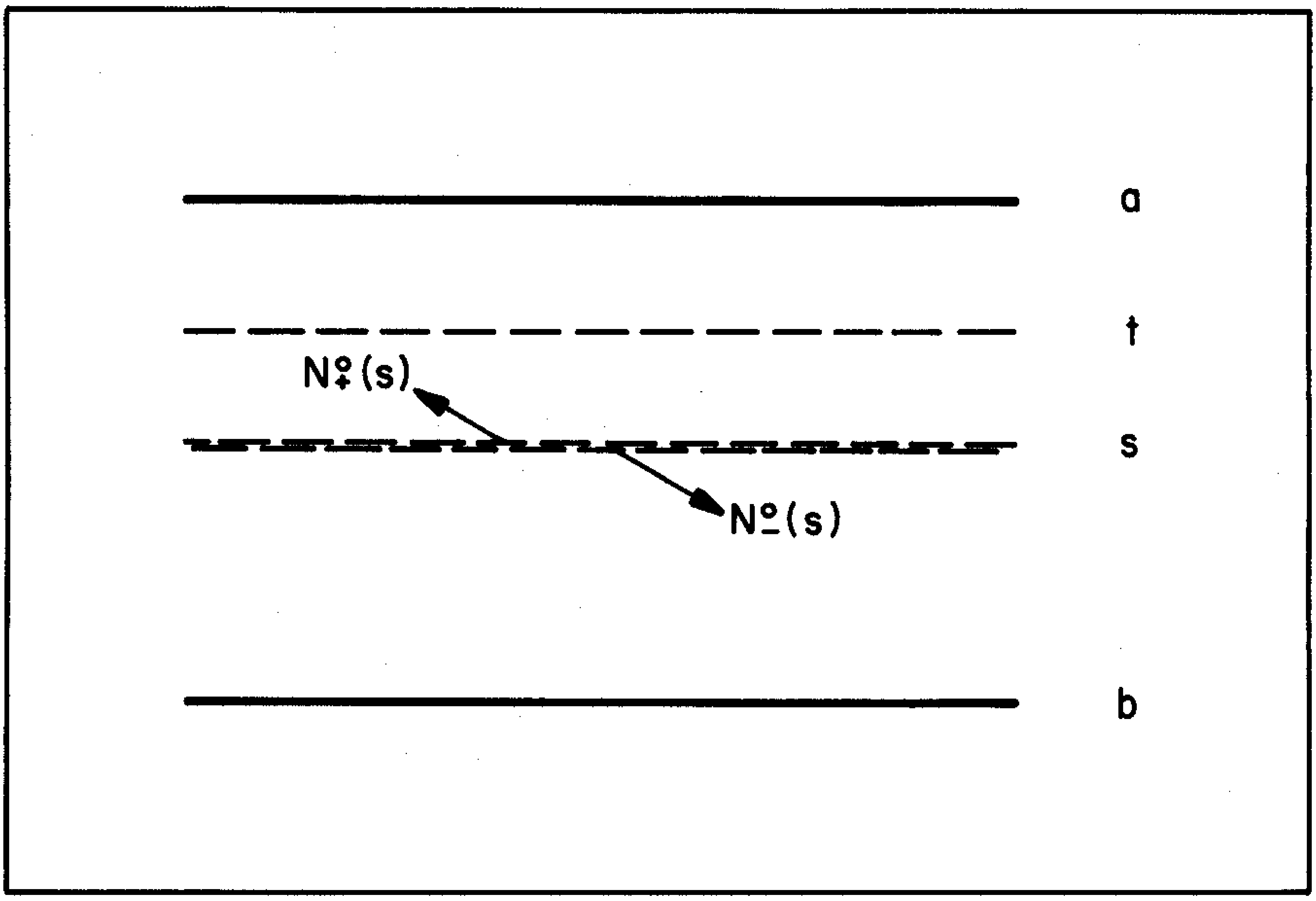

FIG. 7.23 An internal source situation in medium X(a,b). The first main case: source level (s) below observation level (t).

components $N^{o}_{+}(s)$ of the source $N^{o}(s)$. The resultant radiance field at level s generated throughout X(a,b) by this source component is, as explained in Sec. 3.9, given by the local Ψ -operator components $\Psi_{++}(s,s;a,b)$ and $\Psi_{+-}(s,s;a,b)$. The first of these gives the resultant upward field, the second gives the resultant downward field. Now consider $\Psi_{++}(s,s;a,b)$. We wish to study the dependence of $\Psi_{++}(s,s;a,b)$ on a, holding s and b fixed. It will turn out that knowledge of this dependence will lead directly to the requisite integral representation of $\Psi(s,s;a,b)$, in terms of a family of invariant imbedding operators for X(a,b).

Thus, imagine a family $\{X(t,b); a \le t \le b\}$ of optical media in which the original medium X(a,b) is imbedded (i.e., X(a,b) is a member of the family). The associated family of the local Ψ -operator is $\{\Psi_{++}(s,s:t,b): a \le t \le b\}$. The effect of varying t in $\Psi_{++}(s,s:t,b)$ can be determined by taking the derivative of $\Psi_{++}(s,s:t,b)$ in the following way:

$$\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \frac{\partial \Psi_{+-}(s,s:t,b)}{\partial t} R(s,b)$$

which is suggested by (6). Furthermore, (13) suggests that we write:

$$\frac{\partial \psi_{+-}(s,s:t,b)}{\partial t} = \frac{\partial}{\partial t}\{[I + \Psi_{++}(s,s:t,b)]R(s,t)\}$$
$$= \frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} R(s,t)$$
$$+ [I + \Psi_{++}(s,s:t,b)] \frac{\partial R(s,t)}{\partial t}$$
$$= \frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} R(s,t)$$
$$-[I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_{+}(t)T(t,s) \quad .$$

The last equality comes from (28) of Sec. 7.1, applied to X(t,s). The minus sign comes from the fact that depth is measured positive from a to b. Returning to the original equation, we have:

$$\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} R(s,t)T(s,b)$$
$$-[I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_{+}(t)T(t,s)R(s,b).$$

The derivative term may therefore be solved for and found to be of the form:

$$-\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} =$$

$$= [I + \Psi_{++}(s,s;t,b)]T(s,t)\rho_{+}(t)T(t,s)R(s,b)[I - R(s,t)R(s,b)]^{-1}$$

Upon examination, this seemingly complex representation collapses into the composition of three highly intuitive forms. One of these is $\mathscr{R}(t,s,b)$, since, by (42) of Sec. 3.7:

$$\mathscr{R}(t,s,b) = T(t,s)R(s,b)[I - R(s,t)R(s,b)]^{-1} \quad .$$

The next term to consider is:

$$[I + \Psi_{++}(s,s:t,b)]T(s,t) \quad . \tag{16}$$

This is an unfamiliar combination of operators. It has not arisen in our work as yet. However, there is a tantalizing asymmetry between it and (3) above. When the variables t,s,b are placed in (3) we have:

$$\mathcal{T}(t,s,b) = T(t,s)[I + \Psi_{--}(s,s:t,b)] \quad .$$

The interpretation of $\mathcal{T}(t,s,b)$ is at this stage of our studies well understood: downward incident radiance at level t generates a light field in X(t,b) and $\mathcal{T}(t,s,b)$ gives the downward component of that field of level s in X(t,b). (See Figure 7.23) The term (16) seems to give a dual interpretation to that of $\mathcal{T}(t,s,b)$. Thus, it says that upward source radiance at level s generates a light field within X(t,b) and (16) gives the upward component of that light field at level t in X(t,b). Hence (16) acts like a complete transmittance operator, but one whose input level (s) and output level (t) are exactly reversed from their customary relative orientations within X(t,b). With these interpretations and the dual kinship of (16) and $\mathcal{T}(t,s,b)$ in mind, let us write "$\mathcal{T}^{\dagger}(s,t,b)$" for (16). Then the differential equation for $\Psi_{++}(s,s:t,b)$ becomes:

— a
— t
— s
— b

$$-\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b) \qquad (17)$$

Integrating each side of (17) over the interval [a,s], and using the fact that:

$$\Psi_{++}(s,s:s,b) = 0 \quad ,$$

we have:

— a
— t
— s
— b

$$\Psi_{++}(s,s:a,b) = \int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)\,dt \qquad a \le t \le s \le b$$

(18)

The simple physical interpretation of (18) should not escape notice. Consider X(t,b). Imagine the source $N_+^{o}(s)$ at level s giving rise to the upward emergent radiance at level t in the space X(t,b). Then imagine a thin incremental layer added to X(t,b) at level t. This thin layer reflects some of the emergent flux via ($\rho_+(t)$) back down into X(t,b). This reflected flux sets up a light field in X(t,b), the upward component of which at level s being given by $\mathcal{R}(t,s,b)$. By letting X(t,b) grow another thin layer at level t, still another incremental light field is added to that at level s. By adding up all such increments, starting from level s and working up to level a, we obtain the total field at level s induced by the upward source component $N_+^{o}(s)$. The analytical representation (18) summarizes all this compactly, as shown. The little ideograph next to (17) and (18) serves to depict the relative positions of the depth variables in X(a,b).

The representation (18) is one of a pair of representations for $\Psi_{++}(s,s:a,b)$, the other arising when we imbed X(a,b) in the family {X(a,t): $a \le t \le b$} of spaces. (See Figure 7.24) Then we consider:

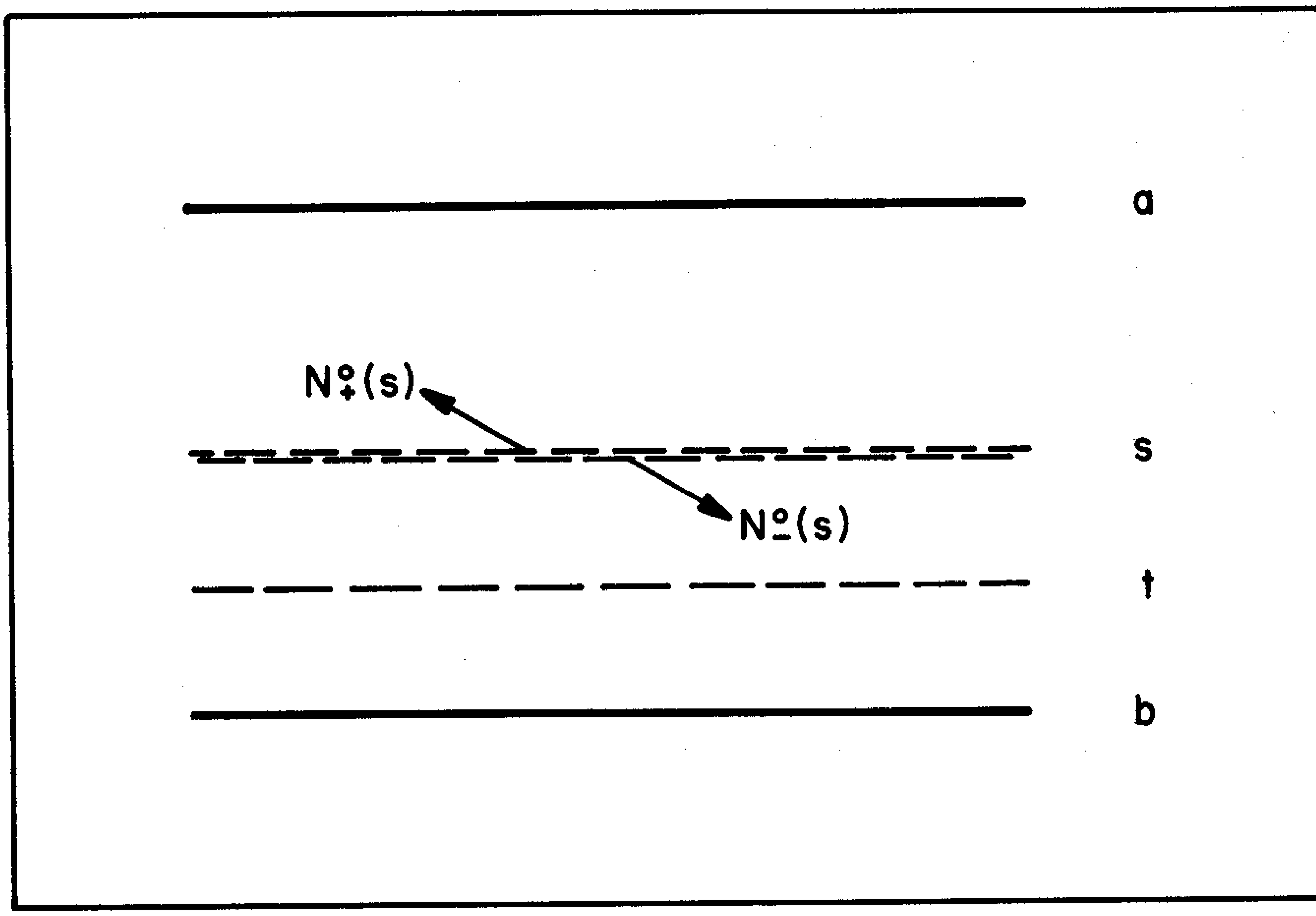

FIG. 7.24 An internal source situation in medium X(a,b). The second main case: source level (s) above observation level (t).

$$\frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} = R(s,a) \frac{\partial \Psi_{-+}(s,s;a,t)}{\partial t} ,$$

using (7). This is analyzed further using (14), which yields:

$$\frac{\partial \Psi_{-+}(s,s:a,t)}{\partial t} = \frac{\partial R(s,t)}{\partial t} [I + \Psi_{++}(s,s:a,t)] + $$
$$+ R(s,t) \frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} .$$

Once again a complex group of terms can be collapsed into a composition of three physically meaningful groups of terms. The operator $\rho_-(t)$ separates the two remaining terms. The last group of terms is simply $\mathcal{T}(t,s,a)$ (cf. (43) of Sec. 3.7). The first group of terms is one of those tantalizing duals to the invariant imbedding operators. This time, by studying (41) of Sec. 3.7 and (1) above, we see that we should write "$\mathcal{R}^\dagger(s,t,a)$" (see (26) below) so that the preceding differential equation for $\Psi_{++}(s,s:a,t)$ becomes:

— a
— s
— t
— b

$$\boxed{\frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} = \mathcal{R}^\dagger(s,t,a)\rho_-(t)\,\mathcal{T}(t,s,a)} \qquad (19)$$

Integrating (19) over the interval [s,b] and using the fact that:

$$\Psi_{++}(s,s:a,s) = 0 \quad ,$$

we have:

— a
— s
— t
— b

$$\boxed{\Psi_{++}(s,s:a,b) = \int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\ dt \qquad a \le s \le t \le b} \tag{20}$$

The physical interpretation of (20) is as follows: consider the medium X(a,t) with upward source $N_+^o(s)$ at level s (Fig. 7.24). The light field in X(a,t) generated by this source has a downward component at level t given by $\mathcal{R}^{\dagger}(s,t,a)$. (If source flux is upward directed at s, then, since $\mathcal{R}^{\dagger}$ is a *reflector*, response flux is downward directed at t.) Adding a thin increment to X(a,t) at level t causes a corresponding increment of reflected radiance (via $\rho_-(t)$) to re-enter X(a,t) and to be completely transmitted by $\mathcal{T}(t,s,a)$ to level s. By adding all such incremental layers on X(a,t) from t = s to t = b, the representation of $\Psi_{++}(s,s:a,b)$ is obtained.

The pattern emerging in these derivations should now be clear. On the basis of this emerging pattern the differential equations for $\Psi_{--}(s,s:t,b)$ and $\Psi_{--}(s,s:a,t)$ and other corresponding integrals can be written down directly without any further detailed derivation. However, the interested reader should verify the formulas so obtained:

— a
— t
— s
— b

$$\boxed{-\frac{\partial \Psi_{--}(s,s:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)} \tag{21}$$

— a
— t
— s
— b

$$\boxed{\Psi_{--}(s,s:a,b) = \int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\ dt \qquad a \le t \le s \le b} \tag{22}$$

These equations are companions to (17) and (18) and go with Fig. 7.23. The following equations are companions to (19) and (20) and go with Fig. 7.24.

— a
— s
— t
— b

$$\boxed{\frac{\partial \Psi_{--}(s,s:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)} \tag{23}$$

— a
— s
— t
— b

$$\boxed{\Psi_{--}(s,s:a,b) = \int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)\ dt \qquad a \le s \le t \le b} \tag{24}$$

We now consolidate the definitions of the dual invariant imbedding operators. For this purpose, we use a general one-parameter setting X(a,b) and an arbitrary level s in X(a,b). The resultant definitions then will be completely dual to their counterparts in (1)-(4). Thus, for $a \le s \le b$ we write:

$$\text{"}\mathcal{R}^{\dagger}(s,a,b)\text{"} \quad \text{for} \quad \Psi_{-+}(s,s:a,b)T(s,a) \tag{25}$$

$$\text{"}\mathcal{R}^{\dagger}(s,b,a)\text{"} \quad \text{for} \quad \Psi_{+-}(s,s:a,b)T(s,b) \tag{26}$$

$$\text{"}\mathcal{T}^{\dagger}(s,a,b)\text{"} \quad \text{for} \quad [I + \Psi_{++}(s,s:a,b)]T(s,a) \tag{27}$$

$$\text{"}\mathcal{T}^{\dagger}(s,b,a)\text{"} \quad \text{for} \quad [I + \Psi_{--}(s,s:a,b)]T(s,b) \; . \tag{28}$$

The reader should study (1)-(4) and (25)-(28) to discover the rhyme and rule which bridges the gap between each of the pairs (1) and (25), (2) and (26), (3) and (27), (4) and (28). We call the four operators in (21)-(25) the *dual invariant imbedding operators*; $\mathcal{R}^{\dagger}$ being a *dual complete reflectance operator*, $\mathcal{T}^{\dagger}$ a *dual complete transmittance operator.*

The set of representations of the local Ψ-operator can be completed without any further derivative operations. To find the representation for $\Psi_{+-}(s,s:a,b)$ we may use either (12) or (13). For example, using (12) and (24) we have:

$$\Psi_{+-}(s,s:a,b) = R(s,a)\left[I + \int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,s,a)\,dt\right] \; .$$

From this, we have at once:

—a
—s
—t
—b

$$\boxed{\begin{aligned}&\Psi_{+-}(s,s:a,b) = R(s,a) + \int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,s,a)\,dt\\ &a \le s \le t \le b\end{aligned}} \tag{29}$$

in which we have used the readily verified fact that:

$$\mathcal{R}^{\dagger}(s,t,a) = R(s,a)\mathcal{T}^{\dagger}(s,t,a) \quad . \tag{30}$$

On the other hand, using (13) and (20), we arrive at the same equation (29), as may be verified by the reader. (The dual relation to (30) is now used, namely (53) of Sec. 3.7.)

Finally $\Psi_{-+}(s,s:a,b)$ is found using either (14) or (15). For example, (14) and (18) yield:

$$\Psi_{-+}(s,s:a,b) = R(s,b)\left[I + \int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)\,dt\right] \; .$$

From this and the fact that:

$$\mathcal{R}^{\dagger}(s,t,b) = R(s,b)\mathcal{T}^{\dagger}(s,t,b) \quad , \tag{31}$$

we have:

—a
—t
—s
—b

$$\Psi_{-+}(s,s:a,b) = R(s,b) + \int_a^s \mathcal{R}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,s,b)\,dt$$

$$a \le t \le s \le b \tag{32}$$

Our goal requires us to find *all* possible representations of the local and global Ψ-operators. Thus, having found $\Psi_{+-}(s,s:a,b)$ in terms of an integral over [s,b], we are led to seek the representation of $\Psi_{+-}(s,s:a,b)$ in terms of an integral over [a,s]. Equation (6) provides the clue: in (18) we should factor R(s,b) from $\mathcal{R}(t,s,b)$ by means of (53) of Sec. 3.7. The result is:

$$\Psi_{++}(s,s:a,b) = \int_a^s \left[\mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\,dt\right]R(s,b) \quad .$$

By (6) we conclude that:

— a
— t
— s
— b

$$\Psi_{+-}(s,s:a,b) = \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\,dt \tag{33}$$

$$a \le t \le s \le b$$

Equation (32) in turn spurs a search for a representation of $\Psi_{-+}(s,s:a,b)$ in terms of an integral over [s,b]. This time (7) makes it quite clear that, by factoring R(s,a) from (20), in this manner:

$$\Psi_{++}(s,s:a,b) = R(s,a)\int_a^b \mathcal{T}^\dagger(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\,dt \quad ,$$

which is possible by (30), we must end up with:

— a
— s
— t
— b

$$\Psi_{-+}(s,s:a,b) = \int_s^b \mathcal{T}^\dagger(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\,dt \tag{34}$$

$$a \le s \le t \le b$$

Integral Representations of the Global Ψ-operators

It remains to derive the integral representations of the operators $\Psi(s,y:a,b)$. The set of relations connecting the global and local Ψ-operators, given in (31)-(34) of Sec. 3.9 are now put to work. Thus we consider first the case $a \le y < s \le b$ using (31) of Sec. 3.9 and (20). We have at once:

—a
—y
—s
—b

$$\Psi_{++}(s,y:a,b) = \mathcal{T}(s,y,a) + \int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\,\mathcal{T}(t,y,a)\,dt \qquad a \le y < s \le b \tag{35}$$

in which we have used the semigroup relation (52) of Sec. 3.7. Using (32) of Sec. 3.9 and (20), the result is:

—a
—y
—s
—b

$$\Psi_{+-}(s,y:a,b) = \mathcal{R}(s,y,a) + \int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\,\mathcal{R}(t,y,a)\,dt \qquad a \le y < s \le b \tag{36}$$

in which the semigroup relation (53) of Sec. 3.7 was used. Using (33) of Sec. 3.9 with (34) above, we have:

— a
— y
— s
— b

$$\Psi_{-+}(s,y:a,b) = \int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\,\mathcal{T}(t,y,a)\,dt \qquad a \le y < s \le b \tag{37}$$

Finally, according to (34) of Sec. 3.9 and (34) above:

— a
— y
— s
— b

$$\Psi_{--}(s,y:a,b) = \int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\,\mathcal{R}(t,y,a)\,dt \qquad a \le y < s \le b \tag{38}$$

The relations (35)-(38) constitute the set of functional relations for the case where the source level s is below the observation level y within X(a,b). For the case where the source level s is above the observation level y within X(a,b) we use the following readily verified set of dual equations to (31)-(34) of Sec. 3.9:

$$\Psi_{--}(s,y:a,b) = [I + \Psi_{--}(s,s:a,b)]\,\mathcal{T}(s,y,b) \tag{39}$$

$$\Psi_{-+}(s,y:a,b) = [I + \Psi_{--}(s,s:a,b)]\,\mathcal{R}(s,y,b) \tag{40}$$

$$\Psi_{+-}(s,y:a,b) = \Psi_{+-}(s,s:a,b)\,\mathcal{T}(s,y,b) \tag{41}$$

$$\Psi_{++}(s,y:a,b) = \Psi_{+-}(s,s:a,b)\,\mathcal{R}(s,y,b) \tag{42}$$

The preceding equations hold for $a \le s < y \le b$. To see how the derivations go in the present case, we use (22) and (39) to find:

—a
—s
—y
—b

$$\Psi_{--}(s,y:a,b) = \mathcal{T}(s,y,b) + \int_a^s \mathcal{R}^\dagger(s,t,b)\rho_+(t)\,\mathcal{T}(t,y,b)\,dt \qquad a \le s < y \le b \tag{43}$$

Equation (22) and (40) yield:

—a
—s
—y
—b

$$\Psi_{-+}(s,y:a,b) = \mathcal{R}(s,y,b) + \int_a^s \mathcal{R}^\dagger(s,t,b)\rho_+(t)\,\mathcal{R}(t,y,b)\,dt \qquad a \le s < y \le b \tag{44}$$

Equation (33) and (41) yield:

— a
— s
— y
— b

$$\Psi_{+-}(s,y:a,b) = \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\,\mathcal{T}(t,y,b)\,dt \qquad a \le s < y \le b \tag{45}$$

Finally, (33) and (42) yield:

— a
— s
— y
— b

$$\Psi_{++}(s,y:a,b) = \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\,\mathcal{R}(t,y,b)\,dt \qquad a \le s < y \le b \tag{46}$$

Incipient Patterns and Nascent Methods

The preceding development of the eight integral representations of the local Ψ-operator and the eight of the global Ψ-operator, followed fairly closely the actual sequence of discovery of the representations. This sequence was reproduced because it seems the most natural didactic path into the present subject matter. We have thus progressed far enough into the forest of integral representations to become acquainted with some of the important individual "trees". It is time, however, to rise above the trees and obtain a glimpse of the entire forest. By doing so we can discern the general pattern of the derivations given so far and thereby organize efficient methods of derivation of the remaining functional relations.

The principal observation to make on all the foregoing activity is on the manner of construction of the Ψ-operators in terms of simpler components; namely, the invariant imbedding operators and their duals and, of course, the local operators $\rho_\pm$ and $\tau_\pm$. We note in particular the way in which a layer in X(a,b) is made to grow from an imbedded core X(s,y) to the entire slab X(a,b), and how during this growth the original simple operators (standard R and T operators) are built up continuously and in a corresponding manner to obtain the Ψ-operator. This mode of construction of X(a,b) is closely related to the Categorical Synthesis Method developed in Ref. [251] for discrete spaces. Figure 7.25 helps describe the synthesis in the present continuous setting.

We begin with a degenerate case X(s,s), i.e., a single parameter surface X_s of X(a,b) which is irradiated by a source radiance $N^o(s)(=(N^o_+(s),N^o_-(s))$. Then a layer X(y,s), $y \le s$, is produced by letting X_y move upward away from X_s. The standard operators R(s,y), T(s,y) are shown in the figure as associated with X(s,y) or X(y,s) ("R(s,y)" denoting a reflectance for upward or downward radiance distribution, as the case may be). The response of X(s,y) or X(y,s) the source $N^o(s)$ at level s is given by these standard operators, since the source is external to the slabs. This completes the first stage of growth: we began with a single surface X_s and continuously built up a slab X(s,y) or X(y,s) from it keeping the irradiation $N^o(s)$ constant, and fixing attention on the response at level y.

The second stage of growth gives rise to four possibilities, as shown in Fig. 7.25. For example, we could start with the upper slab X(y,s) and let it grow to be X(a,s), $a \le y$. During this growth process, the source $N^o(s)$ is retained and we still want to know what the response at layer y is. We have just the conceptual tools to give us the answer, namely the complete reflectance and transmittance operators $\mathcal{R}(s,y,a)$, $\mathcal{T}(s,y,a)$. Alternatively, we could let X(y,s) grow into X(y,b), $s \le b$, with the source $N^o(s)$ remaining at level s. Now the source becomes "submerged" or imbedded in X(y,b) and the conceptual tools which will give us the response at level y are the *dual* invariant imbedding operators defined in (25) and (27) (with y = a). The two other remaining cases in the

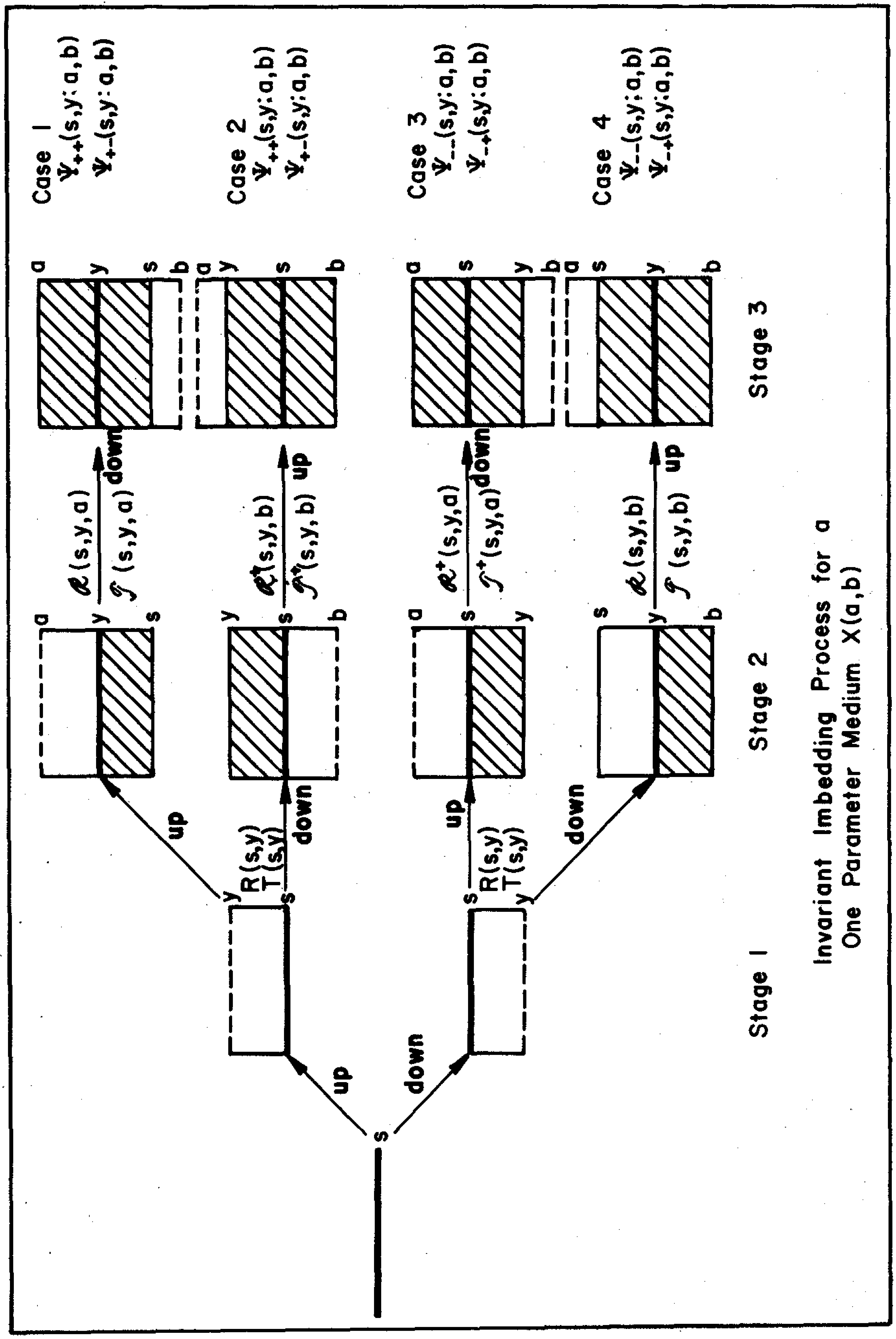

FIG. 7.25 A complete classification of the invariant imbedding process for a one-parameter medium X(a,b). Starting with the surface at level s, successive possibilities of growth of slabs are depicted in stages 1, 2, and 3.

second stage of growth are explained similarly.

The third stage of growth also gives rise to four distinct possibilities. For example, in case 1, starting with X(a,s) we let it grow downward to become X(a,b), $s \le b$, still retaining the source at level s and still inquiring as to the response at level y. The interaction operator that describes this situation is the indicated global Ψ-operator. The remaining three cases are described similarly.

The results of the third stage of growth fall uniformly within the scope of the global Ψ-operator and thereby the most general source-response irradiation configuration within X(a,b) can be described using $\Psi(s,y:a,b)$. Thus, under the suitable confluence of the variables s,y,a,b, the invariant imbedding operators, their duals, and the standard slab operators are all forthcoming from $\Psi(s,y:a,b)$. This will be shown in detail later.

The overview of the preceding derivations of the eight integral representations of the local Ψ-operator is now before us. The settings of the derivations initially took place in the third stage of growth of Fig. 7.25 for the degenerate instance $s = y$. For example, Fig. 7.23, which goes with the derivation of (18), (22), (32), (33), falls under the degenerate instances of cases 2 and 4 in stage three. Finally, Fig. 7.24, which goes with the derivation of (20), (24), (29), (34), falls under the degenerate instances of cases 1 and 3 in stage 3.

The overview of the derivations of the global Ψ-operators in (35)-(46) given by Fig. 7.25 is quite interesting. Had we not made the systematic analysis of all growth possibilities, we might have missed the eight remaining possibilities beyond (35)-(46). To see this in detail, first note that representations (35)-(38) all fall under case 1 of stage three, and that representatives (43)-(46) all fall under case 4. Note further that cases 1 and 4 spring from the two associated invariant imbedding cases in stage 2. The two remaining cases yet to be derived spring from the *dual invariant imbedding contexts* of stage 2 and are depicted as cases 2 and 3 in Fig. 7.25. This will be done below.

A final facet of the overview obtained by means of Fig. 7.25 is that we should expect to find somewhere in the forest of functional relations currently under study a set of functional relations for the dual invariant imbedding operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$ analogous in all respects to those for the invariant imbedding operators $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{M}$ obtained in Sec. 7.5. The dual invariant imbedding operators encountered during the original versions of the derivations above appear to exist as intimate neighbors of the original operators and we should expect every property of the invariant imbedding operators $\mathcal{R}$ and $\mathcal{T}$ to have some 'dual' property in the other camp made up of the operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$. Some of these relations for the dual operators will be derived subsequently.

These observations permit us to see the incipient patterns of similarity, in the functional relations at all stages of construction, forming during the invariant imbedding process on X(a,b), and also permit us to become aware of the possibility of a systematic method of derivation of the various integral representations of $\Psi(s,y:a,b)$. In the remaining space of this section we shall round out the family of integral and differential representations obtained so far by working on cases 2 and 3 in the third growth stage of Fig. 7.25. Furthermore, we shall look briefly into the matter of the functional relations for the dual invariant imbedding operators. Finally, we shall be able to make a thorough critique of the equations (35), (36) of Sec. 6.7, the equations which inspired the research leading to the results of the present section.

For a systematic imbedding procedure developed in complete detail in the discrete space context analogous to that depicted in Fig. 7.25, the reader may consult the Categorical Analysis method, Chapter X, Ref. [251].

Dual Integral Representations of the Global Ψ-operators

We now derive the dual integral representations to (43)-(46). Recall from our discussion of Fig. 7.25 that (43)-(46) are covered by case 4 in stage 3 of the growth pattern of X(a,b). The dual case to this is case 2. Therefore we are to consider the situation where the source level is below the observation level, i.e., we have $a \le y < s \le b$. Equations (31)-(34) of Sec. 3.9 are therefore called up for use. Consider the component $\Psi_{++}(s,y:a,b)$. During the third stage of growth we have for case 2, with the help of (31) of Sec. 3.9:

$$-\frac{\partial \Psi_{++}(s,y:t,b)}{\partial t} = -\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t}\,\mathcal{T}(s,y,t) - [I + \Psi(s,s:t,b)]\,\frac{\partial \mathcal{T}(s,y,t)}{\partial t} .$$

The derivative in the first term is given by (17); the derivative in the second term is given by (18) of Sec. 7.5. The resultant rate of change equation is:

$$\begin{aligned}-\frac{\partial \Psi_{++}(s,y:t,b)}{\partial t} &= \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)\,\mathcal{T}(s,y,t) + \\ &\quad + [I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_{+}(t)\mathcal{R}(t,y,s) \\ &= \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\,[\mathcal{R}(t,s,b)\,\mathcal{T}(s,y,t)+\mathcal{R}(t,y,s)]\end{aligned}$$

The latter equation follows by use of (27). The final step uses (69) of Sec. 7.4, and we have the desired result:

— a
— t
— y
— s
— b

$$-\frac{\partial \Psi_{++}(s,y:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b) \qquad (47)$$

$$a \leq t \leq y < s \leq b$$

Observe that (47) is identical to (17) in all respects save one: (47) has "y" in place of "s" in Ψ_{++} *and* in $\mathcal{R}$. This shows that the differential equation governing $\Psi_{++}(s,y:t,b)$ as a function of t holds for all y, $t \leq y \leq b$. This fact also holds for the differential equations governing the remaining three components of $\Psi(s,y:t,b)$. The requisite integral representation now follows from (47) by integrating from a to y and using the fact that:

$$\Psi_{++}(s,y:y,b) = \mathcal{T}^{\dagger}(s,y,b) \quad .$$

The result is:

—a
—y
—s
—b

$$\Psi_{++}(s,y:a,b) = \mathcal{T}^{\dagger}(s,y,b) + \int_{a}^{y} \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b)\,dt$$

$$a \leq y < s \leq b \qquad (48)$$

This equation is the dual to (43). The dual to (44) is based on the differential equation:

— a
— t
— y
— s
— b

$$-\frac{\partial \Psi_{+-}(s,y:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b) \qquad (49)$$

$$a \leq t \leq y < s \leq b$$

which is derived analogously to (47), using (32) of Sec. 3.9. The corresponding integral representation is:

— a
— y
— s
— b

$$\Psi_{+-}(s,y:a,b) = \int_{a}^{y} \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\,\mathcal{T}(t,y,b)\,dt \qquad (50)$$

$$a \leq y < s \leq b$$

in which we have used the fact that:

$$\Psi_{+-}(s,y:y,b) = 0 \quad .$$

The dual representation to (45) is based on the differential equation:

— a
— t
— y
— s
— b

$$-\frac{\partial \Psi_{-+}(s,y:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b) \qquad a \le t \le y < s \le b \tag{51}$$

The corresponding integral representation is:

— a
— y
— s
— b

$$\Psi_{-+}(s,y:a,b) = \mathcal{R}^{\dagger}(s,y,b) + \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b)\, dt \qquad a \le y < s \le b \tag{52}$$

using the fact that:

$$\Psi_{-+}(s,y:y,b) = \mathcal{R}^{\dagger}(s,y,b) \quad .$$

Finally, the dual to (46) is based on:

— a
— t
— y
— s
— b

$$-\frac{\partial \Psi_{--}(s,y:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b) \qquad a \le t \le y < s \le b \tag{53}$$

Whence:

— a
— y
— s
— b

$$\Psi_{--}(s,y:a,b) = \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b)\, dt \qquad a \le y < s \le b \tag{54}$$

in which was used the fact that:

$$\Psi_{--}(s,y:y,b) = 0 \quad .$$

It remains to derive four pairs of differential-integral representations of the components of $\Psi(s,y:a,b)$ for case 3 of stage 3, as depicted in Fig. 7.25. However, the details will be left as an instructive exercise for the interested student. The results should be dual to (35)-(38) (which is case 1 of stage 3) in the same general way that the preceding integral relations were dual to (43)-(46). The derivations must be carried out for the case $s < y$, so that one begins with (39)-(42). It should be observed that the sets (43)-(46) and their duals just derived are sufficient to completely describe the internal-source problem within X(a,b). Hence the derivations of the dual relations to (35)-(38) is an academic matter. Nevertheless a full understanding of the present method of derivation of the integral representations of $\Psi(s,y:a,b)$ is contingent on a complete list of the dual relations; for this reason they are appended below:

From:

$$\frac{\partial \Psi_{--}(s,y:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a) \quad , \qquad (55)$$

we have the dual to (35):

$$\Psi_{--}(s,y:a,b) = \mathcal{T}^{\dagger}(s,y,a) + \int_{y}^{b} \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a)\, dt \quad . \qquad (56)$$

From:

$$\frac{\partial \Psi_{-+}(s,y:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a) \quad , \qquad (57)$$

we have the dual to (36):

$$\Psi_{-+}(s,y:a,b) = \int_{y}^{b} \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a)\, dt \quad . \qquad (58)$$

From:

$$\frac{\partial \Psi_{+-}(s,y:a,t)}{\partial t} = \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a) \quad , \qquad (59)$$

we have the dual to (37):

$$\Psi_{+-}(s,y:a,b) = \mathcal{R}^{\dagger}(s,y,a) + \int_{y}^{b} \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a)\, dt \quad . \qquad (60)$$

From:

$$\frac{\partial \Psi_{++}(s,y:a,t)}{\partial t} = \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a) \quad , \qquad (61)$$

we have the dual to (38):

$$\Psi_{++}(s,y:a,b) = \int_{y}^{b} \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a)\, dt \quad . \qquad (62)$$

All these preceding equations (55)-(62) are valid for $a \le s < y \le t \le b$, and so may be envisioned by means of the ideograph:

— a
— s
— y
— t
— b

Logical Descendents of $\Psi(s,y:a,b)$

In our survey of the dynamics of the internal-source problem, as depicted in Fig. 7.25, we encountered during the building up of the medium X(a,b) through three stages, all the various interaction operators ranging from the standard operator for a slab (stage 1) through the invariant imbedding operators and their duals (stage 2), and culminating finally with the Ψ-operators. It follows that, given a general Ψ-operator, all these special interaction operators should be recoverable from $\Psi(s,y:a,b)$ under suitable choice of the parameters s,y,a,b. Various special cases were already encountered in the preceding work of this section. We now list these special instances of Ψ for convenient reference. One immediate use of the list is to reexamine the preceding representations and see how the Ψ-operator is built up from its most rudimentary special cases--an operation which on first view is reminiscent of pulling one's self up by one's bootstraps.

The setting for the present discussion is a general one-parameter space X(a,b) with an arbitrary source on level s, $a \le s \le b$. We shall consider two cases: first the case summarized as (s=a or s=b), and then the case summarized as (a<s<b).

The invariant imbedding operators are, by their definitions, concerned with media X(a,b) which are source free. To simulate this, we set s = a or s = b in $\Psi(s,y:a,b)$. If we use (31)-(34) of Sec. 3.9 or (39)-(42) above, then the results are:

$$\Psi(a,y:a,b) = \begin{bmatrix} \Psi_{++}(a,y:a,b) & \Psi_{+-}(a,y:a,b) \\ \Psi_{-+}(a,y:a,b) & \Psi_{--}(a,y:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \mathcal{R}(a,y,b) & \mathcal{T}(a,y,b) \end{bmatrix} \tag{63}$$

$$\Psi(b,y:a,b) = \begin{bmatrix} \Psi_{++}(b,y:a,b) & \Psi_{+-}(b,y:a,b) \\ \Psi_{-+}(b,y:a,b) & \Psi_{--}(b,y:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{T}(b,y,a) & \mathcal{R}(b,y,a) \\ 0 & 0 \end{bmatrix}$$

Setting y = a in (63), we have:*

$$\Psi(a,a:a,b) = \begin{bmatrix} \Psi_{++}(a,a:a,b) & \Psi_{+-}(a,a:a,b) \\ \Psi_{-+}(a,a:a,b) & \Psi_{--}(a,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R(a,b) & 0 \end{bmatrix} . \tag{65}$$

Further y = a in (64) yields:

$$\Psi(b,a:a,b) = \begin{bmatrix} \Psi_{++}(b,a:a,b) & \Psi_{+-}(b,a:a,b) \\ \Psi_{-+}(b,a:a,b) & \Psi_{--}(b,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} T(b,a) & 0 \\ 0 & 0 \end{bmatrix} . \tag{66}$$

Setting y = b in (63):

$$\Psi(a,b:a,b) = \begin{bmatrix} \Psi_{++}(a,b:a,b) & \Psi_{+-}(a,b:a,b) \\ \Psi_{-+}(a,b:a,b) & \Psi_{--}(a,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & T(a,b) \end{bmatrix} . \tag{67}$$

Setting y = b in (64):*

$$\Psi(b,b:a,b) = \begin{bmatrix} \Psi_{++}(b,b:a,b) & \Psi_{+-}(b,b:a,b) \\ \Psi_{-+}(b,b:a,b) & \Psi_{--}(b,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & R(b,a) \\ 0 & 0 \end{bmatrix} . \tag{68}$$

A dual collection follows from $\Psi(s,a:a,b)$ by letting the source level s be inside X(a,b) (i.e., $a<s<b$) and limiting the response level y to be a or b. Thus we have:

---

*The operators Ψ_{++} and Ψ_{--} in these cases by convention (Sec. 3.9) must be interpreted as local Ψ-operators; hence the presence of the zero entries where formally one would have expected identity entries (cf. (20), (23) of Sec. 3.9).

$$\Psi(s,a:a,b) = \begin{bmatrix} \Psi_{++}(s,a:a,b) & \Psi_{+-}(s,a:a,b) \\ \Psi_{-+}(s,a:a,b) & \Psi_{--}(s,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{T}^{\dagger}(s,a,b) & 0 \\ \mathcal{R}^{\dagger}(s,a,b) & 0 \end{bmatrix} \quad . \tag{69}$$

$$\Psi(s,b:a,b) = \begin{bmatrix} \Psi_{++}(s,b:a,b) & \Psi_{+-}(s,b:a,b) \\ \Psi_{-+}(s,b:a,b) & \Psi_{--}(s,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathcal{R}^{\dagger}(s,b,a) \\ 0 & \mathcal{T}^{\dagger}(s,b,a) \end{bmatrix} \quad . \tag{70}$$

The remaining four possibilities dual to (65)-(68) cannot be obtained directly from (69) and (70) by letting $s = a$ or $s = b$.

However, it is easy to see directly from the definitions (26)-(28) that:

$$\mathcal{T}^{\dagger}(a,a,b) = I \tag{71}$$

$$\mathcal{R}^{\dagger}(a,a,b) = R(a,b) \tag{72}$$

$$\mathcal{T}^{\dagger}(b,a,b) = T(b,a) \tag{73}$$

$$\mathcal{R}^{\dagger}(b,a,b) = 0 \tag{74}$$

$$\mathcal{R}^{\dagger}(a,b,a) = 0 \tag{75}$$

$$\mathcal{T}^{\dagger}(a,b,a) = T(a,b) \tag{76}$$

$$\mathcal{R}^{\dagger}(b,b,a) = R(b,a) \tag{77}$$

$$\mathcal{T}^{\dagger}(b,b,a) = I \tag{78}$$

Differential Equations for the Dual Operators

Our analysis of the three stages of the invariant imbedding process as depicted in Fig. 7.25 showed that certain dual operators to the invariant imbedding operators arose in stage two. The physical properties displayed by these dual operators throughout the discussions of this section indicated that we should expect the duality to be a thorough-going one--one that went deeper than their defining forms (26)-(28) or their physical interpretations pictorially summarized in Fig. 7.25. We now embark on a verification of that expectation by deriving the differential equations for the $\mathcal{R}^{\dagger}$ and $\mathcal{T}^{\dagger}$ operators arising in case 3 of stage 2 of the imbedding process. Case 2 of that stage, which proceeds analogously, will be left as an exercise for the reader who wishes to fix the method of derivation firmly in mind. The net result of the present activity will be a set of differential equations for the operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$ which parallel in all essential respects those of the kind exemplified by (8)-(21) of Sec. 7.5 for the invariant imbedding operators and which can be used in practical numerical computations leading to approximate matricial forms of $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$. The model for the reduction of the ensuing differential equations to matricial forms, amenable to numerical computations, was developed in Sec. 7.7.

In view of the introductory remarks, we fix attention on case 3 of stage 2, depicted in Fig. 7.25. In that case the slab $X(s,y)$ is allowed to grow upward or, in the terminology of invariant imbedding theory, we imbed $X(s,y)$ in the family $\{X(t,y): a \le t \le s\}$ of spaces, and then consider the rates of change of $\mathcal{T}^\dagger(s,t,y)$, and $\mathcal{R}^\dagger(s,t,y)$. From (27), on setting $b = y$, $a = t$, and on differentiating with respect to t, we have:

$$-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} = -\frac{\partial \Psi_{++}(s,s:t,y)}{\partial t} T(s,t) - -[I + \Psi_{++}(s,s:t,y)] \frac{T(s,t)}{\partial t} .$$

Using (17) (in which we set $b = y$) and using (27) of Sec. 7.1, suitably adapted to the present setting, we have:

$$\begin{aligned}-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} &= \mathcal{T}^\dagger(s,t,y)\rho_+(t)\mathcal{R}(t,s,y)T(s,t) + \\ &+[I+\Psi_{++}(s,s:t,y)]T(s,t)[\tau_+(t)+\rho_+(t)R(t,s)] \\ &= \mathcal{T}^\dagger(s,t,y)[\rho_+(t)\mathcal{R}(t,s,y)T(s,t)+\rho_+(t)R(t,s)+\tau_+(t)]\end{aligned}$$

Using (69) of Sec. 7.4 (in which we set $a = y = t$, $z = s$, $b = y$) we have:

$$R(t,y) = R(t,s) + \mathcal{R}(t,s,y)T(s,t) .$$

In this way we arrive at:

— a — t — s — y — b	$$-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} = \mathcal{T}^\dagger(s,t,y)[\tau_+(t) + \rho_+(t)R(t,y)] \qquad (79)$$ $a \le t \le s \le y \le b$

The initial condition for $\mathcal{T}^\dagger(s,t,y)$ is (according to (71)):

$$\mathcal{T}^\dagger(s,s,y) = I . \qquad (80)$$

Thus, when integrating (79), we start with $t = s$. The value of the derivative at that initial point is known, being:

$$-\frac{\partial \mathcal{T}^{\dagger}(s,s,y)}{\partial t} = [\tau_{+}(s) + \rho_{+}(s)R(s,y)] \quad ,$$

where R(s,y) is known from the constructions of stage 1, depicted in Fig. 7.25.

Next, we return to (25), set b = y, a = t, and differentiate with respect to t:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} = -\frac{\partial \Psi_{-+}(s,s{:}t,y)}{\partial t}\, T(s,t) - \Psi_{-+}(s,s{:}t,y)\,\frac{\partial T(s,t)}{\partial t}$$

Using (32), we can find the derivative of $\Psi_{-+}(s,s{:}a,t)$ using the fundamental theorem of calculus (now applied to operator integrals). The remaining derivative is the same as before. The net result is:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} =$$

$$= \mathcal{R}^{\dagger}(s,t,y)\rho_{+}(t)\mathcal{R}(t,s,y)T(s,t) +$$

$$+ \Psi_{-+}(s,s{:}t,y)T(s,t)[\tau_{+}(t) + \rho_{+}(t)R(t,s)]$$

$$= \mathcal{R}^{\dagger}(s,t,y)[\rho_{+}(t)\mathcal{R}(t,s,y)T(s,t) + \rho_{+}(t)R(t,s) + \tau_{+}(t)]$$

Whence:

— a
— t
— s
— y
— b

$$-\frac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} = \mathcal{R}^{\dagger}(s,t,y)[\tau_{+}(t) + \rho_{+}(t)R(t,y)] \qquad a \le t \le s \le y \le b \quad . \quad (81)$$

The initial condition for $\mathcal{R}^{\dagger}(s,t,y)$ is (according to (72)):

$$\mathcal{R}^{\dagger}(s,s,y) = R(s,y) \quad .$$

Thus when integrating (81), we start with t = s. The value of the derivative at the initial point is known, being:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,s,y)}{\partial s} = R(s,y)[\tau_{+}(s) + \rho_{+}(s)R(s,y)]$$

where R(s,y) is known from the constructions of stage 1, depicted in Fig. 7.25.

Together, (79) and (81) determine the dual invariant imbedding operators for case 3, stage 2 of the imbedding process on X(a,b). Yet the complete set of equations for that case requires four equations, as a perusal of the originals (18)-(21) of Sec. 7.5 would indicate. Evidently (81) and (79) are, respectively, the dual counterparts to (20) and (21) of Sec. 7.5. Recall that the operators $\rho(t)$ and $\tau(t)$ in Sec. 7.5

were constructed in an isotropic medium, so that subscript signatures were not needed there, since $\rho_+(t) = \rho_-(t)$, and $\tau_+(t) = \tau_-(t)$. In the present setting isotropy was not assumed, principally to point up the interesting and thoroughgoing dualities under study in this section. For, without isotropy, we have generally distinct pairs of local operators $\rho_+(t)$, $\rho_-(t)$ and $\tau_+(t)$, $\tau_-(t)$. The subscripts on the local operators also help one to more readily write and read equations associated with upward (+) and downward (-) radiant fluxes. The remaining two equations for case 3, stage 2 are obtained from (26) and (28) on setting a = t, b = y and differentiating with respect to t:

$$-\frac{\partial \mathcal{R}^\dagger(s,y,t)}{\partial t} = -\frac{\partial \Psi_{+-}(s,s:t,y)}{\partial t}\, T(s,y)$$

$$= \mathcal{T}^\dagger(s,t,y)\rho_+(t)\,\mathcal{T}(t,s,y)T(s,y)$$

whence:

— a
— t
— s
— y
— b

$$-\frac{\partial \mathcal{R}^\dagger(s,y,t)}{\partial t} = \mathcal{T}^\dagger(s,t,y)\rho_+(t)T(t,y) \qquad a \le t \le s \le y \le b \qquad (82)$$

Similarly:

— a
— t
— s
— y
— b

$$-\frac{\partial \mathcal{T}^\dagger(s,y,t)}{\partial t} = \mathcal{R}^\dagger(s,t,y)\rho_+(t)T(t,y) \qquad a \le t \le s \le y \le b \qquad (83)$$

These equations are subordinate to (79) and (81) in the sense that they are powerless to support computations for their respective operators. Equations (79) and (81) are the *autonomous* equations for the present case. Once these are solved, (82) and (83) may be used to find $\mathcal{R}^\dagger(s,y,t)$ and $\mathcal{T}^\dagger(s,y,t)$.

A Colligation of the Component Ψ-operator Equations

We now reach one of the goals of this section by means of a study of equations (35), (36) of Sec. 6.7, as derived by Elliott in the neutron transport context in Ref. [88]. Specifically, our goal is to place the equations into their proper perspective within the domain of invariant imbedding techniques, to see their domain of validity, and to indicate their proper generalizations. In order to do this efficiently we must bind together the relatively massive collection of integral representations for the components of the global Ψ-operator obtained so far. To this task we now turn.

First of all we observe that equations (35), (36) of Sec. 6.7 are associated with two cases of location of the source level c relative to the observation level z. We can see these two cases in perspective by means of Fig. 7.25. Thus we find that the proper setting of Elliott's equations is in Stage 3 of the invariant imbedding process for the semi-infinite medium $X(0,\infty)$ ($a = 0$, $b = \infty$ in $X(a,b)$). In particular cases 2 and 3 correspond, respectively, to the source level below the observation level ($y<s$) and the source level above the observation level ($s<y$).

In case 2 the medium $X(y,b)$ with internal source at level s, $y<s\le b$ is imbedded in the family $\{X(t,b): a\le t\le y\}$ of spaces. The source at level s is described by the incident source radiance distribution $N^o(s)$ which has an upward component $N_+^o(s)$ and a downward component $N_-^o(s)$ defined at each point of the parameter surface X_s. We are interested in the response radiance distribution $N(y)$ over X_y, where $N(y) = (N_+(y),N_-(y))$. By (15) of Sec. 3.9 we have:

$$N(y) = N^o(s)\Psi(s,y:a,b) \tag{84}$$

where now, by (48), (50), (52), and (54), $\Psi(s,y:a,b)$ can be given a specific representation in terms of integral operations on the invariant imbedding operators and their duals:

$$\Psi(s,y:a,b) = \begin{bmatrix} \mathcal{T}^\dagger(s,y,b) + \int_a^y \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \int_a^y \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \\ \mathcal{R}^\dagger(s,y,b) + \int_a^y \mathcal{R}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \int_a^y \mathcal{R}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \end{bmatrix}$$

This formidable structure reduces to a rather simple and intuitively interesting integral expression by means of (63) and (69). With these two equations as a base, it is easy to cast the preceding matrix into the form:

—a
—y
—s
—b

$$\boxed{\Psi(s,y:a,b) = \Psi(s,y:y,b) + \int_a^y \Psi(s,t:t,b)\,\mathcal{K}(t)\,\Psi(t,y:t,b)\,dt} \tag{85}$$

$a\le y<s\le b$ case 2, stage 3, Fig. 7.25

Here $\mathcal{K}(t)$ is the local interaction operator defined in (7) of Sec. 7.1. Equation (85) is the desired generalization of (35) of Sec. 6.7. We can pair off the corresponding functions in (35) of Sec. 6.7 and (85) as follows: $\Psi(s,y:0,\infty)$ pairs off with $f_s(y,\omega)$ (replacing "c" by "s" and "z" by "y" in (35) of Sec. 6.7). Further, $\Psi(s,y:y,\infty)$ pairs off with $f_o(s-y;\omega)$. This pairing is understood more clearly by noting at this point a certain reciprocity property of $f_o(z,\omega)$ proved in Ref. [88]. This property is the following:

$$f_o(z,\omega) = f_z(0,\omega)$$

for every depth z in $X(0,\infty)$.

In words, this states that the Fourier transform of the scalar irradiance field at depth 0 produced by a point source at depth s (represented by "$f_o(s,\omega)$") is equal to that at depth s produced by a point source at depth 0 (represented by "$f_s(0,\omega)$"). In this way we see how $\Psi(s,y:y,\infty)$ pairs off with $f_o(s-y,\omega)$ (= $f_{s-y}(0,\omega)$). Further, the operator $\Psi(s,t:t,\infty)$ pairs off with $f_o(s-t,\omega)$, and $\Psi(t,y:t,\infty)$ pairs off with $f_o((y-s)+(s-t),\omega) = f_o(y-t,\omega)$. Finally, $\mathcal{X}(t)$ pairs off with $s/h\eta$. The pairings between the terms of (85), and (35) of Sec. 6.7, of course, cannot be exact, for the obvious reason that (35) of Sec. 6.7 is a vastly simpler equation than (85). Yet, remarkably, the *general forms* of the two equations are identical and this is an attestation of the *invariant* nature of the semigroup formulation of transport processes with respect to both the changes in the superficial geometric structure of the media within which they evolve, and also with respect to the type of radiometric concept used in the formulation.

Having dispatched case 2, we now consider case 4 in stage 3 of Fig. 7.25, according to our present purposes. Equations (43)-(46) may be used to obtain the following matricial form of $\Psi(s,y:a,b)$, s<y:

$$\Psi(s,y:a,b) =$$

$$= \begin{bmatrix} \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \\ \mathcal{R}(s,y,b) + \int_a^s \mathcal{R}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \mathcal{T}(s,y,b) + \int_a^s \mathcal{R}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \end{bmatrix}$$

By means of (63) and (69) we can reduce this to the intuitively meaningful form:

— a
— s
— y
— b

$$\Psi(s,y:a,b) = \Psi(s,y:s,b) + \int_a^s \Psi(s,t:t,b)\,\mathcal{X}(t)\,\Psi(t,y:t,b)\,dt$$

$a \le s < y \le b$ Case 4, stage 3, Fig. 7.25

(86)

Equation (86) is the general correspondent to (36) of Sec. 6.7. We have now reached the first of our main goals of this section, in the form of (85) and (86), and we pause to make some observations on their structure and further observations on their physical interpretations.

The first things to observe about (85) and (86) are some of their common features: they both use the same integrand in their integral terms; however, the limits differ and this difference reflects the two cases of source-observation levels in X(a,b). Both equations yield an expression for $\Psi(s,y:a,b)$ using invariant imbedding operators and their duals (cf., (63), (69)). These operators are found in stage 2 of the invariant imbedding process on X(a,b). Hence (85) and (86), in the framework of Fig. 7.25, are simply instructions on how to construct the operators of stage 3 from those of stage 2.

Asymmetries of the Ψ-operator

Next we observe a most interesting dissimilarity between (85) and (86). This occurs in the form of the term added to the integral. In (86) the added term is made up of the invariant imbedding operators $\mathscr{R}(s,y,b)$ and $\mathscr{T}(s,y,b)$. In (85) the added term is made up of the *dual* invariant imbedding operators $\mathscr{R}^\dagger(s,y,b)$, $\mathscr{T}^\dagger(s,y,b)$. The diagrammatic representations of X(a,b) next to the equations helps to immediately recall the physical interpretations of these various operators. Now, in the scalar irradiance context, one form of counterpart to $\Psi(s,y:s,b)$ is $f_0(y-s,\omega)$ and that of $\Psi(s,y:y,b)$ is, as we have seen, $f_{s-y}(0,\omega)$. Elliott has shown, in a homogeneous isotropic half space with scalar irradiance as the radiometric quantity described by $f_s(y,\omega)$, that the reciprocity relation:

$$f_0(y-s,\omega) = f_{y-s}(0,\omega) \qquad (87)$$

holds when $y>s$, say. This being so, our attention turns immediately to $\Psi(s,y:s,b)$ and $\Psi(s,y:y,b)$ and we ask: under what conditions on the geometry of X(a,b) and its inherent optical properties do we have:

$$\Psi(s,y:s,b) = \Psi(y,s:s,b) \qquad (88)$$

— s
— y
— b

as a valid equation?

The diagrammatic insert under the equation shows the geometric context in which the question is asked. It should be observed that this is equivalent to the equation:

$$\Psi(s,y:s,b) = \Psi(s,y:y,b) \qquad (89)$$

— s   — y
— y   — s
— b   — b

in the geometric context where names of the levels s and y are interchanged in the same medium as shown by the diagrams below (89).

To see the conditions under which (88) holds, it is sufficient to examine each of the four operator equations within (88). Thus, consider, for example, the equation arising from the "++" components of (88):

$$\Psi_{++}(s,y:s,b) = \Psi_{++}(y,s:s,b) \quad . \tag{90}$$

From this we see at once that (88) cannot generally hold on the operator level since the left side is always zero, while the right is generally not zero. This establishes the fact that the simple scalar condition (87) has no *exact* counterpart in the general operational transport formulations we are now considering. However, we still may inquire as to the other pairs of components in (88). Those pairs that are not zero--are they ever equal? Or: are the *sums* of the components of the left side equal to the *sums* of the components on the right side of (88)? The latter question is prompted by energy conservation considerations. The latter question will be considered subsequently in Chapter 8 in a setting where the question makes physical sense (Example 10, Sec. 8.7). For the present we examine the former question out of simple curiosity.

The diagram below (88) suggests that if we are to find a corresponding pair of nonzero components in (88), it would be those with the signature "-+". (Cf. (63), (69).) Consider then, for possible validity, the statement:

$$\Psi_{-+}(s,y:s,b) = \Psi_{-+}(y,s:s,b)$$

which is equivalent to:

$$\mathcal{R}(s,y,b) = \mathcal{R}^{\dagger}(y,s,b) \quad .$$

By (1) and (25) this is equivalent to:

$$R(y,b)[I-R(y,s)R(y,b)]^{-1}T(y,s) = T(s,y)[I-R(y,b)R(y,s)]^{-1}R(y,b)$$

which in turn is equivalent to:

$$[I-R(y,s)R(y,b)]T(s,y)R(y,b) = R(y,b)T(y,s)[I-R(y,s)R(y,b)] \quad .$$

For this to be valid, it is sufficient to have commutation freely possible between R(y,s),R(y,b) and T(s,y), T(y,s) along with

$$T(s,y) = T(y,s) \tag{91}$$

and, among other things:

$$R(y,s)R(y,b)T(s,y) = T(y,s)R(y,s)R(y,b) \quad . \tag{92}$$

At this point, our studies of Sec. 7.12 may be used to help clear the air of present question. The polarity theorem asserts that a *plane-parallel medium* X(a,b) must be isotropic and separable in order that (91) hold. This is not too stringent a requirement on the medium and its inherent optical properties. However, if X(a,b) is not plane-parallel, it is generally the case that (91) no longer holds, no matter how regular its inherent optical properties. That commutativity and condition (92) are also to hold--i.e., to have a reciprocity condition--is hopeless in general. One exception occurs in the scalar context, i.e., when the R(y,b), R(y,s) and T(s,y) are real valued functions of s, y, b and not matrices or integral operators (as in the present discussion).

In this way we see that (35) and (36) of Sec. 6.7 cannot be directly generalized to the operator level without loss of the rather special reciprocity condition (87). This is a small loss in view of the fact that (85) and (86) are capable, as they stand, of solving in principle the most general point source problems on continuous one-parameter optical media. Their complementary counterparts associated with cases 1 and 3 in stage 3 of Fig. 7.25 are also capable of performing this service. The derivation of the associated equations are left to the reader as an important exercise (cf., (108)-(111) below.

A Royal Road to the Internal-Source Functional Relations

It was perhaps somewhat of an anticlimax for the attentive reader to see the four operator equations of case 2, stage 3 (in Fig. 7.25), so hard-won through the early portions of this section, unceremoniously collapsed into the simple operator equation (85). Still another such revelation may have occurred when (86) was reached. Be that as it may, the relative simplicity of (85) and (86), compared with the system of their progenitors, attests to the correctness of the deductions and to the power of the invariant imbedding approach which gave us the general Ψ-operator concept. But yet the very simplicity of these results invites an attempt of a correspondingly simple derivation of (85) and (86). We shall now indicate the outlines of such a derivation. We shall be very careful not to add all the rigorous details or else we shall simply retrace the work of this section. Thus we shall embark on a 'royal road' to (85) and (86), in the sense that it is ostensibly well-paved with no long steep grades, and along which the analytic and algebraic pitfalls have been filled and smoothed with rhetoric.

We choose as a setting case 2 of stage 3 in Fig. 7.25. The present derivation begins with a partition of X(a,b) by the internal surface X_t, $a \le t \le y$. The only source on or in X(a,b) is at level s, $y < s \le b$; and this is of an arbitrary nature. Having partitioned X(a,b) into two parts X(a,t), X(t,b), we isolate X(t,b) and consider all sources incident on it. There are two incident sources on X(t,b): the hypothesized internal source at level s, and the externally incident flux at level t coming from X(a,t), and which is part of the integral light field set up throughout X(a,b) by the source at level s. Therefore, by the interaction principle (cf. (38), Sec. 3.9), we have two operators Ψ(s,y:t,b) and Ψ(t,y:t,b) associated with X(t,b) such that the response field N(y) at level y in X(t,b) is given by:

$$N(y) = N^o(s)\Psi(s,y:t,b) + N(t)\Psi(t,y:t,b) \quad . \qquad (93)$$

A detailed analysis by means of the principles of invariance on X(a,b) shows that the radiance distribution over X_t is the result of two activities: the overall transmission of the effects of $N^o(s)$ within X(t,b) up to X_t *and* the response at level t of the total interaction of this transmitted flux as it oscillates between X(a,b) and X(t,b). For the first of these we have:

$$N^o(s)\Psi(s,t:t,b) \quad ,$$

and accounting for the second of these we have:

$$N(t) = [N^o(s)\Psi(s,t:t,b)]\Psi(t,t:a,b) \quad . \tag{94}$$

Using this in (93) we see that:

$$N(y) = N^o(s)[\Psi(s,y:t,b) + \Psi(s,t:t,b)\Psi(t,t:a,b)\Psi(t,y:t,b)] \tag{95}$$

Since we have also:

$$N(y) = N^o(s)\Psi(s,y:a,b) \quad ,$$

the conclusion is:

$$\boxed{\begin{gathered}\Psi(s,y:a,b) = \Psi(s,y:t,b) + \Psi(s,t:t,b)\Psi(t,t:a,b)\Psi(t,y:t,b)\\ a \le t \le y, \ t \le s \le b\end{gathered}} \tag{96}$$

Equation (96) is the desired functional relation for case 2. An examination of its derivation shows that it is actually quite general, holding also for case 4 and within an arbitrary one-parameter space X(a,b) with a source at level s. Equation (96) is a *finite algebraic counterpart* to (85) and (86). Observe in particular how Ψ(s,y:a,b) can be calculated from knowledge of Ψ(s,y:t,b) (the operator for a smaller medium X(t,b) within X(a,b)) and the invariant imbedding operators Ψ(s,t:t,b), Ψ(t,y:t,b), and the local Ψ-operator Ψ(t,t:a,b).

The next step is to use (96) to form the difference quotient:

$$\frac{\Psi(s,y:a,b) - \Psi(s,y:t,b)}{t-a} = \Psi(s,t:t,b)\,\frac{\Psi(t,t:a,b)}{t-a}\,\Psi(t,y:t,b)$$

and go to the limit as $t \to a$. The left side becomes $-\partial\Psi(s,y:a,b)/\partial a$; the right side can be reduced with the aid of (63), (69) and the set (20)-(23) of Sec. 3.9. The result is:

$$\boxed{-\frac{\partial\Psi(s,y:a,b)}{\partial a} = \Psi(s,a:a,b)\mathcal{K}(a)\Psi(a,y:a,b)} \tag{97}$$

where $\mathcal{X}(a)$ is defined in (7) of Sec. 7.1. The reader may also find (21) of Sec. 7.3 helpful in the verification of (97).

Equation (85) is now readily forthcoming from (97): in (97) replace "a" by "t" and integrate over all t from a to y. To obtain (86), integrate the modified (97) from a to s. Once the integral expressions have been obtained, they may be dissected to release their associated quartets of operator equations. In this way eight of the earlier integral expressions are obtained--namely those for cases 2 and 4 in stage 3 of Fig. 7.25.

The preceding derivation has demonstrated the rich analytic harvest that can be yielded up by (97). However, the potentialities of (97) have by no means been exhausted. Suppose we return to Fig. 7.25 and now move over to stage 2 in the invariant imbedding process for X(a,b). Equation (97) will now be associated with cases 2 and 4 in stage 2. For example, in case 2 we have a = y and we are therefore led to consider derivatives of the form:

$$\frac{\partial \Psi(s,y:y,b)}{\partial y} \quad .$$

It will be helpful to note that $\Psi(s,y:y,b)C_+ = \Psi(s,y:y,b)$. By the rules of elementary calculus, the preceding derivative is to be interpreted as:

$$\lim_{a\to y} \frac{\partial \Psi(s,a:y,b)}{\partial a} C_+ + \lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial a} C_+ \quad .$$

The matrix C_+ serves to keep the second columns of the two matrices zero, as required. This type of derivative operation has been considered earlier in connection with the standard reflectance and transmittance operators (cf. (23), (24) of Sec. 7.5). The first of these derivatives has been studied earlier ((33) of Sec. 7.5) and we have:

$$\lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial y} = \lim_{a\to y} \Psi(s,y:a,b)\mathcal{X}(a)$$

$$= \Psi(s,y:y,b)\,\mathcal{X}(y) \quad .$$

The second derivative is obtained from (97):

$$\lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial a} = -\Psi(s,y:y,b)\mathcal{X}(y)\Psi(y,y:y,b) \quad .$$

Hence:

$$\boxed{-\frac{\partial \Psi(s,y:y,b)}{\partial y} = \Psi(s,y:y,b)\,\mathcal{X}(y)\left[\Psi(y,y:y,b) - C_+\right]} \qquad (98)$$

by noting that $\Psi(y,y:y,b)C_+ = \Psi(y,y:y,b)$. Equation (98) is the generic differential equation for the invariant imbedding operators in case 2 of stage 2 in Fig. 7.25. These operators were already obtained piecemeal in (79)-(83). In like manner, the operators of stage 1 of Fig. 7.25 may be obtained from (98) (and its complement associated with cases 1 and 3 of stage 2) by suitable confluence of variables (namely $b\to s$). This is left to the reader as an important exercise.

It is of interest to note that (97) and (98) may be cast into a form which explicitly exhibits the invariant imbedding operators. First recall that we have written:

$$\text{"}\mathcal{M}(a,y,b)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathcal{T}(b,y,a) & \mathcal{R}(b,y,a) \\ \mathcal{R}(a,y,b) & \mathcal{T}(a,y,b) \end{bmatrix} . \tag{99}$$

With this and (69) and (70) as guides, we write:

$$\text{"}\mathcal{M}^{\dagger}(s,a,b)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathcal{T}^{\dagger}(s,a,b) & \mathcal{R}^{\dagger}(s,b,a) \\ \mathcal{R}^{\dagger}(s,a,b) & \mathcal{T}^{\dagger}(s,b,a) \end{bmatrix} . \tag{100}$$

Then (97) becomes:

$$\boxed{-\frac{\partial\Psi(s,y:a,b)}{\partial a} = \mathcal{M}^{\dagger}(s,a,b)C_+\,\mathcal{K}(a)C_-\,\mathcal{M}(a,y,b)} \tag{101}$$

where C_+ and C_- are defined in (4), (5) of Sec. 7.4. Turning now to (98) for the purpose of converting it into invariant imbedding operator form, we note that:

$$\Psi(s,y:y,b) + \Psi(s,b:y,b) = \mathcal{M}^{\dagger}(s,y,b) \tag{102}$$

for $y\le s\le b$. This may be checked by recalling (100), (69), and (70). To find the derivative of $\mathcal{M}^{\dagger}(s,y,b)$ with respect to y, we need only find those of $\Psi(s,y:y,b)$ and $\Psi(s,b:y,b)$ with respect to y. Equation (98) gives us one of these; and (97) gives us the other on making the permissible substitutions: $(y\to b)$, $(a\to y)$ in (97). With these observations, we have:

$$-\frac{\partial\mathcal{M}^{\dagger}(s,y,b)}{\partial y} = -\left[\frac{\partial\Psi(s,y:y,b)}{\partial y} + \frac{\partial\Psi(s,b:y,b)}{\partial y}\right]$$

$$+ \Psi(s,y:y,b)\,\mathcal{K}(y)\,[\Psi(y,y:y,b) - C_+]$$

$$+ \Psi(s,y:y,b)\,\mathcal{K}(y)\,[\Psi(y,b:y,b)]$$

$$= \Psi(s,y:y,b)\,\mathcal{K}(y)\,[\Psi(y,y:y,b) + \Psi(y,b:y,b) - C_+]$$

Recall (102) and note that:

$$\mathcal{M}^{\dagger}(s,y,b)C_{+} = \Psi(s,y:y,b) \quad .$$

Then the net result is:

$$-\frac{\partial \mathcal{M}^{\dagger}(s,y,b)}{\partial y} = \mathcal{M}^{\dagger}(s,y,b)C_{+}\mathcal{X}(y)[\mathcal{M}^{\dagger}(y,y,b) - C_{+}] \qquad y \le s \le b \tag{103}$$

If this operator equation is opened up, the four resultant component equations have precisely the forms of (79), (81), (82), (83). The dual to (103) is (11) of Sec. 7.5, as a perusal of cases 1, 2 of stage 2 in Fig. 7.25 would indicate.

It is a relatively simple matter to derive the complementary functional relations to (96) and (97). Toward this end, we partition X(a,b) into pieces X(a,t), X(t,b) such that $a \le s \le t$, $y \le t \le b$. This partition goes with cases 1 and 3 of stage 3 in Fig. 7.25. Using the preceding derivation of (96) as a pattern, the reader may show that:

$$\Psi(s,y:a,b) = \Psi(s,y:a,t) + \Psi(s,t:a,t)\Psi(t,t:a,b)\Psi(t,y:a,t) \qquad a \le s \le t,\ y \le t \le b \tag{104}$$

Forming the difference quotient via (98):

$$\frac{\Psi(s,y:a,b) - \Psi(s,y:a,t)}{b-t} = \Psi(s,t:a,t)\,\frac{\Psi(t,t:a,b)}{b-t}\,\Psi(t,y:a,t)$$

we go to the limit as $t \to b$. The result is:

$$\begin{aligned}\frac{\partial \Psi(s,y:a,b)}{\partial b} &= \Psi(s,b:a,b)\,\mathcal{X}(b)\Psi(b,y:a,b) \\ &= \mathcal{M}^{\dagger}(s,b,a)C_{-}\mathcal{X}(b)C_{+}\mathcal{M}(b,y,a)\end{aligned} \quad . \tag{105}$$

Equation (105) can be used to generate the eight functional equations governing cases 1, 3 of stage 3 in Fig. 7.25. To do so, replace "b" by "t" in (105) and integrate over all t from s to b for case 1, and from y to b for case 3. Two integral expressions will result which are the complements of (85) and (86). These results will be summarized below.

Summary and Prospectus

The problem of internal sources in an arbitrary optical medium X has presented the opportunity to develop in the present section the full strength of the invariant imbedding technique as applied to radiative transfer--or generally, linear transport--phenomena. We shall now summarize this technique.

Let X be an arbitrary optical medium with internal sources. Let "X(a,b)" denote the parametrization of X, i.e., the representation of X as partitioned into a family $\{X_y: a \le y \le b\}$ of surfaces X_y, each indexed by a real number y drawn from a closed interval [a,b] of real numbers. In the case of plane-parallel media the X_y are planes parallel to the boundary planes. In the case of an arbitrary X, any slicing up of X by a one-parameter family of surfaces will do for the present summary (re: (17) of Sec. 7.11). For simplicity of exposition (and without any attendant loss of generality), we let the sources in X be confined to a single surface X_s. For if the sources are several discrete sources or are continuous and are confined to an interval of depths, the resultant light field is obtained by a suitable superposition operation, since the theory is completely linear.

The general invariant imbedding process begins with X_s and imbeds it in a family $\{X_z: y \le z \le s\}$ of surfaces for the *upward case* or in the family $\{X_z: s \le z \le y\}$ for the *downward case*. This is stage 1 of the imbedding process and serves to develop the four standard R and T operators associated with the slabs X(y,s) and X(s,y). The theory of these operators has been the primary concern of various earlier sections of this chapter and in Chapter 3.

Stage 2 of the invariant imbedding process gives rise to the advent of the invariant imbedding operators $\mathcal{R}$, $\mathcal{T}$ and their duals $\mathcal{R}^{\dagger}$, $\mathcal{T}^{\dagger}$. The operators $\mathcal{R}$ and $\mathcal{T}$ have been studied at some length in this chapter and Chapter 3. The dual operators are newcomers to the scene and fulfill the role of completing with $\mathcal{R}$ and $\mathcal{T}$ the full description of Stage 2 of the invariant imbedding process in X(a,b). The theory of the dual operators was shown in the present section to be parallel in all essential respects to that of the original invariant imbedding operators.

Stage 3 culminates the imbedding process and is the setting for the derivation of the functional relations for the global Ψ-operators. There are two generic differential equations which go with Stage 3. The first is (97), repeated here for convenience:

$$-\frac{\partial \Psi(s,y:a,b)}{\partial a} = \Psi(s,a:a,b)\,\mathcal{X}(a)\,\Psi(a,y:a,b) \qquad (106)$$

and which governs cases 2 and 4 of Stage 3. The second equation is (105):

$$\frac{\partial \Psi(s,y:a,b)}{\partial b} = \Psi(s,b:a,b)\,\mathcal{X}(b)\,\Psi(b,y:a,b) \tag{107}$$

and which governs cases 1 and 3 of stage 3.

Equations (106) and (107) are perhaps two of the most productive equations in the invariant imbedding theory of radiative transfer phenomena, in the sense that they yield differential equations for the various R, T, $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{M}$ operators (cf., (63)-(78)). Thus, as was shown at great length above, they hold within themselves the means toward the differential functional relations of all three stages of an invariant imbedding process on X(a,b), and this includes in particular the differential equations for the operators $\mathcal{M}(a,y,b)$ and $\mathcal{M}^{\dagger}(y,a,b)$, and hence the differential equations for the standard R and T operators R(a,b), R(b,a), T(a,b), T(b,a). Their algebraic progenitors are (96) and (104).

Each equation (106) and (107) gives rise, by means of an integration, to a pair of integral representations, depending on whether s<y or y<s. Going down the four cases in stage 3, as depicted in Fig. 7.25, the associated integral representations are:

$$\Psi(s,y:a,b) = \Psi(s,y:a,s) + \int_s^b \Psi(s,t:a,t)\,\mathcal{X}(t)\,\Psi(t,y:a,t)\,dt \qquad \text{From (107) Case 1 } y<s \tag{108}$$

$$\Psi(s,y:a,b) = \Psi(s,y:y,b) + \int_a^y \Psi(s,t:t,b)\,\mathcal{X}(t)\,\Psi(t,y:t,b)\,dt \qquad \text{From (106) Case 2 } y<s \tag{109}$$

$$\Psi(s,y:a,b) = \Psi(s,y:a,y) + \int_y^b \Psi(s,t:a,t)\,\mathcal{X}(t)\,\Psi(t,y:a,t)\,dt \qquad \text{From (107) Case 3 } s<y \tag{110}$$

$$\Psi(s,y:a,b) = \Psi(s,y:s,b) + \int_a^s \Psi(s,t:t,b)\,\mathcal{X}(t)\,\Psi(t,y:t,b)\,dt \qquad \text{From (106) Case 4 } s<y \tag{111}$$

in which $a \le s \le b$, $a \le y \le b$. These four operator equations blossom into the sixteen operator equations scattered here and there throughout this section and which completely describe how to find Ψ(s,y:a,b) in every case of stage 3 using the operators ($\mathcal{R}$, $\mathcal{R}^{\dagger}$; $\mathcal{T}$, $\mathcal{T}^{\dagger}$) which belong to the results of stage 2.

The invariant imbedding process for X(a,b), as depicted in Fig. 7.25, is closely related to the *Categorical Analysis Method* described in detail in Ref. [251]. In that work the geometric setting is a discrete space rather than a continuous space, and the analysis is thereby permitted to descend directly to the point level in the medium. Together, the invariant imbedding process for one-parameter continuous media X(a,b) as summarized in Fig. 7.25, and the Categorical Analysis Method for discrete media X_n as given in Ref. [251] present a potentially complete means of solving all steady state internal source problems in radiative transfer theory. Be that as it may, much work remains yet to be done in exploring the many special cases arising in particular geometries and physical settings. Thus many opportunities for original research lie in the relatively unexplored new territory of radiative transfer theory surveyed in this section.

Final Observations on the Relations Between the Operators $\mathcal{M}(v,x:u,w)$ and $\Psi(s,y:a,b)$

The two most general radiative transfer operators considered in this work are the $\mathcal{M}$-operator $\mathcal{M}(v,x:u,w)$ introduced for the purpose of formulating the generalized invariant imbedding relation (51) of Sec. 3.7, and the internal source operator Ψ in the form $\Psi(s,y:a,b)$, introduced in (15) of Sec. 3.9 for the study of internal sources. The latter operator we have discussed at length in this section; the former operator was discussed in Sec. 3.7 and at length in Secs. 7.4 and 7.5. Here, by way of summary, is the manner in which one may view these operators conceptually and analytically: the $\mathcal{M}$-operator is to be used in source-free settings; the Ψ-operator is to be used in settings with sources. The $\mathcal{M}$-operator is sufficiently fundamental so that Ψ may be characterized in terms of it, as in (20)-(23) and (31)-(34) of Sec. 3.9. On the other hand, the $\mathcal{M}$-operator can also be represented by and built up from special cases of Ψ, as shown in this section (cf., (63)-(78), in particular). An important relation between them is summarized in (56) of Sec. 7.4. The $\mathcal{M}$-operator enjoys deep algebraic and differential properties, as shown in Secs. 7.4, 7.5; the operator Ψ enjoys a sweeping analytical power as shown in this section and summarized in (106)-(111). Therefore each operator, $\mathcal{M}$ or Ψ, is sufficient to carry radiative transfer theory by itself; and each has algebraic and analytic properties worthy of independent mathematical study.

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7.14 Invariant Imbedding and Integral Transform Techniques

In this the final section of the present chapter on invariant imbedding techniques we shall briefly consider one of the more serious types of problems which, if left unchecked, may keep invariant imbedding techniques from reaching their full practical utility. This is the problem of the exploding variable-population. To see what is meant, consider the following observations.

There was a time when the radiative transferist was content with solving the Wiener-Hopf equation which described energy density in a homogeneous source-free infinitely deep medium. (See, e.g., (1) of Sec. 6.7 in which X is one-dimen-