geometric setting is a discrete space rather than a continuous space, and the analysis is thereby permitted to descend directly to the point level in the medium. Together, the invariant imbedding process for one-parameter continuous media X(a,b) as summarized in Fig. 7.25, and the Categorical Analysis Method for discrete media $X_n$ as given in Ref. [251] present a potentially complete means of solving all steady state internal source problems in radiative transfer theory. Be that as it may, much work remains yet to be done in exploring the many special cases arising in particular geometries and physical settings. Thus many opportunities for original research lie in the relatively unexplored new territory of radiative transfer theory surveyed in this section.

### Final Observations on the Relations Between the Operators $\mathcal{M}(v,x:u,w)$ and $\Psi(s,y:a,b)$

The two most general radiative transfer operators considered in this work are the $\mathcal{M}$-operator $\mathcal{M}(v,x:u,w)$ introduced for the purpose of formulating the generalized invariant imbedding relation (51) of Sec. 3.7, and the internal source operator $\Psi$ in the form $\Psi(s,y:a,b)$, introduced in (15) of Sec. 3.9 for the study of internal sources. The latter operator we have discussed at length in this section; the former operator was discussed in Sec. 3.7 and at length in Secs. 7.4 and 7.5. Here, by way of summary, is the manner in which one may view these operators conceptually and analytically: the $\mathcal{M}$-operator is to be used in source-free settings; the $\Psi$-operator is to be used in settings with sources. The $\mathcal{M}$-operator is sufficiently fundamental so that $\Psi$ may be characterized in terms of it, as in (20)-(23) and (31)-(34) of Sec. 3.9. On the other hand, the $\mathcal{M}$-operator can also be represented by and built up from special cases of $\Psi$, as shown in this section (cf., (63)-(78), in particular). An important relation between them is summarized in (56) of Sec. 7.4. The $\mathcal{M}$-operator enjoys deep algebraic and differential properties, as shown in Secs. 7.4, 7.5; the operator $\Psi$ enjoys a sweeping analytical power as shown in this section and summarized in (106)-(111). Therefore each operator, $\mathcal{M}$ or $\Psi$, is sufficient to carry radiative transfer theory by itself; and each has algebraic and analytic properties worthy of independent mathematical study.

## 7.14 Invariant Imbedding and Integral Transform Techniques

In this the final section of the present chapter on invariant imbedding techniques we shall briefly consider one of the more serious types of problems which, if left unchecked, may keep invariant imbedding techniques from reaching their full practical utility. This is the problem of the exploding variable-population. To see what is meant, consider the following observations.

There was a time when the radiative transferist was content with solving the Wiener-Hopf equation which described energy density in a homogeneous source-free infinitely deep medium. (See, e.g., (1) of Sec. 6.7 in which X is one-dimensional, i.e., the real line, and $h_\eta = 0$.) This was a highly idealized problem, deliberately idealized so that the Wiener-Hopf equation was a singular integral equation of the first kind in one dimension and eventually dispatched by a technique which has now become classical. There was also a time when it was a major breakthrough to have solved the radiative transfer problem on homogeneous plane-parallel media with isotropic scattering and stratified light fields (re: Sec. 6.4). The breakthrough was possible because of some judicious application of the spherical harmonic method to reduce the integro-differential equation of transfer to a set of coupled differential equations.

However, when these victories on the beachhead were over, there remained the more difficult high ground to take, and progress was correspondingly slower: the spaces arising in practice became odd-shaped and inhomogeneous, scattering became unmanageably anisotropic and heterochromatic, sources were encountered in the hitherto inviolate interiors of media, and matters were made worse by giving all physical quantities rapid temporal variations. The number of variables needed to fully describe the new radiometric environments grew from *one* (the depth location for scalar irradiance or radiant density) to *five* (steady-state monochromatic radiance transfer) to *seven* (time-dependent heterochromatic radiance transfer) to *twenty-eight* (when polarized light fields were considered). Clearly, the halcyon days of the subject were in the past and further progress with new real problems seemed to require new techniques and concepts, not only to solve them but to formulate them in the first place!

The advent of the principles of invariance (circa 1943) helped further high ground to be taken on the island of radiative transfer theory. The work of Ambarzumian and Chandrasekhar showed the potentialities of the concept of the global approach to transfer problems. This approach was subsequently considerably extended with the advent of the principles of invariance (Ref. [43]). The remaining high ground was surveyed using the invariant imbedding relation, Ref. [233], and its generalizations attained in Ref. [248], culminating in the interaction principle of Ref. [251] and the further results developed in the present work; the net result being compact formulations of transfer problems by means of one-point boundary value settings. The latter type settings are, as we have seen repeatedly throughout this chapter, simply elaborations of the basic integration problem:

$$\frac{dy(x)}{dx} \begin{cases} = f(x)y(x) + g(x) \\ = g\bigl(y(x),x\bigr) \end{cases} \qquad (*)$$

over an interval [a,b] given f and g on [a,b] and given the value y(a). The first equation is for linear, the second for nonlinear problems. In this way the invariant imbedding approach, the logical outgrowth to the principle of invariance approach, reduces radiative transfer problems to their simplest conceivable mathematical form.

Now the theoretical bases of the solution procedures of equations of the kind (*) and small finite systems of such equations are straightforward and usually dispatched in an introductory course in ordinary differential equations. The only serious difficulty such a system can present is on a practical and not a conceptual or theoretical level: the functions f, g, and y may no longer be scalar-valued, but matrix-valued, whose entries themselves are operators in integral or matricial form, and where the combinations f(x)y(x) or g(y(x),x) are no longer simple products but compositions of operators or functions f(x), y(x). In short, an uncontrollable, almost explosive increase in the number of variables needed to describe the domains and ranges of f, g, and y and their combinations could render (*) worthless from a practical point of view.

Since formulations of the kind (*) are quite clearly the simplest analytic forms into which the manifold problems of radiative transfer theory can be cast, the next major task on the practical front that faces radiative transfer theory is the successful handling of the variable-population explosion associated with (*). One immediate measure that can be taken is the judicious application of Laplace or Fourier transform techniques, or more generally, the application of integral transform techniques to the transport equations and the various principles. These integral transform methods have as their primary purpose in applied mathematics the reduction of the number of variables in a given physical formulation. For example, time derivatives can be transformed away and spatial or frequency convolution integrals can be transformed away resulting in two very important variable-reducing operations which can be effected by suitably chosen integral kernel transforms. In the case of the equation of transfer in radiative transfer theory, these two measures take on quite practical significance: the possibility of transforming away time derivatives means that a time-dependent problem can be reduced to a steady-state problem, solved in that context, and the solution so found, transformed back to the original setting. The possibility of transforming away convolution integrals could mean that various special heterochromatic transfer problems (in which scattering takes place from one wavelength to another) can be reduced to monochromatic transfer problems, solved in that context, and the solution transformed back to the original setting. Furthermore, since the multidimensional spatial settings used in the statement of the invariant imbedding relations (and their special principle of invariance forms) employ convolution integrals, relatively complex three-dimensional settings can occasionally be reduced to more tractable one-dimensional settings for a solution interim.

We now illustrate under what conditions integral kernel transforms can be used to reduce the number of variables in transport problems. We shall choose three examples for this purpose: the case of time-dependent radiative transfer; the case of heterochromatic radiative transfer; and the case of multidimensional radiative transfer on a plane-parallel medium with a non-stratified light field. For the benefit of readers not acquainted with the notions of integral kernel transforms, we precede the illustrations with a few introductory comments on this subject.

## An Integral-Transform Primer

To introduce the notion of an integral kernel transform in sufficient detail for our present purposes requires remarkably little mathematical machinery. From the welter of formulas and theorems of transform theory we prescind the idea of a real or complex valued integral operation:

$$\int_X [\ ]\ K(x,\omega)\ dx \tag{1}$$

which can act on a real valued function f defined on a set X. Thus:

$$\int_X f(x)K(x,\omega)\ dx$$

is a number, denoted by "$\hat{f}(\omega)$" or by "$\mathcal{F}[f,\omega]$" which is the result of integrating the product of the functions f and $K(\cdot,\omega)$ over X. The function K determines the form of the integral transform (1). The numbers $\omega$ are drawn from some set $\Omega$; the significance of $\Omega$ is immaterial for the present discussion. On the other hand X will take various familiar forms: the real line R extending from $-\infty$ to $\infty$, or the half line $R^+$ from 0 to $\infty$, or the xy-plane (i.e., the cartesian product $R \times R$), etc. The real or complex valued function K on $X \times \Omega$ is the *kernel* of (1). One may think of $X \times \Omega$, i.e., the set of all pairs $(x,\omega)$ with x in X and $\omega$ in $\Omega$, as a 'plane' with X and $\Omega$ as axes. We shall assume K to be continuous on $X \times \Omega$.

It is remarkable that the only formal property needed for K in the present discussion is that, for some $x_o$, we have $K(x_o,\omega) \neq 0$; and that:

$$K(x+y,\omega) = K(x,\omega)K(y,\omega) \tag{2}$$

which is the *group property* of K. When x,y are drawn from the set R of real numbers, we can formally deduce from (2) the differential property:

$$\frac{\partial K(x,\omega)}{\partial x} = K(x,\omega)g(\omega) \tag{3}$$

much in the way we deduced the differential property of beam transmittance in (2) of Sec. 3.11. Indeed, the similarity is not accidental since $K(x,\omega)$ is an exponential function over X. For example, $K(x,\omega)$ can be $\exp\{-i\omega x\}$, or $\exp\{\omega x\}$, or $\exp\{1/2\ x\ (\omega-1/\omega)\}$ for the cases of Fourier, Laplace, and Hankel-type transforms, respectively. There are other kernels K used in practice which do not have the group property (2) (e.g., Mellin and Euler kernels) but we shall consider only kernels with property (2).

Now the group property of K endows the theory of the integral kernel transform (1) with a most useful theorem, which is attained as follows. Suppose we write:

$$\text{"}f*g(y)\text{"} \quad \text{for} \quad \int_X f(x)g(y-x)\,dx \tag{4}$$

where f and g are functions on X for which the integral exists. The function $f*g$ is the *convolution* of f and g. The value of $f*g$ at y in X is denoted by "$f*g(y)$", as shown. Now let f and g be defined on X, and zero elsewhere. Then observe that if X is either $R^+$, R, $R^+ \times R^+$, $R \times R$, or $R \times \ldots \times R$ to n factors:

$$\begin{aligned}\int_X f*g(y)K(y,\omega)\,dy &= \int_X \left[\int_X f(x)g(y-x)\,dx\right] K(y,\omega)\,dy \\ &= \int_X f(x)\left[\int_X g(y-x)K(y,\omega)\,dy\right] dx \\ &= \int_X f(x)\left[\int_X g(u)K(x+u,\omega)\,du\right] dx \\ &= \int_X f(x)K(x,\omega)\left[\int_X g(u)K(u,\omega)\,du\right] dx \\ &= \hat{f}(\omega)\hat{g}(\omega) \quad .\end{aligned} \tag{5}$$

That is:

$$\boxed{\mathcal{F}[f*g;\omega] = \int_X (f*g)(y)K(y,\omega)\,dy = \hat{f}(\omega)\hat{g}(\omega) = \mathcal{F}[f;\omega]\,\mathcal{F}[g;\omega]} \tag{6}$$

In words: *the transform of the convolution* $f*g$ *of two functions on* X *is equal to the numerical product of their transforms.* In this way a complicated operation (the functional product (4)) is seen to be replaceable by a vastly simpler operation (the numerical product (5)) by means of integral transform operations. Statement (6) is the *convolution theorem* for the operator (1). It is by far the single most important property of integral kernel transforms whose kernels obey (2).

The second most important property of integral kernel transforms whose kernels obey (2) is the *derivative property*. We shall need this property only for the case where X is $R^+$ or some interval [a,b] of $R^+$ (which could be all of $R^+$). Thus consider the transform of the derivative f' of a function f on [a,b]:

$$\int_a^b f'(x)K(x,\omega)\ dx$$

which via integration by parts and (3) becomes:

$$f(x)K(x,\omega)\Big|_a^b - \int_a^b f(x)K'(x,\omega)\ dx =$$

$$= [f(b)K(b,\omega)-f(a)K(a,\omega)] - g(\omega)\int_a^b f(x)K(x,\omega)\ dx \quad .$$

From this we conclude that:

$$\mathcal{F}[f';\omega] = [f(b)K(b,\omega)-f(a)K(a,\omega)] - \mathcal{F}[f;\omega]g(\omega) \ . \qquad (7)$$

For example, if $a = 0$ and $b = \infty$, then:

$$\boxed{\mathcal{F}[f';\omega] = -[f(0)+\hat{f}(\omega)g(\omega)]} \quad . \qquad (8)$$

This follows from the necessity of having $\lim_x f(x)K(x,\omega) = 0$ as $x\to\infty$ for an integrable function f, and from (2) which implies:

$$K(0,\omega) = 1$$

for every $\omega$ in $\Omega$. To see this, set $y = 0$ in (2) and note that for every x, $K(x,\omega)$ cannot be zero. Equation (8) shows how the integral transform can do away with derivatives. It is true that the price one pays for ridding the scene of f' is the linear combination of $\hat{f}(\omega)$ and f(0), but this *algebraic* combination is usually more tractable than the generally *transcendental* object f' (i.e., one whose definition requires, in addition to the usual algebraic operations, the operation of limit).

One final matter, and that is the explication of the concept of the inverse of $\mathcal{F}$. We need only remark here that the inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is generally of the form:

$$\int_Y [\ \ ]H(x,\omega)\ d\omega$$

where x is in X, and Y is some subset of $\Omega$. Matters can often be arranged so that $\mathcal{F}^{-1}$ exists in the mathematical sense. For example, in the Fourier transform case, if $K(x,\omega) = 1/\sqrt{2\pi}\ \exp\{-i\omega x\}$, then $H(x,\omega) = 1/\sqrt{2\pi}\ \exp\{i\omega x\}$, or if $K(x,\omega) = \exp\{-i\omega x\}$, then $H(x,\omega) = 1/2\pi\ \exp\{i\omega x\}$, and in either case $Y = X = R$. Further, in the Laplace transform case, if $K(x,\omega) = \exp\{-x\omega\}$, then $H(x,\omega) = K(-x,\omega)/2\pi i$ ; and if $X = R$, then for some real $\gamma$, $Y = \{\gamma+i\omega:\ \omega\varepsilon R\}$. The inverse operation

$\mathcal{F}^{-1}$ undoes what $\mathcal{F}$ does:

$$\text{If } \mathcal{F}[f;\omega] = \hat{f}(\omega), \text{ then } \mathcal{F}^{-1}[\hat{f};x] = f(x) \quad .$$

For example, in the one-dimensional Fourier Integral setting:

$$\hat{f}(\omega) = \mathcal{F}[f;\omega] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$\mathcal{F}^{-1}[\hat{f};x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega \quad . \tag{9}$$

In the two-dimensional Fourier Integral setting:

$$\hat{f}(\omega_1,\omega_2) = \mathcal{F}[f;\omega_1,\omega_2] = \int_{-\infty}^{\infty}\int f(x_1,x_2)e^{-i[\omega_1 x_1+\omega_2 x_2]}\,dx_1 dx_2$$

$$f(x_1,x_2) = \mathcal{F}^{-1}[\hat{f};x_1,x_2] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int \hat{f}(\omega_1,\omega_2)e^{i[\omega_1 x_1+\omega_2 x_2]}\,d\omega_1 d\omega_2 \tag{10}$$

## Time-Dependent Radiative Transfer

Let us begin our studies of integral transform techniques with the time-dependent equation of transfer (4) of Sec. 3.15, to which is appended a source term $N_\eta$:

$$\frac{1}{v}\frac{\partial N(x,\xi,t)}{\partial t} + \frac{dN(x,\xi,t)}{dr} =$$

$$= -\alpha(x,\xi,t)N(x,\xi,t)+\int_{\Xi} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi')+N_\eta(x,\xi,t) \tag{11}$$

We require for the present discussion that $N(x,\xi,t) = 0$ for $t \leq 0$. We choose the following form of the integral operator (1):

$$X = R^+ = [0,\infty]$$

$$K(t,\omega) = e^{-t\omega}, \text{ so that } g(\omega) = -\omega$$

$\Omega$ is the set of complex numbers

with $\omega$ a positive real number in $R^+$. Let us write:

$$\text{"}\hat{N}(x,\xi,\omega)\text{"} \quad \text{for} \quad \int_0^{\infty} N(x,\xi,t)e^{-t\omega}\,dt \quad .$$

Applying the operator:

$$\int_0^\infty [\ ]e^{-t\omega}dt$$

to each side of (11) and using its linearity property, namely:

$$\int_0^\infty [cf(t) + dg(t)]e^{-t\omega}dt = c\hat{f}(\omega) + d\hat{g}(\omega)$$

we see that by (8) the time derivative term becomes:

$$\frac{1}{v}\int_0^\infty \frac{\partial N(x,\xi,t)}{\partial t} e^{-t\omega}dt = \frac{\omega}{v}\hat{N}(x,\xi,\omega) \quad .$$

The spatial derivative term becomes:

$$\int_0^\infty \frac{dN(x,\xi,t)}{dr} e^{-t\omega}dt = \frac{d\hat{N}(x,\xi,\omega)}{dr} \quad .$$

On the right side we have, for the first term:

$$-\int_0^\infty \alpha(x,\xi,t)N(x,\xi,t)e^{-t\omega}\, dt \quad .$$

At this point we realize that, for the Laplace transform to be effective in the present case, we require $\alpha(x,\xi,\cdot)$ be a constant function of time (i.e., constant on $R^+$). Hence we may study time-dependent transfer problems in which the radiance field is truly time-varying but this variation is of a transient nature traceable to the finite speed v of propagation of radiant flux throughout the medium and not to the time-dependence of the inherent optical properties of the medium. We therefore assume for the remainder of this discussion that $\alpha$ and $\sigma$ are independent of time, and shall write "$\alpha(x,\xi)$" for $\alpha(x,\xi,0)$ and "$\sigma(x;\xi';\xi)$" for $\sigma(x;\xi';\xi;0)$. With this agreement the preceding transformed term becomes:

$$-\alpha(x,\xi)\hat{N}(x,\xi,\omega) \quad .$$

Altogether, the transformed terms of (11) become:

$$\frac{d\hat{N}(x,\xi,\omega)}{dr} = -\left(\alpha(x,\xi)+\frac{\omega}{v}\right)\hat{N}(x,\xi,\omega) + \int_\Xi \hat{N}(x,\xi,\omega)\sigma(x;\xi';\xi)\,d\Omega(\xi') + \hat{N}_\eta(x,\xi,\omega) \tag{12}$$

Equation (12) is the Laplace-transformed equation of transfer. It has clearly the gestalt of a steady state equation of transfer with source term, and with a slightly-odd volume attenuation function. It seems that under transformation from transient to steady state, the volume attenuation function has been altered artificially by a fixed amount $\omega/v$. Inspection of (12) thus yields the following observation: *The entire theory of the steady state field (all classical and invariant imbedding techniques) can be applied to (12)*--the variable $\omega$ is a fixed, passive complex variable, dangling throughout all the subsequent steady state proceedings like a useless appendix, but ready to play its role in the final movements of the solution procedure. The return to physical setting is made by means of the inverse operation:

$$N(x,\xi,t) = \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} \hat{N}(x,\xi,\omega) e^{t\omega}\, d\omega \quad . \qquad (13)$$

This inversion operation can be written as a real integral and performed numerically on computers. The details of the real integral representation may be found, e.g., in Ref. [46].

The reader can obtain practice with the Laplace transformation procedure by transforming (5), (6) of Sec. 7.2 directly into their 'steady state' forms. The results should agree with the local forms derived from (12).

The time-dependent invariant imbedding relation in its forms (27), (28) of Sec. 7.2 can be transformed directly by means of the Laplace integral operation. A typical term of the relation is:

$$\int_E N_+(z,t')\, \mathcal{T}(z,y,x,t',t)\, dt' \qquad (14)$$

where we now choose E to be $R^+$ and require $N_+(z,t') = 0$ for $t \le 0$. We assume that:

$$\mathcal{T}(z,y,x,t',t) = \mathcal{T}(z,y,x,u',u)$$

whenever $t-t' = u-u'$. This is tantamount to assuming that $\alpha$ and $\sigma$ are time independent, and is the corresponding assumption on the global level of radiative transfer that one must make before the Laplace transform method can be invoked. With this agreement, we shall write:

$$\text{"}\mathcal{T}(z,y,x,t-t')\text{"} \quad \text{for} \quad \mathcal{T}(z,y,x,t',t)$$

so that (14) becomes:

$$\int_{R^+} N_+(z,t')\, \mathcal{T}(z,y,x,t-t')\, dt' = N_+(z) * \mathcal{T}(z,y,x) \quad . \qquad (15)$$

Therefore, operating on this convolution of $N_+(z)$ and the complete transmittance operator $\mathcal{T}(z,y,x)$, we have, by (6):

$$\mathcal{F}[N_+(z) * \mathcal{T}(z,y,x;\omega] = \hat{N}_+(z,\omega)\,\hat{\mathcal{T}}(z,y,x,\omega) \quad . \qquad (16)$$

By our remarks in Sec. 7.2, especially those after (22) of Sec. 7.2, we realize that $\hat{\mathcal{T}}(z,y,x,\omega)$ is itself an integral operator over $\Xi_+$ and the parameter surface $X_z$, so that we are back again in the steady state context. Once $\hat{N}_+(z,\omega)$ has been found, the physically meaningful radiance $N(z,t)$ can be recovered using (13). Or, again, $\mathcal{T}(z,y,x,t-t')$ can be recovered using (13). This is all new territory and is free to be explored by interested students of the subject.

The complete set of transformed invariant imbedding relations (27),(28) of Sec. 7.2 is given as:

$$\hat{N}_+(y,\omega) = \hat{N}_+(z,\omega)\,\hat{\mathcal{T}}(z,y,x,\omega) + \hat{N}_-(x,\omega)\,\hat{\mathcal{R}}(x,y,z,\omega) \qquad (17)$$

$$\hat{N}_-(y,\omega) = \hat{N}_-(x,\omega)\,\hat{\mathcal{T}}(x,y,z,\omega) + \hat{N}_+(z,\omega)\,\hat{\mathcal{R}}(z,y,x,\omega). \qquad (18)$$

From these we can find the Laplace-transformed principles of invariance in the usual manner and all the functional relations for the transformed R and T operators paralleling those of Sec. 7.1. Or then, again, these functional relations may be obtained directly by transforming principles I'-IV' of Sec. 7.2.

## Heterochromatic Radiative Transfer

When radiative transfer takes place across the spectrum of frequencies in addition to spatial and directional transfer, we have heterochromatic radiative transfer. The volume transpectral scattering function $\hat{\sigma}$ is designed to do for the heterochromatic context what $\sigma$ does in the monochromatic setting. The function $\hat{\sigma}$ is defined in Ex. 3, Sec. 3.17, and discussed in detail in Sec. 19 of Ref. [251]. Suppose the radiance field is of frequency $\nu$ and is in steady state (either real or the pseudo steady state, in $\omega$-space, of a Laplace transformed N-field). For our present purposes we need only note that in addition to the term:

$$N_*(x,\xi,\nu) = \int_{\Xi} N(x,\xi',\nu)\sigma(x;\xi';\xi,\nu)\, d\Omega(\xi')$$

we have the term:

$$N_s(x,\xi,\nu) = \int_{\Xi}\int_{\Lambda} N(x,\xi',\nu')\hat{\sigma}(x;\xi';\xi;\nu',\nu)\, d\Omega(\xi')dl(\nu')$$

where "$\Lambda$" denotes the spectrum of frequencies, and is mathematically simply another name for $R^+$. Now if it is possible to find a mapping T on $R^+$ onto itself so that:

$$\hat{\sigma}\big(x;\xi';\xi;T(\nu'),T(\nu)\big) = \hat{\sigma}\big(x;\xi';\xi;T(\mu'),T(\mu)\big)$$

whenever

$$T(\nu) - T(\nu') = T(\mu) - T(\mu') \quad ,$$

then we may view:

$$\int_{\Lambda} N(x,\xi',\nu')\hat{\sigma}\big(x;\xi';\xi;T(\nu'),T(\nu)\big)\ dl(\nu')$$

as a convolution of $N(x,\xi',\cdot)$ and $\hat{\sigma}$. However, even if it is possible to find such a mapping T (in neutron transport theory such a mapping exists and can be used to develop the so-called 'Fermi-age theory') some reflection would show that the Laplace transform method will not be applicable in the present case. Thus, unless very limited regions in $\Lambda \times \Lambda$ are considered, outside of which $\hat{\sigma}$ is zero and inside which the following translation condition holds:

$$\hat{\sigma}(x;\xi';\xi;\nu',\nu) = \hat{\sigma}(x;\xi';\xi;\mu',\mu) \tag{19}$$

whenever $\nu-\nu' = \mu-\mu'$, the Laplace transform method fails to simplify the heterochromatic radiative transfer problem. It is also physically unlikely that (19) will hold in the usual settings. Nevertheless, assuming (19) holds, then the convolution theorem yields:

$$\hat{N}_s(x,\xi,\omega) = \int_{\Xi} \hat{N}(x,\xi,\omega)\hat{\hat{\sigma}}(x;\xi';\xi;\omega)\ d\Omega(\xi')$$

and the transformed $N_*$-term is:

$$\hat{N}_*(x,\xi,\omega) = \int_{R^+}\left[\int_{\Xi} N(x,\xi',\nu)\sigma(x;\xi';\xi,\nu)\ d\Omega(\xi')\right]e^{-\nu\omega}\ d\nu \quad .$$

Since $\sigma$ generally depends on $\nu$, we encounter a difficulty similar to that with the term $\alpha(x,\xi,t)N(x,\xi,t)$ in (11) when the time-dependent case is considered. It is simply too much to ask $\sigma$ to be generally independent of $\nu$ (whereas it was not too much of a sacrifice in accuracy to ask $\sigma$ to be independent of time during the course of a given transfer process). Hence we conclude that except in the most special of settings, the integral transform method is generally of no use in the solution of heterochromatic radiative transfer problems. We shall instead rely on such methods as given in Refs. [136], [137] or more generally in Ref. [288] or Ref. [251] to solve the general heterochromatic radiative transfer problem.

## Multidimensional Radiative Transfer

Multidimensional radiative transfer problems arise most frequently in practice in plane-parallel media in which the optical properties are all well behaved--stratified with depth or altitude or simply constant--but in which the light field is not stratified with depth and which is generally variable laterally over the plane surfaces parallel to the boundaries. The practical instances of these problems arise when clouds induce a checkerboard pattern of light and dark over a horizontal plane in an otherwise homogeneous body of air, sea, or lake, or when an isotropic point source or narrow beam of flux is present within or near these natural media. The "multidimensional" aspect of these settings consists in the full three spatial variables being required in addition to the two direction variables to describe the radiance field in such media.

To see how the integral transform methods are of use in such radiometric situations as just described, recall first of all the discussion of the point source problem for scalar irradiance in Sec. 6.7. Then, consider a general internal or external source problem on a stratified plane-parallel medium $X(a,b)$. The theory of Example 3 of Sec. 3.9 and the work of Sec. 7.13 showed how this problem can be solved using invariant imbedding techniques, and of how the operators in the solution procedure could always be reduced to suitable assemblies of the standard R and T operators. Therefore we are to consider the R and T operators in their full generality as given in (8)-(11) of Sec. 3.6. To fix ideas, consider the operator $R(a,b)$ acting on the incident radiance function $N_-(a)$ over the surface $X_a$. Then by (12) of Sec. 3.6:

$$N_-(a)R(a,b) = \int_{\Xi}\int_{X_a} N(x',\xi')S(X;x',\xi';x,\xi)\ dA(x')d\Omega(\xi')$$

where "X" denotes $X(a,b)$. If the medium $X(a,b)$ is stratified over horizontal planes, then:

$$S(X;x',\xi';x,\xi) = S(X;y',\xi';y,\xi)$$

whenever $x-x' = y-y'$, where $x$, $x'$, $y$, $y'$ are points in $X_a$. Thus, e.g., $x'$ is an ordered triple of the form $(x_1,x_2,x_3)$ with $x_3 = a$. Assuming stratification, we can write:

$$\text{"}S(X;x-x';\xi';\xi)\text{"} \quad \text{for} \quad S(X;x',\xi';x,\xi)$$

so that:

$$N_+(a) = N_-(a)R(a,b) = \int_{\Xi}\left[\int_{X_a} N(x',\xi')S(X;x-x';\xi';\xi)\ dA(x')\right] d\Omega(\xi') \tag{20}$$

Next we choose the following form of the integral operator (1):

$$X = R \times R$$

$$K(x,\omega) = \exp\{-i(ix_1+jx_2)\cdot(i\omega_1+j\omega_2)\}$$

$$\Omega = R \times R$$

where now clearly we need $x = (ix_1+jx_2)$ and $\omega = (i\omega_1+j\omega_2)$. See, e.g., (23) of Sec. 6.7. Hence (1) now becomes a two-dimensional Fourier transform. Applying the resultant integral transform to (20), we have by (6):

$$\hat{N}_+(a,\omega) = \int_{\Xi_-} \hat{N}(a,\xi',\omega)\hat{S}(X;\omega;\xi';\xi)\ d\Omega(\xi') \qquad (21)$$

$$= \hat{N}(a,\omega)R(a,b;\omega)$$

which is vastly simpler to deal with than (20). In (21), "X" stands for X(a,b). Comparing the S-operator in equation (21) with the corresponding operator for stratified plane-parallel media with *stratified light field* ((31), (32) of Sec. 3.7), we see that we have returned to the fully stratified context and can apply the theory of stratified light fields to the following set of Fourier-transformed principles of invariance for X(a,b) (obtained from I, II of Example 3, Sec. 3.7 by applying the present Fourier transform operator):

$$\hat{N}_+(y,\omega) = \hat{N}_+(z,\omega)\hat{T}(z,y;\omega) + \hat{N}_-(y,\omega)\hat{R}(y,z;\omega) \qquad (22)$$

$$\hat{N}_-(y,\omega) = \hat{N}_-(x,\omega)\hat{T}(x,y;\omega) + \hat{N}_+(y,\omega)\hat{R}(y,x;\omega) \quad . \qquad (23)$$

Of course in actual practice we can drop the carets and the omegas so as to work with simpler notation. Equations (22) and (23) serve to show that the general structure of Fourier-transformed principles of invariance in the nonstratified case are the same as those of the stratified case, under the present assumptions.

## Conclusion

Sufficient examples have now been given to show some of the power and the limitations of the integral transform method in radiative transfer theory in general, and particularly in conjunction with the operator equations of the invariant imbedding technique. Spatio-temporal inhomogeneities of the medium and heterochromatic radiative transfer severely limit the applicability of the integral transform techniques. Much work therefore remains to be done in the time-dependent and multidimensional problems.

<< Table of Contents << Title Page >> Next Page >>

## 7.15 Bibliographic Notes for Chapter 7.

The steady state functional relations for the standard R and T operators in Sec. 7.1 are based on the work in Ref. [234]. The time-dependent functional relations for R and T in Sec. 7.2 are drawn from Ref. [235]. The partition relations of Sec. 7.3 are continuous-operator versions of similar matricial relations developed in Ref. [251]. The algebraic studies of Sec. 7.4 grew out of Refs. [248] and [249]. We draw attention to some interesting related results in electrical network theory and diffusion theory found independently by Redheffer in Refs. [252]-[259]. Also the work of Reid is of interest in the present invariant imbedding studies [261], [262]. The