transform to (20), we have by (6):

$$\hat{N}_+(a,\omega) = \int_{\Xi_-} \hat{N}(a,\xi',\omega)\hat{S}(X;\omega;\xi';\xi)\ d\Omega(\xi') \qquad (21)$$

$$= \hat{N}(a,\omega)R(a,b;\omega)$$

which is vastly simpler to deal with than (20). In (21), "X" stands for X(a,b). Comparing the S-operator in equation (21) with the corresponding operator for stratified plane-parallel media with *stratified light field* ((31), (32) of Sec. 3.7), we see that we have returned to the fully stratified context and can apply the theory of stratified light fields to the following set of Fourier-transformed principles of invariance for X(a,b) (obtained from I, II of Example 3, Sec. 3.7 by applying the present Fourier transform operator):

$$\hat{N}_+(y,\omega) = \hat{N}_+(z,\omega)\hat{T}(z,y;\omega) + \hat{N}_-(y,\omega)\hat{R}(y,z;\omega) \qquad (22)$$

$$\hat{N}_-(y,\omega) = \hat{N}_-(x,\omega)\hat{T}(x,y;\omega) + \hat{N}_+(y,\omega)\hat{R}(y,x;\omega) \quad . \qquad (23)$$

Of course in actual practice we can drop the carets and the omegas so as to work with simpler notation. Equations (22) and (23) serve to show that the general structure of Fourier-transformed principles of invariance in the nonstratified case are the same as those of the stratified case, under the present assumptions.

Conclusion

Sufficient examples have now been given to show some of the power and the limitations of the integral transform method in radiative transfer theory in general, and particularly in conjunction with the operator equations of the invariant imbedding technique. Spatio-temporal inhomogeneities of the medium and heterochromatic radiative transfer severely limit the applicability of the integral transform techniques. Much work therefore remains to be done in the time-dependent and multidimensional problems.

7.15 Bibliographic Notes for Chapter 7.

The steady state functional relations for the standard R and T operators in Sec. 7.1 are based on the work in Ref. [234]. The time-dependent functional relations for R and T in Sec. 7.2 are drawn from Ref. [235]. The partition relations of Sec. 7.3 are continuous-operator versions of similar matricial relations developed in Ref. [251]. The algebraic studies of Sec. 7.4 grew out of Refs. [248] and [249]. We draw attention to some interesting related results in electrical network theory and diffusion theory found independently by Redheffer in Refs. [252]-[259]. Also the work of Reid is of interest in the present invariant imbedding studies [261], [262]. The analytic properties of the invariant imbedding operators in Sec. 7.5 appear to be new. The examples of numerical solutions for R(a,b) given in Sec. 7.6 are based on the work of Bellman, Kalaba and Prestrud in Ref. [15]. The general solution procedures of Secs. 7.11 are new, along with the developments of Sec. 7.13 concerned with the general internal source problem.

For a study of the internal source problem in the context of neutron transport theory, see the work of Elliott [88], and that of Bellman, Kalaba, and Wing [17].

Further discussion of the theory of polarized radiance fields as developed by Chandrasekhar and applied in natural hydrosols may be found in [108], [157]. Also the work of Sekera [284], although applied to the atmosphere, illustrates further the applications of Chandrasekhar's approach to the theory of polarized light fields.

The general functional equation approach of this chapter may be divided into the integral and differential approaches, and the hybrid integro-differential approach. The tap root of integral equation formulations of radiative transfer theory lies in the work of King [138], and that of the differential approach is the work of Schuster [279]. A brief, readable account of these two approaches, which places them in historical perspective, was given by Duntley [70]. Important future developments of the theory rest in using functional analysis along the lines developed throughout this chapter, particularly using the notions of semigroup theory. See, e.g., the discussion of the equation of evolution in [326], and recall the closing remarks of [216].