

and those for the radiance distributions as studied, e.g., in Chapter 7. The similarity is a thorough-going one which may, indeed, be used as a heuristic guide in pursuing either subject matter, using the other as a base. In this connection, it may be noted that the original forms of the invariant imbedding relations and internal source relations studied in Chapter 7 were tentatively found in the irradiance context by means of informal scratch pad calculations; the simplicity of their derivation and resultant forms in that context encouraged the rigorous search for the associated full fledged operator equations for radiance sprinkled throughout that chapter.

We shall be guided in our discussions in the initial sections of this chapter (8.1, 8.2) and once again in Section 8.7 by the close conceptual connections between the functional equations for irradiance and radiance fields. In this way we can effect a smooth transition from the general formulations of Chapter 7 to the simpler settings of this chapter and at the same time gain some insight into the unity of the theory engendered by the invariance concepts. However, for the main sections of the chapter (8.3-8.6) the discussions will for the most part dwell on specific models which have been tried and found useful in the daily tasks of obtaining numerical estimates and rule-of-thumb algebraic approximations to the magnitudes of light fields, and the properties of their associated optical media.

Throughout this chapter we shall work with an arbitrary plane-parallel medium $X(a,b)$ and adopt the reference frame for $X(a,b)$, as defined in Section 2.4. Furthermore the steady state irradiances $H(z,\pm)$ defined in Section 2.4, along with their attendant radiometric concepts, will be adopted without further explanation. The medium $X(a,b)$ will be assumed free of internal sources unless specifically noted otherwise, and arbitrarily stratified with a stratified light field, so that both radiometric and optical properties depend only on depth z within $X(a,b)$, $a < z < b$. Arbitrary sources are incident on the upper and lower boundaries of $X(a,b)$. (In real media, however, it is customary in practice to have no sources incident on the lower boundary.) The explicit retention of a source on the lower boundary will have the effect of keeping the resultant theoretical imbedding relations in their full symmetric form.

8.1 Invariant Imbedding Relation for Irradiance Fields

Our point of departure for the present discussion is the set of principles of invariance (7), (8) of Section 3.7. We recall that these statements were deduced from an application of the interaction principle to an arbitrary subslab $X(x,z)$ of a plane parallel medium of the type $X(a,b)$, schematically depicted in Fig. 8.1. The results may be written:

$$I. \quad H(y,+) = H(z,+) T(z,y) + H(y,-) R(y,z) \quad (1)$$

$$II. \quad H(y,-) = H(x,-) T(x,y) + H(y,+) R(y,x) \quad (2)$$

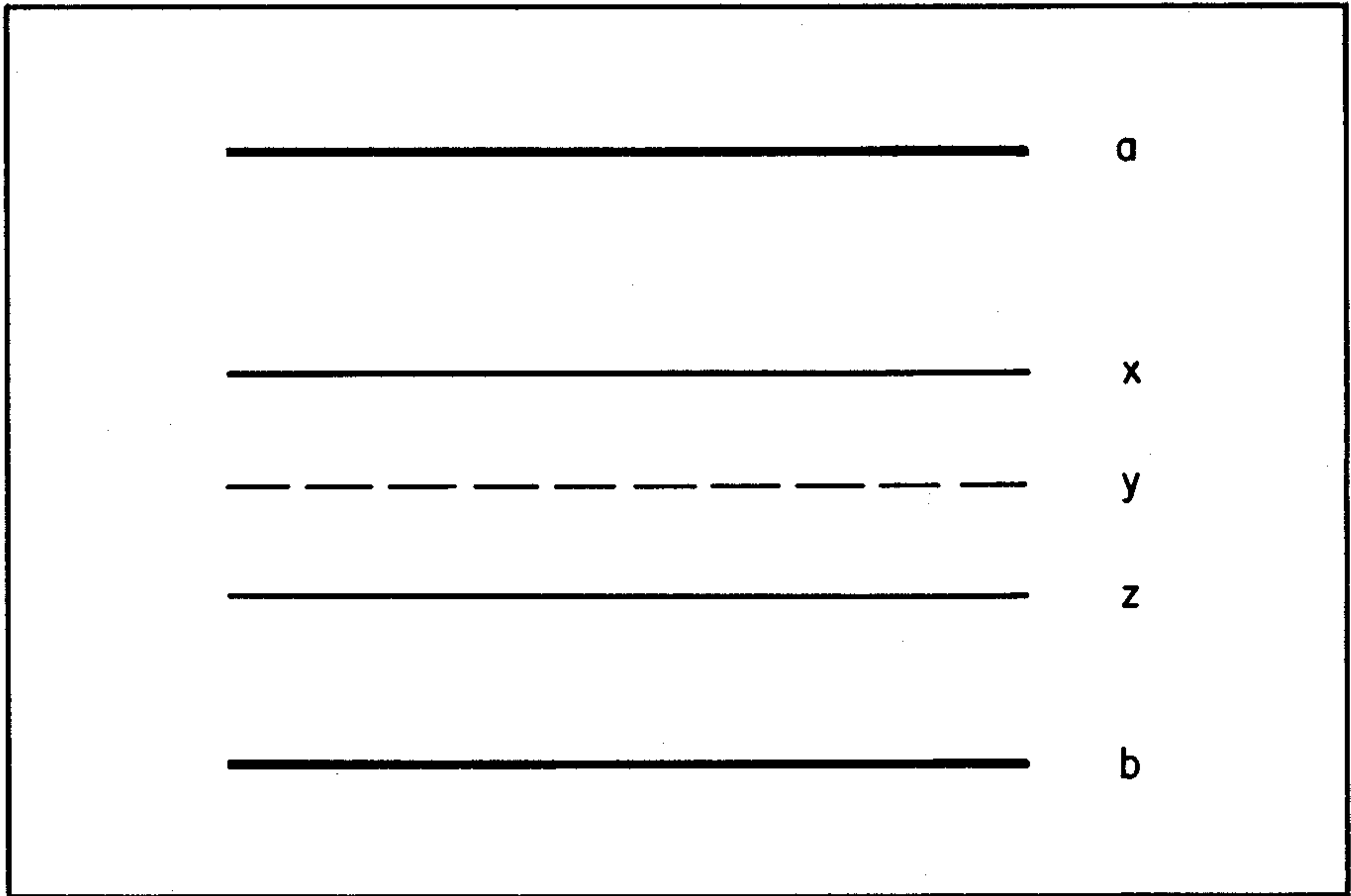


FIG. 8.1 The setting for the principles of invariance governing irradiance fields on plane-parallel media.

where $a \leq x \leq y \leq z \leq b$. The four numbers $T(x,y)$, $R(y,x)$, $T(z,y)$, $R(y,z)$ are the various *transmittances* (T) and *reflectances* (R) of the pieces $X(x,y)$ and $X(y,z)$ of the partitioned slab $X(x,z)$. The present goal is to derive the invariant imbedding relation for $X(a,b)$ in the irradiance context. Our activity will parallel very closely that in examples 4 and 5 of Section 3.9, thereby casting light on those earlier computations and in turn adding evidence to the belief that the manner of approach to the invariant imbedding relation, at least on the algebraic level, is independent of the geometry of the medium and the radiometric concepts used in the approach.

Most of the work toward attaining the invariant imbedding relation is already contained in the results (9), and (10) of Section 3.17; for if we now apply those equations to the subslab $X(x,z)$ of the present setting (by letting $z = x$, $b = z$) and write:

$$"R(x,y,z)" \quad \text{for} \quad \frac{T(x,y)R(y,z)}{1-R(y,x)R(y,z)} \quad (3)$$

$$"T(x,y,z)" \quad \text{for} \quad \frac{T(x,y)}{1-R(y,x)R(y,z)} \quad (4)$$

$$"R(z,y,x)" \quad \text{for} \quad \frac{T(z,y)R(y,x)}{1-R(y,x)R(y,z)} \quad (5)$$

$$" \mathcal{T}(z, y, x) " \text{ for } \frac{T(z, y)}{1 - R(y, x)R(y, z)}, \quad (6)$$

then those equations can be written:

$$H(y, +) = H(z, +)\mathcal{T}(z, y, x) + H(x, -)\mathcal{R}(x, y, z) \quad (7)$$

$$H(y, -) = H(x, -)\mathcal{T}(x, y, z) + H(z, +)\mathcal{R}(z, y, x) \quad (8)$$

Equations (7) and (8) present an excellent opportunity for the reader to become acquainted with the invariant imbedding relation in a relatively simple setting. It was for this reason that the irradiance example was presented first in Chapter 3. In this chapter we shall have more opportunity to explore the irradiance context of the invariant imbedding concepts.

The operators \mathcal{R} and \mathcal{T} defined in (3) through (6) above are the *complete reflectance* and *complete transmittance* factors, respectively. Observe that the following special cases of \mathcal{R} and \mathcal{T} hold:

$$\mathcal{R}(x, x, y) = R(x, y) \quad (9)$$

$$\mathcal{T}(x, y, y) = T(x, y) \quad (10)$$

$$\mathcal{R}(x, y, y) = 0 \quad (11)$$

$$\mathcal{T}(x, x, y) = 1 \quad (12)$$

These statements may be obtained directly from (3) and (4) upon suitable substitutions, and by appeal to (13) and (14) of Sec. 3.7. A complementary set of four equations can be obtained from (5) and (6). It follows that the invariant imbedding equations (7) and (8) contain the principles of invariance (1) and (2) as special cases. The equations (7) and (8) can be cast into matrix form by writing:

$$" \mathcal{M}(x, y, z) " \text{ for } \begin{bmatrix} \mathcal{T}(z, y, x)\mathcal{R}(z, y, x) \\ \mathcal{R}(x, y, z)\mathcal{T}(x, y, z) \end{bmatrix} \quad (13)$$

so that we have:

$$(H(y, +), H(y, -)) = (H(z, +), H(x, -))\mathcal{M}(x, y, z) \quad (14)$$

This is the required *invariant imbedding relation* for the irradiance context. It should be observed that we are adopting in the irradiance context, without essential change, the notation for standard and complete operators used earlier in Chapter 3 for the radiance (operator) context. This affords a great economy of terminology, retains a useful and suggestive notation, and serves to strengthen the conceptual unity of radiative transfer theory attained by means of the invariant imbedding techniques. There shall be no confusion arising from this practice, for it very rarely happens that the

irradiance field and the radiance field are simultaneously under study in a given one-parameter medium, since these are two very distinct levels of description of given radiative transfer phenomena: the irradiance description presently under study is a simple numerical description of a light field while the radiance description is a more detailed functional description of the light field.

As it stands, the invariant imbedding relation (14) is the general form of the solution to the radiative transfer problem in $X(a,b)$ for the irradiance field: knowledge of $\mathcal{M}(x,y,z)$ for every three successive levels x,y,z in a subslab $X(x,z)$ of $X(z,b)$ allows one to compute the irradiance field $(H(y,+),H(y,-))$ at every level y in $X(x,z)$ knowing the incident radiance field $(H(z,+),H(x,-))$ on $X(x,z)$. The complete reflectance and transmittance operators \mathcal{R} and \mathcal{T} in $\mathcal{M}(x,y,z)$ depend in turn on the standard operators R and T as shown in (3) through (6). Therefore the complete solution of the irradiance transfer problem in $X(a,b)$ devolves on knowledge of the standard R and T factors, in exact analogy to the radiance transfer problem studied in detail in Chapter 7. Consequently, knowing the R and T , or better still the \mathcal{R} and \mathcal{T} factors, for a medium $X(a,b)$, we can write down by sight the answer to every question about $H(y,\pm)$ for every level y in $X(a,b)$. We shall in the course of this chapter obtain methods for the determination of the standard R and T factors and the complete factors \mathcal{R} and \mathcal{T} . For the present we go on to formulate further equations governing the irradiance field.

8.2 General Irradiance Equations

The global description of the irradiance field in a plane parallel medium $X(a,b)$ as given by the invariant imbedding relation (14) of Sec. 8.1 will now be supplemented by a local description in the form of a pair of differential equations for the irradiances $H(z,\pm)$ as a function of depth z in $X(a,b)$. The approach we shall take at present is through the principles of invariance (1) and (2) of Sec. 8.1.

The idea of the derivation is quite simple: We isolate for attention the subslab $X(x,z)$ of $X(a,b)$ and form the difference quotient:

$$\frac{H(z,-) - H(x,-)}{z - x} \quad (1)$$

We then let x approach z and determine the associated limit of (1). The physical meaning of this activity should be carefully noted at the outset, as it will repeatedly suggest the subsequent moves in the sequence of explorations below. Thus (1) is the average rate of change of the downward irradiance field over the depth interval from x to z . As $z - x$ is made smaller and smaller (and hence $X(x,z)$ thinner and thinner) we increasingly localize the factors governing this average rate of change until, in the limit, we should have a completely local description of the change of the downward irradiance field at depth z .