

irradiance field and the radiance field are simultaneously under study in a given one-parameter medium, since these are two very distinct levels of description of given radiative transfer phenomena: the irradiance description presently under study is a simple numerical description of a light field while the radiance description is a more detailed functional description of the light field.

As it stands, the invariant imbedding relation (14) is the general form of the solution to the radiative transfer problem in $X(a,b)$ for the irradiance field: knowledge of $\mathcal{M}(x,y,z)$ for every three successive levels x,y,z in a subslab $X(x,z)$ of $X(z,b)$ allows one to compute the irradiance field $(H(y,+),H(y,-))$ at every level y in $X(x,z)$ knowing the incident radiance field $(H(z,+),H(x,-))$ on $X(x,z)$. The complete reflectance and transmittance operators \mathcal{R} and \mathcal{T} in $\mathcal{M}(x,y,z)$ depend in turn on the standard operators R and T as shown in (3) through (6). Therefore the complete solution of the irradiance transfer problem in $X(a,b)$ devolves on knowledge of the standard R and T factors, in exact analogy to the radiance transfer problem studied in detail in Chapter 7. Consequently, knowing the R and T , or better still the \mathcal{R} and \mathcal{T} factors, for a medium $X(a,b)$, we can write down by sight the answer to every question about $H(y,\pm)$ for every level y in $X(a,b)$. We shall in the course of this chapter obtain methods for the determination of the standard R and T factors and the complete factors \mathcal{R} and \mathcal{T} . For the present we go on to formulate further equations governing the irradiance field.

8.2 General Irradiance Equations

The global description of the irradiance field in a plane parallel medium $X(a,b)$ as given by the invariant imbedding relation (14) of Sec. 8.1 will now be supplemented by a local description in the form of a pair of differential equations for the irradiances $H(z,\pm)$ as a function of depth z in $X(a,b)$. The approach we shall take at present is through the principles of invariance (1) and (2) of Sec. 8.1.

The idea of the derivation is quite simple: We isolate for attention the subslab $X(x,z)$ of $X(a,b)$ and form the difference quotient:

$$\frac{H(z,-) - H(x,-)}{z - x} \quad (1)$$

We then let x approach z and determine the associated limit of (1). The physical meaning of this activity should be carefully noted at the outset, as it will repeatedly suggest the subsequent moves in the sequence of explorations below. Thus (1) is the average rate of change of the downward irradiance field over the depth interval from x to z . As $z - x$ is made smaller and smaller (and hence $X(x,z)$ thinner and thinner) we increasingly localize the factors governing this average rate of change until, in the limit, we should have a completely local description of the change of the downward irradiance field at depth z .

Following the program just outlined, we set $y = z$ in (2) of Sec. 8.1, the result being:

$$H(z, -) = H(x, -)T(x, z) + H(z, +)R(z, x) .$$

This representation of $H(z, -)$ is used in (1) to obtain:

$$\frac{H(z, -) - H(x, -)}{z - x} = H(x, -) \frac{[T(x, z) - 1]}{z - x} + H(z, +) \frac{R(z, x)}{z - x} \quad (2)$$

Now as x approaches z , the difference $z - x$ approaches zero, and the slab $X(x, z)$ becomes increasingly thinner, so that its downward transmittance $T(x, z)$ approaches 1 and its upward reflectance $R(z, x)$ approaches zero (cf. (9)-(12) of Sec. 7.3). Therefore the quotients on the right side of (2) have a chance of going to well-defined limits. Indeed, our discussion in Sec. 7.3 (see e.g., (9)-(12) of that section) prepares the ground for the following definitions; we write:

$$" \tau(z, -) " \quad \text{for} \quad \lim_{x \rightarrow z} \frac{T(x, z) - 1}{z - x} \quad (3)$$

$$" \rho(z, +) " \quad \text{for} \quad \lim_{x \rightarrow z} \frac{R(z, x)}{z - x} \quad (4)$$

Then, if we write as usual:

$$\frac{dH(z, -)}{dz} \quad \text{for} \quad \lim_{x \rightarrow z} \frac{H(z, -) - H(x, -)}{z - x} \quad (5)$$

equation (2) yields:

$$\boxed{\frac{dH(z, -)}{dz} = \tau(z, -)H(z, -) + \rho(z, +)H(z, +)} \quad (6)$$

This is the equation governing the downward irradiance field $H(z, -)$. In a similar way, setting $y = x$ in (1) of Sec. 8.1, forming the difference quotient:

$$-\frac{H(x, +) - H(z, +)}{(x - z)} = H(z, +) \left[\frac{T(z, x) - 1}{z - x} \right] + H(x, -) \frac{R(x, z)}{z - x}$$

and writing:

$$" \tau(z, +) " \quad \text{for} \quad \lim_{x \rightarrow z} \frac{T(z, x) - 1}{z - x}$$

$$" \rho(z, -) " \quad \text{for} \quad \lim_{x \rightarrow z} \frac{R(x, z)}{z - x}$$

and

$$\frac{dH(z, +)}{dz} \quad \text{for} \quad \lim_{x \rightarrow z} \frac{H(x, +) - H(z, +)}{x - z} ,$$

the preceding equation yields:

$$\boxed{-\frac{dH(z,+)}{dz} = \tau(z,+)H(z,+) + \rho(z,-)H(z,-)} \quad (7)$$

This is the equation governing the upward irradiance field $H(z,+)$.

The notation in (6) and (7) is designed to point up the fundamental similarity of these equations to the principles of invariance (1) and (2) of Sec. 8.1. On the basis of this similarity (6) and (7) are also called the *local forms* of the principles of invariance, and $\tau(z,\pm)$ and $\rho(z,\pm)$ are the *local transmittance* and *local reflectance factors*, respectively for upward (+) and downward (-) irradiance. The unity of the invariant imbedding approach is underscored when (6) and (7) are compared with (5) and (6) of Sec. 7.1. In fact on the basis of this comparison, we are moved to write:

$$"H(z)" \quad \text{for} \quad (H(z,+), H(z,-)) \quad (8)$$

$$"K(z)" \quad \text{for} \quad \begin{bmatrix} -\tau(z,+) & \rho(z,+) \\ -\rho(z,-) & \tau(z,-) \end{bmatrix} \quad (9)$$

so that (6) and (7) can be written:

$$\boxed{\frac{dH(z)}{dz} = H(z)K(z)} \quad (10)$$

Equation (10) is the *vector form* of the general irradiance equations, and is the irradiance counterpart to (9) of Sec. 7.1.

The practical distinction between (10) above and (9) of Sec. 7.1 should be kept firmly in mind: (10) is a vector equation whose components are *numbers* while (9) of Sec. 7.1 is a vector equation whose components are *functions* and thus one level higher in the conceptual hierarchy. However, both the functional and numerical components obey many of the same algebraic relations and, generally speaking, *whenever a functional equation deduced from (9) of Sec. 7.1 is valid, then there exists a corresponding valid counterpart deducible from (10)*. It is almost as if the local forms of the theorems of irradiance fields are the one dimensional shadows of the corresponding theorems of radiance fields; similarly for deductions from the *global* forms of the principles of invariance and their irradiance correspondents. Caution should be exercised in attempting to extend results the other way, i.e., from the irradiance level (10) to the radiance level (9) of Sec. 7.1, or from (1) and (2) of Sec. 8.1 back to the principles in example 3 of Sec. 3.7. While no universal rule exists to guide extensions from the irradiance to the radiance level,

it is clear that if essential use is made of the commutativity of the *numerical factors* R and T while gaining a result, then the associated result need not exist on the radiance level, since commutativity of the R and T *operators* does not hold in general. Some examples using this observation were discussed in Sec. 7.13 (see (91) of Sec. 7.14).

We shall turn to the applications of (10) in the discussions of Sec. 8.7; for the present we continue to explore its analytic structure. Before going on to do so, we pause and note one rather interesting similarity between (10) above and the fundamental dynamical equation of quantum mechanics:

$$i\hbar \frac{d}{dt} |\psi\rangle = |\psi\rangle \hat{H} \quad (10a)$$

where $|\psi\rangle$ is a state vector which pairs with our $H(z)$ and \hat{H} (or $-(i/\hbar)\hat{H}$) is the Hamiltonian matrix operator which pairs with our $\mathcal{K}(z)$. As a result of this pairing we see that the mathematics of time-dependent atomic systems is homomorphic to (i.e., of the same kind as) that for steady irradiance fields in stratified media (cf. also (46) of Sec. 8.6 and the remarks following (91) of Sec. 3.7). One more connection between (10) above (and its generalization (46) of Sec. 8.6) and the mathematical structure of different fields of physics may be noted. This concerns the formulations by Brillouin* of the transmission line equations for two phase and polyphase electric fields. The use of Pauli and Dirac matrices to compactly represent circuit equations for such fields can evidently be carried over with only slight modifications to the radiative transfer context. However, in the present work we shall develop the algebra of radiative transfer on the basis of the invariant imbedding point of view introduced in Chapter 7.

8.3 Two-Flow Equations: Undecomposed Form

We now add another block to the foundations for the model constructions of irradiance fields to be given below by deriving the classical two-flow equations for irradiance which correspond to (6) and (7) of Sec. 8.2. The primary distinction between (6) and (7) above and the two-flow equations below lies in the structure of the coefficients of $H(z, \pm)$ in the respective equations. In order to arrive at the two-flow equations we shall analyze the local transmittance factors $\tau(z, \pm)$ and the local reflectance factors $\rho(z, \pm)$ into further parts and relate these parts directly to the radiance distributions and the inherent optical properties α, σ of the optical medium $X(a, b)$. In this way we will be able to make direct contact with certain well-known models of irradiance fields starting with the Schuster progenitors of classical radiative transfer theory, down through the variations wrought by Ryde, Gurevic, Duntley and others during the decades that followed. The historical details of

*Brillouin, L., *Wave Propagation in Periodic Structures*. Dover Publications, New York (1953).