it is clear that if essential use is made of the commutativity of the *numerical factors* R and T while gaining a result, then the associated result need not exist on the radiance level, since commutativity of the R and T *operators* does not hold in general. Some examples using this observation were discussed in Sec. 7.13 (see (91) of Sec. 7.14).

We shall turn to the applications of (10) in the discussions of Sec. 8.7; for the present we continue to explore its analytic structure. Before going on to do so, we pause and note one rather interesting similarity between (10) above and the fundamental dynamical equation of quantum mechanics:

$$i\hbar \frac{d\,|\psi>}{dt} = |\psi>\hat{H} \qquad (10a)$$

where $|\psi>$ is a state vector which pairs with our $H(z)$ and H (or $-(i/\hbar)\hat{H}$) is the Hamiltonian matrix operator which pairs with our $\mathcal{H}(z)$. As a result of this pairing we see that the mathematics of time-dependent atomic systems is homomorphic to (i.e., of the same kind as) that for steady irradiance fields in stratified media (cf. also (46) of Sec. 8.6 and the remarks following (91) of Sec. 3.7). One more connection between (10) above (and its generalization (46) of Sec. 8.6) and the mathematical structure of different fields of physics may be noted. This concerns the formulations by Brillouin* of the transmission line equations for two phase and polyphase electric fields. The use of Pauli and Dirac matrices to compactly represent circuit equations for such fields can evidently be carried over with only slight modifications to the radiative transfer context. However, in the present work we shall develop the algebra of radiative transfer on the basis of the invariant imbedding point of view introduced in Chapter 7.

8.3 Two-Flow Equations: Undecomposed Form

We now add another block to the foundations for the model constructions of irradiance fields to be given below by deriving the classical two-flow equations for irradiance which correspond to (6) and (7) of Sec. 8.2. The primary distinction between (6) and (7) above and the two-flow equations below lies in the structure of the coefficients of $H(z,\pm)$ in the respective equations. In order to arrive at the two-flow equations we shall analyze the local transmittance factors $\tau(z,\pm)$ and the local reflectance factors $\rho(z,\pm)$ into further parts and relate these parts directly to the radiance distributions and the inherent optical properties α, σ of the optical medium $X(a,b)$. In this way we will be able to make direct contact with certain well-known models of irradiance fields starting with the Schuster progenitors of classical radiative transfer theory, down through the variations wrought by Ryde, Gurevic, Duntley and others during the decades that followed. The historical details of

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*Brillouin, L., *Wave Propagation in Periodic Structures*. Dover Publications, New York (1953).

the manifold forms of the two-flow equations are reserved for discussion in the bibliographic notes appended to this chapter. For the present we go on to a modern development and analysis of the two-flow equations.

Our starting point for the present derivations is the steady state equation of transfer for radiance in a source-free isotropic stratified optical medium, i.e., we begin with (3) of Sec. 3.15 in the form:*

$$\xi\cdot\nabla N(z,\xi) = -\alpha(z)N(z,\xi) + \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad . \tag{1}$$

Here "z" denotes depth in the present stratified plane parallel medium $X(a,b)$, and the term "$\xi\cdot\nabla$" is defined in (6) of Sec. 3.15. If ever sources are to be taken into account, one need only add "$N_\eta(z,\xi)$" to the right side of (1). The net result on all subsequent equations is the addition of irradiance terms $H_\eta(z,\pm)$ to the right sides of the respective equations. The time-dependent version of (1) is obtained, as usual, by adding the time derivative term $(1/v)\,\partial N/\partial t$ to the left side.

The next step of the derivation is to integrate the terms of each side of (1) over the set Ξ_+ of upward directions. Taking the terms one by one, we begin with the derivative term:

$$\int_{\Xi_+} \xi\cdot\nabla N(z,)\,d\Omega(\xi) = -\int_{\Xi_+} \xi\cdot\mathbf{k}\,\frac{d}{dz}\,N(z,\xi)\,d\Omega(\xi)$$

$$= -\frac{d}{dz}\int_{\Xi_+} \xi\cdot\mathbf{k}N(z,\xi)\,d\Omega(\xi) = -\frac{dH(z,+)}{dz} \tag{2}$$

The first equality rests on the form of ∇ in a Cartesian coordinate frame in which z is measured positive in the direction $-\mathbf{k}$, as is the case in the terrestrial reference frame for hydrologic optics presently in use.** The stratified light field condition is also used to obtain the first equality, for under that condition the x and y derivatives of N vanish. The last equality is based on (9) of Sec. 2.4 and (8) of Sec. 2.5.

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*The isotropic assumption plays no essential role in this derivation in the sense that the structure of the resultant formulas (9) and (10) below are the same for the anisotropic case.

**If ∇ is to be used in a coordinate frame other than Cartesian then, in general (2) yields $\nabla\cdot H(x,+)$, where $H(x,+)$ is that contribution to $H(x)$ by radiances in the directions of Ξ_+. See [221].

Starting now on the right side of (1), we integrate the linear term over Ξ_+ :

$$\int_{\Xi_+} \alpha(z)N(z,\xi)d\Omega(\xi) = \alpha(z)h(z,+) \tag{3}$$

in which we have used (7) and (11) of Sec. 2.7. Next, the integrated integral term in (1) becomes:

$$\begin{aligned}\int_{\Xi_+} N_*(z,\xi)d\Omega(\xi) &= \\ &= \int_{\Xi_+} \left[\int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)d\Omega(\xi')\right] d\Omega(\xi) \\ &= \int_{\Xi_+} \left[\int_{\Xi_+} N(z,\xi')\sigma(z;\xi';\xi)d\Omega(\xi')\right. \\ &\quad \left. + \int_{\Xi_-} N(z,\xi')\sigma(z;\xi';\xi)d\Omega(\xi')\right] d\Omega(\xi)\end{aligned} \tag{4}$$

In the last equality, we have merely split the integration over Ξ into two parts: over Ξ_+ and over Ξ_- .

Equations (2), (3), and (4) are as far as we can go, blindly and mechanically. The next step in the derivations of the two-flow equations requires a strong sense of direction of the goal, namely two differential equations for the irradiances $H(z,\pm)$. Now, equation (2) shows us we are on the right track; but equation (3) shows us that we must perform some analytic legerdemain in order to obtain $H(z,+)$ from $h(z,+)$; and equation (4) shows us that the requisite analytic trickery must be thoroughgoing and boldly done. Some experimentation shows that we may profitably write:

$$\text{"}D(z,\pm)\text{"} \quad \text{for} \quad \frac{h(z,\pm)}{H(z,\pm)} \tag{5}$$

$$\text{"}\alpha(z,\pm)\text{"} \quad \text{for} \quad \alpha(z)D(z,\pm) \tag{6}$$

and:

$$\text{"}f(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H(z,\pm)} \int_{\Xi_\pm} \left[\int_{\Xi_\pm} N(z,\xi')\sigma(z;\xi';\xi)d\Omega(\xi')\right] d\Omega(\xi) \tag{7}$$

$$\text{"}b(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H(z,\pm)} \int_{\Xi_{\mp}} \left[\int_{\Xi_{\pm}} N(z,\xi')\sigma(z,\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{8}$$

It follows at once that (3) becomes:

$$\int_{\Xi_+} \alpha(z)N(z,\xi)\,d\Omega(\xi) = \alpha(z,+)H(z,+)$$

and (4) becomes:

$$\int_{\Xi_+} N_*(z,\xi)\,d\Omega(\xi) = f(z,+)H(z,+) + b(z,-)H(z,-) \quad ,$$

so that with (2), these results assemble into the requisite equation for $H(z,+)$:

$$\boxed{-\frac{dH(z,+)}{dz} = [f(z,+) - \alpha(z,+)]H(z,+) + b(z,-)H(z,-)} \tag{9}$$

Integrating (1) over Ξ_- , and using similar tactics to those described, we have:

$$\boxed{\frac{dH(z,-)}{dz} = [f(z,-) - \alpha(z,-)]H(z,-) + b(z,+)H(z,+)} \tag{10}$$

Equations (9) and (10) are the requisite two-flow equations for the irradiance fields $H(z,\pm)$. Comparison with (6) and (7) of Sec. 8.2 yields the important connections:

$$\boxed{\tau(z,\pm) = f(z,\pm) - \alpha(z,\pm)} \tag{11}$$

$$\boxed{\rho(z,\pm) = b(z,\pm)} \tag{12}$$

$f(z,\pm)$ and $b(z,\pm)$ are respectively, the *forward* and *backward scattering functions* for the irradiances $H(z,\pm)$; $\alpha(z,\pm)$ are the *attenuation functions* for $H(z,\pm)$, respectively. The functions $D(z,\pm)$ are called the *distribution functions* for $H(z,\pm)$. We also note the interesting connection between the values $\alpha(z,\pm)$ and those of $a(z,\pm)$ and $s(z,\pm)$ where we have written:

$$\text{"}a(z,\pm)\text{"} \quad \text{for} \quad a(z)D(z,\pm) \tag{13}$$

$$\text{"}s(z,\pm)\text{"} \quad \text{for} \quad s(z)D(z,\pm) \quad . \tag{14}$$

The connection of interest is:

$$\alpha(z,\pm) = a(z,\pm) + s(z,\pm) \quad , \tag{15}$$

which follows at once from the definition of the volume absorption function $a(z,\xi)$ given in (4) of Sec. 4.2; $a(z,\pm)$ and $s(z,\pm)$ are, respectively the *absorption* and *total scattering functions* for the irradiances $H(z,\pm)$. Equation (15) parallels the basic connection:

$$\alpha(z) = a(z) + s(z) \tag{16}$$

among the volume attenuation, absorption and total scattering functions in $X(a,b)$. Furthermore, from (3) of Sec. 4.2 and (7) and (8) above we have:

$$s(z,\pm) = f(z,\pm) + b(z,\pm) \tag{17}$$

which, combined with (15) yields:

$$\boxed{\alpha(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm)} \tag{18}$$

Equation (18) shows that the attenuation function for $H(z,\pm)$ is generally the sum of three terms: the absorption, forward and backward scattering functions. Using this connection, the two-flow equations (9) and (10) may be cast into their alternate forms:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [a(z,\pm) + b(z,\pm)]H(z,\pm) + b(z,\mp)H(z,\mp)} \tag{19}$$

We pause to examine the meanings of the terms in (19) and to sample the strong intuitive flavor of the two-flow equations. Choosing the upper signs in (19), we have the differential equation for $H(z,+)$ which states that the rate of change of the upward flow of radiant energy per unit area consists of three terms representing the simultaneous activity of the following three processes in $X(a,b)$:

(i) The decrease of $H(z,+)$ by absorption of $H(z,+)$ per unit length of travel.

(ii) The decrease of $H(z,+)$ by backscattering of $H(z,+)$ per unit length of travel.

(iii) The increase of $H(z,+)$ by backscattering of $H(z,-)$ per unit length of travel.

A similar interpretation may be assigned to the downward irradiance field $H(z,-)$ by replacing "+" by "-" throughout (i)-(iii) above. The minus sign before the derivative of $H(z,+)$ adjusts the vertical measurements to be positive upward for that equation, and is the vestige of the general convention to measure r positive in the direction ξ in the general equation of transfer (3) of Sec. 3.15.

Equilibrium Form of the Two-Flow Equations

We can cast the two-flow equations (19) into a form which points up even more strikingly the intuitive features of the irradiance field and which underscores still further their similarities to the radiance equations. We have in mind the introduction of the irradiance counterparts to the equilibrium form of the equation of transfer (4) of Sec. 4.3. Toward this end let us write:

$$\text{"}H_q(z,\pm)\text{"} \quad \text{for} \quad \frac{b(z,\mp)H(z,\mp)}{a(z,\pm)+b(z,\pm)} \tag{20}$$

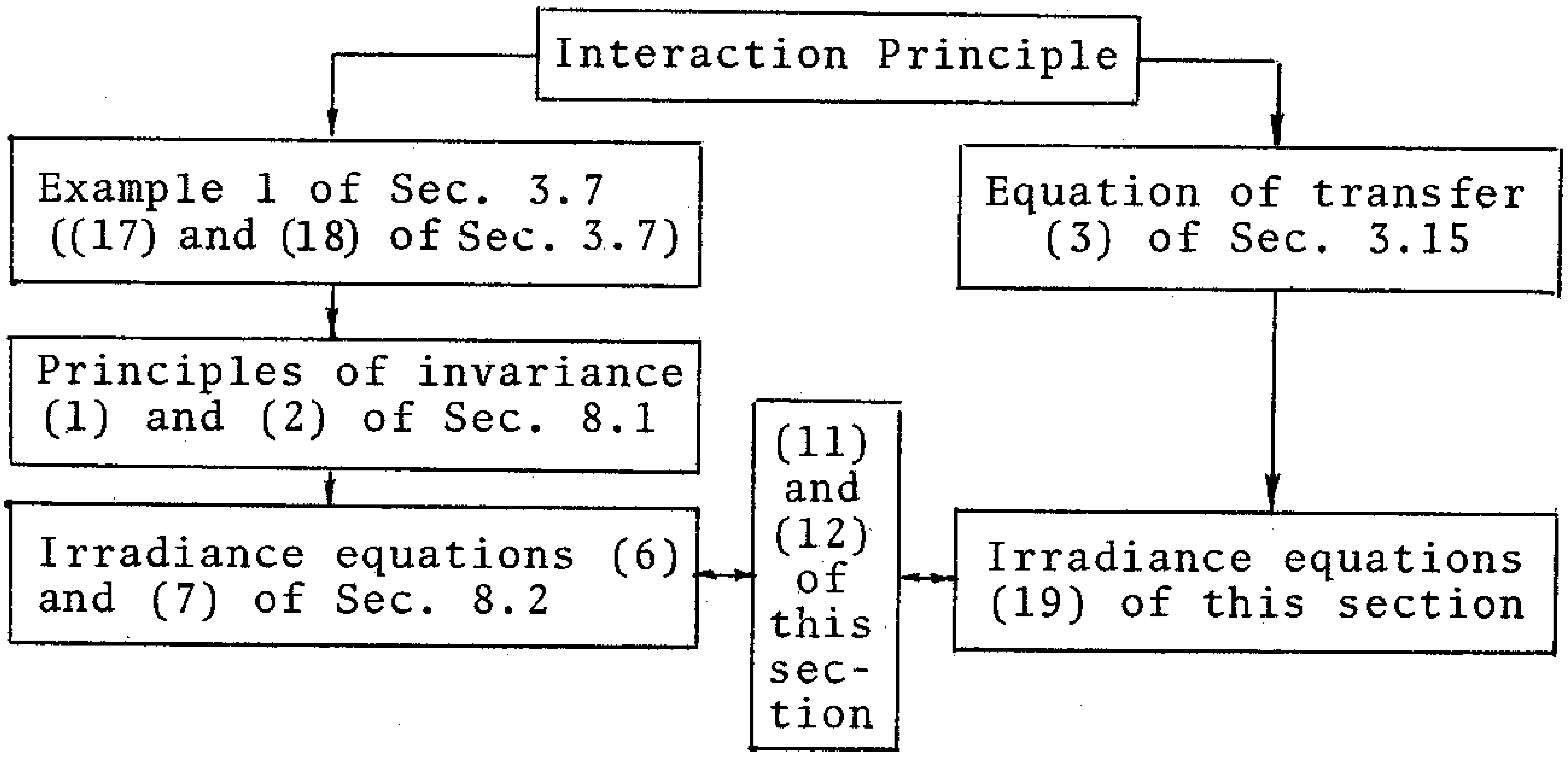
so that (19) becomes:

$$\mp\frac{dH(z,\pm)}{dz} = -\,[a(z,\pm)+b(z,\pm)]\,[H(z,\pm)-H_q(z,\pm)] \tag{21}$$

$H_q(z,\pm)$ is the *equilibrium irradiance* for $H(z,\pm)$. The reason for this nomenclature is obvious on inspection of (21). Consider $H(z,-)$: If $H(z,-) < H_q(z,-)$, then the derivative of $H(z,-)$ is positive and $H(z,-)$ is increasing toward $H_q(z,+)$. On the other hand, if $H_q(z,-) < H(z,-)$, then the derivative of $H(z,-)$ is negative and $H(z,-)$ is decreasing toward $H_q(z,-)$. Thus $H(z,-)$ relentlessly pursues $H_q(z,-)$. Unlike the race between N and N_q along a horizontal line of sight, that between $H(z,-)$ and $H_q(z,-)$ is never finished in real hydrosols. Reasons for this will become clear during our discussions in Chapters 9, 10, and 11.

Ontogeny of the Two-Flow Equations

Before going on to further derivations and to the solution procedures for the two-flow equations we pause to take note, for the student of radiative transfer theory, of the ontogenetic features of the two-flow equations. Historically, they trace back to Schuster's basic paper of 1905 (Ref. [279]). Logically, they rest on the interaction principle of Chapter 3. The logical route to them may be traversed in two distinct ways. The following diagram illustrates these routes schematically:



The bridge between the two forms of the irradiance equations is the pair of relations (11) and (12) which can be rigorously proved from their definitions and the interaction principle. However the simple visual match between the set (9) and (10) and (6) and (7) of Sec. 8.2 which suggests (11) and (12) will suffice for the purposes of this work. Interested students may attempt the direct proof of (11) and (12). It should be noted that the rigorous proof is not trivial and, if successfully done, has important related results in alternate modes of approach to radiative transfer theory (e.g., see step seven in Sec. 126 of Ref. [251]).

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8.4 Two-Flow Equations: Decomposed Form

In this section we retrace the main steps of the preceding section with the goal in mind of deriving the two-flow equations for the decomposed light field in $X(a,b)$. The immediate basis for the derivation rests in (7) of Sec. 5.2. See also (19) through (22) of Sec. 5.1 wherein are also defined the notions underlying the idea of a decomposed light field. A suitable prerequisite for the present derivations are the discussions between (1) and (7) of Sec. 5.2, and between (56) and (62) of Sec. 6.6. The ultimate basis of the present discussion is (5) of Sec. 3.13.

Starting with (7) of Sec. 5.2:

$$\xi\cdot\nabla N_*(z,\xi) = -\alpha(z)H_*(z,\xi) + \int_{\Xi} N_*(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') + N^1_*(z,\xi) \tag{1}$$

where

$$N^1_*(z,\xi) = \int_{\Xi} N^0(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad , \tag{2}$$

we integrate each side of (1) over Ξ_+. The derivative term becomes:

$$\int_{\Xi_+} \xi\cdot\nabla N_*(z,\xi)\,d\Omega(\xi) \doteq -\frac{d}{dz}\int_{\Xi_+} \xi\cdot\mathbf{k}N_*(z,\xi)\,d\Omega(\xi) \quad .$$

This motivates us to write:

$$\text{"}H_*(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi\cdot\mathbf{k}\,|\,N_*(z,\xi)\,d\Omega(\xi) \tag{3}$$

We call $H_*(z,\pm)$ the *diffuse upward (+) or downward (-) irradiance.* Similarly for $a \le z \le b$, we write:

$$\text{"}H^0(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi\cdot\mathbf{k}\,|\,N^0(z,\xi)\,d\Omega(\xi) \quad , \tag{4}$$