The bridge between the two forms of the irradiance equations is the pair of relations (11) and (12) which can be rigorously proved from their definitions and the interaction principle. However the simple visual match between the set (9) and (10) and (6) and (7) of Sec. 8.2 which suggests (11) and (12) will suffice for the purposes of this work. Interested students may attempt the direct proof of (11) and (12). It should be noted that the rigorous proof is not trivial and, if successfully done, has important related results in alternate modes of approach to radiative transfer theory (e.g., see step seven in Sec. 126 of Ref. [251]).

8.4 Two-Flow Equations: Decomposed Form

In this section we retrace the main steps of the preceding section with the goal in mind of deriving the two-flow equations for the decomposed light field in $X(a,b)$. The immediate basis for the derivation rests in (7) of Sec. 5.2. See also (19) through (22) of Sec. 5.1 wherein are also defined the notions underlying the idea of a decomposed light field. A suitable prerequisite for the present derivations are the discussions between (1) and (7) of Sec. 5.2, and between (56) and (62) of Sec. 6.6. The ultimate basis of the present discussion is (5) of Sec. 3.13.

Starting with (7) of Sec. 5.2:

$$\xi\cdot\nabla N^*(z,\xi) = -\alpha(z)H^*(z,\xi) + \int_\Xi N^*(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') + N^1_*(z,\xi) \qquad (1)$$

where

$$N^1_*(z,\xi) = \int_\Xi N^o(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad , \qquad (2)$$

we integrate each side of (1) over Ξ_+. The derivative term becomes:

$$\int_{\Xi_+} \xi\cdot\nabla N^*(z,\xi)\,d\Omega(\xi) = -\frac{d}{dz}\int_{\Xi_+} \xi\cdot\mathbf{k}N^*(z,\xi)\,d\Omega(\xi) \quad .$$

This motivates us to write:

$$\text{"}H^*(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi\cdot\mathbf{k}\,|\, N^*(z,\xi)\,d\Omega(\xi) \qquad (3)$$

We call $H^*(z,\pm)$ the *diffuse upward (+) or downward (-) irradiance.* Similarly for $a \le z \le b$, we write:

$$\text{"}H^o(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi\cdot\mathbf{k}\,|\, N^o(z,\xi)\,d\Omega(\xi) \quad , \qquad (4)$$

so that by (5) of Sec. 3.13:

$$\boxed{H(z,\pm) = H^{o}(z,\pm) + H^{*}(z,\pm)} \tag{5}$$

We call $H^{o}(z,\pm)$ the *residual* (or reduced) *upward* (+) or *downward* (-) irradiance. Equation (5) exhibits the *decomposition of the irradiance fields*. The definitions (5) through (8) of Sec. 8.3 can now be repeated for the *diffuse* and *residual* irradiances by the simple expedient of placing a star (*) or zero (o) superscript on the radiometric quantities involved For example in the case of diffuse irradiance we write:

$$\text{"}D^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{h^{*}(z,\pm)}{H^{*}(z,\pm)} \tag{6}$$

$$\text{"}\alpha^{*}(z,\pm)\text{"} \quad \text{for} \quad \alpha(z)D^{*}(z,\pm) \tag{7}$$

$$\text{"}f^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H^{*}(z,\pm)} \int_{\Xi_{\pm}} \left[\int_{\Xi_{\pm}} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{8}$$

$$\text{"}b^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H^{*}(z,\pm)} \int_{\Xi_{\mp}} \left[\int_{\Xi_{\pm}} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{9}$$

Of course in (6) we have written:

$$\text{"}h^{*}(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_{\pm}} N^{*}(z,\xi)\,d\Omega(\xi) \tag{10}$$

In this regard, see (19) and (22) of Sec. 5.1, which prevent unnecessary listings of definitions associated with n-ary concepts.

Continuing with the methodical integration of the terms of (1) over Ξ_{+}, the first term on the right becomes:

$$-\int_{\Xi_{+}} \alpha(z)N^{*}(z,\xi)\,d\Omega(\xi) = -\,\alpha(z)h^{*}(z,+) = -\,\alpha(z,+)H^{*}(z,+) \tag{11}$$

The integral term becomes:

$$\int_{\Xi_{+}} \left[\int_{\Xi} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) = f^{*}(z,+)H^{*}(z,+) + b^{*}(z,-)H^{*}(z,-) \quad .$$

Finally, in like manner (and using the notation conventions agreed upon above):

$$\int_{\Xi_+} \left[\int_{\Xi} N^{\circ}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] \, d\Omega(\xi)$$

$$= f^{\circ}(z,+)H^{\circ}(z,+) + b^{\circ}(z,-)H^{\circ}(z,-) \quad .$$

Assembling these results, we have:

$$- \frac{dH^*(z,+)}{dz} = [f^*(z,+) - \alpha^*(z,+)]H^*(z,+) + b^*(z,-)H^*(z,-) + f^{\circ}(z,+)H^{\circ}(z,+) + b^{\circ}(z,-)H^{\circ}(z,-) \tag{12}$$

In a similar manner we derive:

$$\frac{dH^*(z,-)}{dz} = [f^*(z,-) - \alpha^*(z,-)]H^*(z,-) + b^*(z,+)H^*(z,+) + f^{\circ}(z,-)H^{\circ}(z,-) + b^{\circ}(z,+)H^{\circ}(z,+) \tag{13}$$

Analogously to (15) and (17) of Sec. 8.3, we have:

$$\alpha^*(z,\pm) = a^*(z,\pm) + s^*(z,\pm) \tag{14}$$

$$\alpha^{\circ}(z,\pm) = a^{\circ}(z,\pm) + s^{\circ}(z,\pm) \tag{15}$$

$$s^*(z,\pm) = f^*(z,\pm) + b^*(z,\pm) \tag{16}$$

$$s^{\circ}(z,\pm) = f^{\circ}(z,\pm) + b^{\circ}(z,\pm) \tag{17}$$

From the preceding four equations, we have:

$$\alpha^*(z,\pm) = a^*(z,\pm) + f^*(z,\pm) + b^*(z,\pm) \tag{18}$$

$$\alpha^{\circ}(z,\pm) = a^{\circ}(z,\pm) + f^{\circ}(z,\pm) + b^{\circ}(z,\pm) \tag{19}$$

In view of (18), equations (12) and (13) may be written alternatively as:

$$\mp \frac{dH^*(z,\pm)}{dz} = - [a^*(z,\pm) + b^*(z,\pm)]H^*(z,\pm) + b^*(z,\mp)H^*(z,\mp) + f^{\circ}(z,\pm)H^{\circ}(z,\pm) + b^{\circ}(z,\mp)H^{\circ}(z,\mp) \quad . \tag{20}$$

The interpretation of the terms of the differential equations (20) for the diffuse component of the light field is analogous to that for (19) of Sec. 8.3. Now, in addition to the increase of $H^*(z,\pm)$ from backscattering of the other stream, there are two additional terms representing increases: the forward and backward scattering effects of the residual irradiance flows.

To round out the system (20), we deduce from (2) of Sec. 5.2 the following pair of equations governing the residual irradiance fields:

$$\boxed{\mp \frac{dH^{o}(z,\pm)}{dz} = -\alpha^{o}(z,\pm)H^{o}(z,\pm)} \quad . \tag{21}$$

This shows that the system (20) is a pair of nonhomogeneous equations with known "source terms" as represented by the scattered residual irradiances. By (21) we can solve for $H^{o}(z,\pm)$ directly, provided $\alpha^{o}(z,\pm)$ are known. These attenuation functions are known once $\alpha(z)$ for $X(a,b)$ is given $a \leq z \leq b$, and the shapes of the incident radiance distributions on the boundaries of $X(a,b)$ are specified. The shape of the incident radiance distribution determines the distribution functions $D^{o}(z,\pm)$. Hence we have at once from (21):

$$H^{o}(y,-) = H^{o}(x,-) \exp\left\{ - \int_x^y \alpha^{o}(z',-)dz' \right\} \tag{22}$$

$$H^{o}(y,+) = H^{o}(z,+) \exp\left\{ - \int_y^z \alpha^{o}(z',+)dz' \right\} \tag{23}$$

for $a \leq x \leq y \leq z \leq b$. Let us write:

$$\text{"}T^{o}(x,y)\text{"} \quad \text{for} \quad \exp\left\{ - \int_x^y \alpha^{o}(z',-)dz' \right\} \tag{24}$$

and:

$$\text{"}T^{o}(y,x)\text{"} \quad \text{for} \quad \exp\left\{ - \int_x^y \alpha^{o}(z',+)dz' \right\} \tag{25}$$

whenever $a \leq x \leq y \leq b$. $T^{o}(x,y)$ is the *transmittance factor for residual irradiance.* From these definitions we can construct the transmittance factors for diffuse irradiance. For we need only write:

$$\text{"}T^*(x,y)\text{"} \quad \text{for} \quad T(x,y) - T^{o}(x,y) \quad , \tag{26}$$

whenever $a \leq x \leq y \leq b$. This defines the *transmittance factor for diffuse downward irradiance.* A similar definition is readily phrased for the upward diffuse irradiance. Definition (26) is the irradiance counterpart to (41) of Sec. 7.1. From (26) follows:

$$\boxed{T(x,y) = T^{\circ}(x,y) + T^{*}(x,y)} \qquad (27)$$

which is the decomposition of the transmittance factor T(x,y) into its residual and diffuse parts.

Principles of Invariance for Diffuse Irradiance

Starting with the principles of invariance (1) and (2) of Sec. 8.1 for irradiance, and using the decomposition (27) of the irradiance transmittance factors along with the irradiance decomposition (5), we can deduce the principles of invariance for the diffuse irradiance field. Thus, from (1) of Sec. 8.1:

$$H^{\circ}(y,+) + H^{*}(y,+) = (H^{\circ}(z,+) + H^{*}(z,+))(T^{\circ}(z,y) + T^{*}(z,y)) + (H^{\circ}(y,-) + H^{*}(y,-))R(y,z)$$

and using (23), we have:

$$\boxed{\begin{aligned} \text{I}^{*} \quad H^{*}(y,+) &= H^{*}(z,+)T(z,y) + H^{*}(y,-)R(y,z) \\ &\quad + H^{\circ}(z,+)T^{*}(z,y) + H^{\circ}(y,-)R(y,z) \end{aligned}} \qquad (28)$$

Similarly:

$$\boxed{\begin{aligned} \text{II}^{*} \quad H^{*}(y,-) &= H^{*}(x,-)T(x,y) + H^{*}(y,+)R(y,x) \\ &\quad + H^{\circ}(z,-)T^{*}(x,y) + H^{\circ}(y,+)R(y,x) \end{aligned}} \qquad (29)$$

Errata

where $a \le x \le y \le z \le b$. These are the two main principles of invariance for diffuse irradiance. The remaining two are obtained from I* and II* in the usual manner. Thus in I* let $y = a$ and $z = b$:

$$\boxed{\text{III}^{*} \quad H^{*}(a,+) = H^{\circ}(b,+)T^{*}(b,a) + H^{\circ}(a,-)R(a,b)} \qquad (30)$$

and setting $x = a$, $y = b$ in II*:

$$\boxed{\text{IV}^{*} \quad H^{*}(b,-) = H^{\circ}(a,-)T^{*}(a,b) + H^{\circ}(b,+)R(b,a)} \qquad (31)$$

In obtaining III* and IV* from I* and II* use was made of the convention, stated in the discussion of (30) of Sec. 7.1, that $N^{*}_{+}(b) = 0$ and $N^{*}_{-}(a) = 0$. These principles for $H^{*}(y,\pm)$ should be compared with I* through IV* of Sec. 7.1 (starting with (52) of Sec. 7.1).

Classical Models for Irradiance Fields

As illustrations of the general ideas developed in this and the preceding section we discuss some of the classical models for irradiance fields studied by early workers in the field of radiative transfer theory.

The classical equations of the two-flow irradiance field as studied by Schuster, Silberstein, Ryde, Duntley, etc., were in each case derived *de novo* for the case of the decomposed light field. By "derived *de novo*" is meant the derivation of the two irradiance equations afresh without any reference to underlying principles of radiative transfer and exactly in the intuitive manner Schuster first derived them in 1905 using physical conservation arguments on radiant flux through a very thin layer of the plane-parallel medium. The geometrical setting was the slab geometry. Furthermore, as far as assumptions on the optical properties of the slab were concerned, homogeneity and isotropy were invariably adopted. The boundaries were nonreflecting, the usual plan being that the equations were first to be solved for this case, and then an interreflection calculation, in infinite series form, was to be undertaken subsequently if desired. The diffuse light field within such models was represented by a uniform radiance distribution over Ξ_+ and Ξ_- at each point of the medium. The incident radiance distributions were invariably of two types: collimated or uniform over Ξ_- or Ξ_+.

Collimated-Diffuse Light Field Models

Our present task is to show how some of the general concepts introduced in Sec. 8.3 and Sec. 8.4 reduce to simpler forms under the applications of the assumptions customarily invoked in the classical two-flow models. For this purpose we assume that collimated radiance is incident on the upper boundary X_a of $X(a,b)$ along the direction ξ^o in Ξ_- and of magnitude N^o. Hence:

$$N^o(a,\xi) = N^o\delta(\xi - \xi^o) \tag{32}$$

for ξ^o, ξ in Ξ_-. No other sources of flux are incident on $X(a,b)$. Furthermore, we assume that the diffuse radiance distribution at each depth z in $X(a,b)$ is uniform over Ξ_+ and over Ξ_-. Hence:

$$N^*(z,\xi) = \begin{cases} f_+(z) & \text{if } \xi \text{ is in } \Xi_+ \\ \\ f_-(z) & \text{if } \xi \text{ is in } \Xi_- \end{cases} \tag{33}$$

for every z, $a \le z \le b$. The actual depth structure of the nonvanishing functions $f_\pm(z)$ is not of interest at the moment.

It follows from (37) of Sec. 7.1 that for ξ in Ξ_-:

$$N^{o}(z,\xi) = N^{o} \exp \left\{ - (z-a)\alpha / \mid \xi \cdot \mathbf{k} \mid \right\} \delta(\xi - \xi^{o}) \quad . \quad (34)$$

The distribution function $D^{o}(z,-)$ associated with $N^{o}(z,\xi)$ is given by:

$$D^{o}(z,-) = \frac{1}{\mid \xi^{o} \cdot \mathbf{k} \mid} \quad , \quad (35)$$

which follows immediately from the definition of $D^{o}(z,-)$. In contrast to $D^{o}(z,-)$, the distribution function for the diffuse irradiance field is, by (3), (6), and (10):

$$D^{*}(z,\pm) = 2 \quad . \quad (36)$$

To see this in more detail, observe first that:

$$\begin{aligned} h^{*}(z,\pm) &= \int_{\Xi_{\pm}} N^{*}(z,\xi)\, d\Omega(\xi) \\ &= \int_{\Xi_{\pm}} f_{\pm}(z)\, d\Omega(\xi) \\ &= f_{\pm}(z) \int_{\Xi_{\pm}} d\Omega(\xi) = 2\pi\, f_{\pm}(z) \quad (37) \end{aligned}$$

Next, note that:

$$\begin{aligned} H^{*}(z,\pm) &= \int_{\Xi_{\pm}} N^{*}(z,\xi) \mid \xi \cdot \mathbf{k} \mid d\Omega(\xi) \\ &= f_{\pm}(z) \int_{\Xi_{\pm}} \mid \xi \cdot \mathbf{k} \mid \, d\Omega(\xi) \\ &= \pi\, f_{\pm}(z) \quad (38) \end{aligned}$$

Then, by (6), (36) follows.

With these particular diffuse radiance distributions extant in $X(a,b)$, we can readily evaluate the attenuation, forward, and backward scattering functions by means of (7) through (9). Thus:

$$\alpha^{*}(z,\pm) = 2\alpha(z) \quad (39)$$

$$f^*(z,\pm) = \frac{1}{\pi} \int_{\Xi_\pm} \left[\int_{\Xi_\pm} \sigma(z;\xi';\xi)\, d\Omega(\xi') \right] d\Omega(\xi) \ , \qquad (40)$$

$$b^*(z,\pm) = \frac{1}{\pi} \int_{\Xi_\mp} \left[\int_{\Xi_\pm} \sigma(z;\xi';\xi)\, d\Omega(\xi') \right] d\Omega(\xi) \ . \qquad (41)$$

The corresponding attenuating functions for the residual irradiance field associated with the collimated radiance distributions are obtained by using the residual-flux analogs to (7) through (19):

$$\alpha^o(z,-) = \alpha(z)/\,|\,\xi^o \cdot \mathbf{k}\,| \ , \qquad (42)$$

$$f^o(z,-) = \frac{1}{|\,\xi^o \cdot \mathbf{k}\,|} \int_{\Xi_-} \sigma(z;\xi^o;\xi)\, d\Omega(\xi) \ , \qquad (43)$$

$$b^o(z,-) = \frac{1}{|\,\xi^o \cdot \mathbf{k}\,|} \int_{\Xi_+} \sigma(z;\xi^o;\xi)\, d\Omega(\xi) \ . \qquad (44)$$

From the residual counterpart to (16):

$$s^o(z,-) = s(z)/\,|\,\xi^o \cdot \mathbf{k}\,| \ , \qquad (45)$$

and similarly:

$$a^o(z,-) = a(z)/\,|\,\xi^o \cdot \mathbf{k}\,| \ . \qquad (46)$$

We have retained the " z " in the notation, even though X(a,b) is assumed homogeneous, to show that arbitrary depth dependence of α , σ does not destroy the otherwise simple relations holding among the attenuation functions for the residual and diffuse components of the light field. In this way we show that it is the *directional* structures of the radiance distributions which complicate the form of the associated attenuating functions, and not the depth dependence of the attenuating functions.

The early principal work on the attenuating functions was done by Ryde and Cooper in Ref. [270]. However, no account was taken of the dependence of these functions on the intricate directional structure of the radiance distributions in real media. As can be seen by an examination of (8) and (9), all dependence of f^* and b^* on N^* is wiped away by the assumption (33). Furthermore in the absence of a rigorous general definition of f and b , as given in (7) and (8) of Sec. 8.3 or (8) and (9) of this section, one was unable to deduce with rigor the various important properties of these functions, and occasionally inaccuracies arose. For example, one of Ryde's principal conclusions about the properties of f and b was that (in the present notation):

$$f^*(z,\pm) + b^*(z,\pm) = f^\circ(z,\pm) + b^\circ(z,\pm)$$

i.e., that:

$$s^*(z,\pm) = s^\circ(z,\pm) \quad .$$

However, it is at once clear from (14) of Sec. 8.3 that for the diffuse and residual cases:

$$s^*(z,\pm) = s(z)D^*(z,\pm) = 2s(z) \tag{47}$$

$$s^\circ(z,\pm) = s(z)D^\circ(z,\pm) = s(z)/\,|\,\xi_o \cdot \mathbf{k}\,| \tag{48}$$

so that $s^\circ = s^*$ if and only if $2 = 1/\,|\,\xi_o \cdot \mathbf{k}\,|$. Thus the distribution function plays an essential role in the correct study of the directional structure of radiance distributions and of the dependencies of the various attenuating functions on the radiance distributions. The connections among the functions $\alpha(z,\pm)$, $s(z,\pm)$, $a(z,\pm)$, $f(z,\pm)$, $b(z,\pm)$ and the directional structure of the light fields was not clearly understood in the early papers of the two-flow theory. The investigators were invariably preoccupied with obtaining a soluble differential equational for some particular special practical problem and sparse attention was addressed to the general logical and physical aspects of the equations.

It was not until the work of Duntley, Ref. [69], that a reasonably clear indication was obtained of the possible existence of a full family of attenuating functions that may be associated with the two-flow equations. Duntley added a new attenuating function to Ryde's list, namely (in our notation) $\alpha^\circ(z,-)$. Under the usual assumptions on N° and N^* it follows from (13) of Sec. 8.3 that:

$$a^\circ(z,-) = a(z)/\,|\,\xi^\circ \cdot \mathbf{k}\,| \tag{49}$$

and that:

$$a^*(z,-) = 2a(z) \quad . \tag{50}$$

Duntley concluded that $a^\circ(z,-)$ and $a^*(z,-)$ should differ by virtue of the difference in directional structure of N° and N^*. However, the simple connection:

$$a^*(z,-) = [2\,|\,\xi^\circ \cdot \mathbf{k}\,|\,]\; a^\circ(z,-) \tag{51}$$

that existed between these two absorption functions was not given, for lack of availability of the concept of the distribution function.

A connection between a° and a^* was noted by Hulburt in Ref. [114] for the special case where $\xi_o = -\mathbf{k}$, so that $a^*(z,-) = 2a^\circ(z,-)$. The preceding relation (51) and Hulburt's special observation are special cases of the general relations:

$$a^*(z,+) = [D^*(z,\pm)/D^\circ(z,\pm)]\; a^\circ(z,\pm) \tag{52}$$

$$s^*(z,\pm) = [D^*(z,\pm) / D^{\circ}(z,\pm)]s^{\circ}(z,\pm) \tag{53}$$

and these in turn are special cases of:

$$\boxed{\alpha^*(z,\pm) = [D^*(z,\pm) / D^{\circ}(z,\pm)]\alpha^{\circ}(z,\pm)} \tag{54}$$

These observations show the importance of the systematic use of the distribution function concept in the two-flow theory of irradiance fields. We shall use this concept repeatedly in subsequent discussions.

Isotropic Scattering Models

The next class of special two-flow equations to be considered is distinguished by the assumption of isotropy imposed on σ. Thus it is assumed that:

$$\sigma(z;\xi';\xi) = s/4\pi \tag{55}$$

for every ξ', ξ in Ξ. Hence all directional structure of σ is suppressed in such models. Let us examine the consequences of such an assumption. In what follows, we shall allow the directional structure of the light field to be arbitrary. We begin with the undecomposed light field. Using (55) in (7) and (8) of Sec. 8.3 we have:

$$f(z,\pm) = \frac{1}{2} D(z,\pm)s \tag{56}$$

$$b(z,\pm) = \frac{1}{2} D(z,\pm)s \tag{57}$$

Further:

$$a(z,\pm) = D(z,\pm)a \tag{58}$$

$$s(z,\pm) = D(z,\pm)s \tag{59}$$

So that

$$\alpha(z,\pm) = D(z,\pm)\alpha \tag{60}$$

From this we see that, under the assumption (55), the burden of the depth dependence of the light field is carried by the distribution functions. The associated forms of (19) of Sec. 8.3 are:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = -\frac{1}{2}[2a + s]D(z,\pm)H(z,\pm) + \frac{1}{2} sD(z,\pm)H(z,\mp)} \tag{61}$$

Since $D(z,\pm)$ clearly depends on the unknown structure of the radiance distributions in $X(a,b)$, equation (61) as it stands has unknown variable coefficients. If the usual assumption is now made that $D(z,\pm)$ are known (or that they vary in some relatively innocuous manner) then the preceding system is solvable. The original Schuster equations were of the form (61) in which the irradiances were diffuse only and such that $D^*(z,\pm) = 2$, and with source terms $h_\eta(z,\pm)$ added. The transition from (61) to its decomposed form is then attained by simply starring all quantities and adding the appropriate source terms (cf. (20)).

Connections with Diffusion Theory

It is of interest to observe that (61) is just two steps away from a steady state diffusion equation for photons. The first step toward the diffusion equation is taken by adding the members of (61), term by term. Thus the left side becomes:

$$-\frac{d}{dz}\,[H(z,+) - H(z,-)] = \nabla \cdot \mathbf{H}(z)$$

By virtue of (10) through (12) of Sec. 2.8 and the stratified light field condition the x and y derivatives in the divergence operation vanish. Furthermore, the sum of the first terms on the right becomes:

$$-\frac{1}{2}\,\left[2a + s\right]\left[D(z,+)H(z,+) + D(z,-)H(z,-)\right] = -\frac{1}{2}\,\left[2a + s\right]h(z)$$

The sum of the second terms on the right is:

$$\frac{1}{2}\,s\,[D(z,+)H(z,+) + D(z,-)H(z,-)] = \frac{1}{2}\,sh(z)$$

The first step concludes as we reduce these sums still further so that the net result is:

$$\nabla \cdot \mathbf{H}(z) = -ah(z) \quad . \tag{62}$$

The second step toward the diffusion equation is to assume that Fick's law (5) of Sec. 6.5 is valid and of the form:

$$\mathbf{H}(z) = -D\nabla h(z) \quad .$$

This assumption, as we saw in Chapter 6, holds relatively accurately in decomposed light fields. Combining this with (62) we have:

$$D\,\nabla^2 h(z) - a\,h(z) = 0 \tag{63}$$

which is the steady-state emission-free version of (7) of Sec. 6.5.

Now that we have seen the connection between (61) and the diffusion equation, we are led to wonder if (19) of Sec. 8.3 has the same property, i.e., of being connectible to diffusion theory via the two generic steps just taken for (61). The answer is in the affirmative. However, we shall leave this matter until Sec. 8.8 in which the vectorial aspects of the irradiance field will be studied in detail.

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8.5 Two-D Models for Irradiance Fields

We arrive now at the heart of the theory of irradiance fields in natural optical media, namely the two-D models in such media. The "two-D" aspect of the models refers to the radiance distributions over $\Xi_{\pm}$ being assigned fixed shapes so that in turn the distribution functions $D(z,\pm)$ are assigned two arbitrary fixed values: $D(\pm)$ for all z. As a result, the exact two-flow equations (19) of Sec. 8.3 and (12) of Sec. 8.4 have known depth-independent attenuation functions and so the solution procedures of those equations reduce to straightforward applications of the theory of second order ordinary differential equations with constant coefficients. The routine solution procedure of the equations can be enriched with digressions into the physical meanings of the various basic terms arising in the procedure, and we shall expend most of our efforts in the present section in such activity.

On the Depth Dependence of the Attenuating Functions

The first matter we shall take up is the depth dependence of the functions $f(z,\pm)$, $b(z,\pm)$, $\alpha(z,\pm)$, and $s(z,\pm)$, in natural optical media. The observations we shall make are designed to lay the ground work for the two-D theory. Thus our present goal is to show that the attenuation functions listed above vary relatively little with depth in homogeneous media. To begin, we consider the depth behavior of $\alpha(z,\pm)$ and $s(z,\pm)$, as this behavior is relatively simple to analyze into its physical and geometrical components. The case of $\alpha(z,\pm)$ is typical, so that we can limit our attention to it. According to (6) of Sec. 8.3 $\alpha(z,\pm)$ is the product of two factors: $\alpha(z)$ and $D(z,\pm)$. Hence the depth-variation of $\alpha(z,\pm)$ is tied to that of $\alpha(z)$ and $D(z,\pm)$. The depth variation of $\alpha(z)$ constitutes the physical component of the depth variation of $\alpha(z,\pm)$ and the distribution function $D(z,\pm)$ constitutes the geometric component of the depth variation of $\alpha(z,\pm)$. If the medium $X(a,b)$ is homogeneous, then $\alpha(z)$ is independent of z, so that any depth variation of $\alpha(z,\pm)$ is contributed by that of $D(z,\pm)$.

Now it is intuitively clear that there is generally a variation of the shape of the radiance distribution with depth z in natural waters or altitude z in the atmosphere. This variation in shape is reproduced by $D(z,\pm)$ in its own characteristic manner. It turns out that in most natural hydrosols, for example oceans and lakes, the depth variation of $D(z,\pm)$ is quite small, and what variation exists is quite regular or mild with depth. Table 1 exhibits a typical set of cases of the depth variation of $D(z,\pm)$ on clear and overcast days. These values were computed from some experimental data collected by J. E. Tyler (Ref. [298]) taken in a homogeneous part of Pend Oreille Lake, Idaho, for wavelength 480