Here, of course,

$$T(-1,0) = t_-(0) \tag{97}$$

$$R(0,-1) = r_+(0) \tag{98}$$

Furthermore, by (15) of Sec. 7.3, we have:

$$R(0,z_1+1) = R(0,z_1) + \mathcal{R}(0,z_1,z_1+1)T(z_1,0) \tag{99}$$

By the semigroup property of $\mathcal{R}$:

$$\mathcal{R}(0,z_1,z_1+1) = \mathcal{T}(0,z_1,z_1+1)R(z_1,z_1+1) \tag{100}$$

Here we have:

$$R(z_1,z_1+1) = r_-(z_1) \tag{101}$$

and once again by (42) of Sec. 3.7:

$$\mathcal{T}(0,z_1,z_1+1) = T(0,z_1)[1-R(z_1,0)R(z_1,z_1+1)]^{-1} \tag{102}$$

In this way we can completely and systematically analyze the factors on the right in (95) using only the main semigroup properties of $\mathcal{T}$ and $\mathcal{R}$ and the composition formulas, (40) through (43) of Sec. 3.7. The remaining three components of $\mathcal{M}(-1,z,z_1+1)$ may be analyzed similarly.

The reader interested in pursuing the radiative transfer theory of "mixed spaces"--i.e., collections of simultaneously interacting surfaces, slabs, and general media, in three and higher dimensional settings--is referred to Ref. [251], which systematically develops this discipline, known as *discrete-space radiative transfer theory.* The preceding activity is a particular instance of application of discrete-space theory. We shall return to this matter in Example 5 of Sec. 8.7 and systematize the preceding boundary effects procedure for irradiance settings.

8.6 One-D and Many-D Models

We have reached a stage in the development of the two-flow models for irradiance fields where we have enough machinery to readily examine two extreme types of irradiance models used in practice and theory. The first of these types is the *one-D* model which is the modern counterpart to the early Schuster-type equations associated with light fields having spherical radiance distributions. The other extreme is the *many-D* model for irradiance fields which pushes the two-flow irradiance theory to its logical limit and can serve as the bridge back to the domain of ideas developed in Chapter 7. The explicit formulations of each of these extreme types is of importance to the subsequent developments of this work and therefore will be studied in some detail.

One-D Models for Undecomposed Irradiance Fields

The one-D model for undecomposed irradiance fields takes as its foundation (8) of Sec. 8.5 and adopts the following assumptions: Let $X(a,b)$ be a plane-parallel medium such that:

(i) $a = 0$, $b = z_1$ with $0 \le z_1 \le \infty$

(ii) $X(0,z_1)$ is separable and its boundaries are transparent.

(iii) The radiance distributions in $X(0,z_1)$ satisfy condition (1) of Sec. 8.5 with $c\, g_+(\xi) = g_-(\xi)$, for some positive constant c.

(iv) $H(0,-)$ is an arbitrary irradiance, and $H(z_1,+) = 0$.

A comparison of these assumptions with the corresponding set (i) through (iv) in Sec. 8.5 just before (8) of that section shows that the main change is in the uniformization of the radiance distribution: It is now to have the same shape over Ξ_+ and Ξ_-. The shape may be spherical, elliptical, or any arbitrary shape imaginable; however, the crucial point is that the shapes are the same in the upper and lower hemispheres of Ξ.

The immediate consequences of assumption (iii) are the following. First we have:

$$D(+) = D(-) \quad , \tag{1}$$

which comes from (2) of Sec. 8.5. Suppose we write "D" for this common value of $D(\pm)$. As a result, (6), (13), and (14) of Sec. 8.3 imply:

$$\alpha(+) = \alpha(-) = \alpha D \tag{2}$$

$$a(+) = a(-) = aD \tag{3}$$

$$s(+) = s(-) = sD \tag{4}$$

Finally, (6) and (7) of Sec. 8.5 show that:

$$f(+) = f(-) \tag{5}$$

$$b(+) = b(-) \tag{6}$$

We shall write "f" and "b" for these common values of $f(\pm)$, $b(\pm)$. These observations make it clear that the general effect of (iii) is to induce a systematic collapse of complexity throughout Sec. 8.5. Some of the resultant condensations will now be surveyed.

The equations for the irradiance field (8) of Sec. 8.5 now take the form:

$$\mp \frac{dH(z,\pm)}{dz} = - [aD + b]H(z,\pm) + bH(z,\mp) \qquad (7)$$

The first discernible effect of the one-D assumption on the solution of these equations is in the structure of k_+ and k_-, as given in (12) of Sec. 8.5. We now have:

$$k_\pm = \pm [(aD + b)^2 - b^2]^{1/2}$$

$$= \pm [aD(aD + 2b)]^{1/2} \quad . \qquad (8)$$

If the radiance field is assumed spherical (as it occurs very nearly deep inside media with high s/α ratio) then $D = 2$ and:

$$k_\pm = \pm 2[a(a + b)]^{1/2} \quad .$$

If the radiance field is assumed nearly collimated vertically (as it occurs very nearly deep inside media with low s/α ratio) then $D = 1$ and:

$$k_\pm = \pm [a(a + 2b)]^{1/2} \qquad (9)$$

Observe that for the one-D theory, having $a = 0$ requires $k_\pm = 0$. However, in the two-D theory, matters need not be so simple (cf. (13) through (16) of Sec. 8.5). For brevity we shall henceforth write:

"k" for k_+

so that:

$$k_- = - k \qquad (10)$$

Observe that in the one-D theory, whenever $b \le f$, we have from (8):

$$k \le \alpha D \qquad (11)$$

which represents a fundamental inequality throughout radiative transfer theory (cf. (7) of Sec. 6.6 and (26) of Sec. 9.2). We next find that $g_\pm(\pm)$ in (10) and (11) of Sec. 8.5 take the forms:

$$g_+(\pm) = 1 \pm \frac{aD}{k} \qquad (12)$$

$$g_-(\pm) = 1 \mp \frac{aD}{k} \qquad (13)$$

In other words:

$$g_+(+) = g_-(-) = 1 + \frac{aD}{k} \quad , \qquad (14)$$

the common value for which we shall write "g_+"; and we shall write "g_-" for the common value of:

$$g_+(-) = g_-(+) = 1 - \frac{aD}{k} \qquad (15)$$

From this and (9) of Sec. 8.5, we see that the solutions of (7) can be written:

$$\boxed{H(z,\pm) = m_+ g_\pm e^{kz} + m_- g_\mp e^{-kz}} \qquad (16)$$

From this in turn we see that the system determinant $\Delta(z_1)$ in (20) of Sec. 8.5 becomes:

$$\Delta(z_1) = \left(1 + \frac{aD}{k}\right)^2 e^{kz_1} - \left(1 - \frac{aD}{k}\right)^2 e^{-kz_1} \qquad (17)$$

so that:

$$\Delta(0) = \frac{4aD}{k} \quad .$$

The irradiance fields, as given by (22) and (23) of Sec. 8.5, now take the forms:

$$H(z,+) = \frac{H(0,-)}{\Delta(z_1)} \left(1 - \frac{a^2D^2}{k^2}\right) \left[e^{k(z_1-z)} - e^{-k(z_1-z)}\right] \qquad (18)$$

$$H(z,-) = \frac{H(0,-)}{\Delta(z_1)} \Delta(z_1 - z) \qquad (19)$$

An important constant in the one-D theory is the reflectance of an infinitely deep medium $X(0,\infty)$. This constant may be obtained from (26) of Sec. 8.5 by going to the limit as $z_1 \to \infty$. Thus if we write:

$$"R_\infty" \text{ for } \lim_{z_1 \to \infty} R(0,z_1) \qquad (20)$$

and observe that:

$$\lim_{z_1 \to \infty} \frac{e^{kz_1}}{\Delta(z_1)} = \frac{1}{g_+^2} \quad ,$$

then (26) of Sec. 8.5 implies:

$$R_\infty = \frac{g_+ g_-}{g_+ g_+} = \frac{g_-}{g_+} = \frac{1 + \frac{a(-)}{k_-}}{1 - \frac{a(+)}{k_-}} \quad .$$

Alternatively, in view of (14) and (15) this may be written:

$$R_\infty = \frac{1 - \frac{aD}{k}}{1 + \frac{aD}{k}} \doteq \frac{k-aD}{k+aD} \tag{21}$$

Another important feature of the one-D model for $X(0,\infty)$ consists in its representations of $H(z,\pm)$. To obtain these representations, it is sufficient to observe that:

$$\lim_{z_1 \to \infty} \frac{\Delta(z_1 - z)}{\Delta(z_1)} = \lim_{z_1 \to \infty} \frac{g_+^2\, e^{k(z_1 - z)}}{\Delta(z_1)} = e^{-kz}$$

Then (19) supplies the requisite representation for $H(z,-)$ in $X(0,\infty)$:

$$\boxed{H(z,-) = H(0,-)e^{-kz}} \quad , \tag{22}$$

and from (18) and (22);

$$\boxed{H(z,+) = H(z,-)R_\infty = H(0,+)e^{-kz}} \quad , \tag{23}$$

which holds in $X(0,\infty)$.

One-D Model for Internal Sources

The reductions from the two-D to the one-D model in the case of internal sources proceeds generally as the undecomposed light field discussion just completed. Of course, the new reduction features should be the one-D forms of the particular source solutions (56) of Sec. 8.5. To illustrate these reductions, suppose that there is a set of sources uniformly distributed throughout $X(0,z_1)$, so that $h_\eta(z,\pm)$ is independent of depth and that $h_\eta(z,+) = h_\eta(z,-)$. Let "$h_\eta$" denote the constant value. Then (50) of Sec. 8.5 becomes:

$$Y(s,\pm) = (\tau - \rho)h_\eta \quad , \tag{24}$$

so that (56) of Sec. 8.5 reduces to:

$$H_p(z,\pm) = - \frac{h_\eta(\tau - \rho)}{2k} \int_0^z \left[e^{k(z-s)} - e^{-k(z-s)}\right] ds \tag{25}$$

One-D Model for Decomposed Irradiances

The reduction of the two-D equations for decomposed irradiance fields proceeds similarly to the undecomposed case, starting with the same conditions (i) through (iii) now applied to N^*. In particular, the attenuation coefficients become:

$$\alpha^*(+) = \alpha^*(-) = \alpha D^* \tag{26}$$

$$a^*(+) = a^*(-) = aD^* \tag{27}$$

$$s^*(+) = s^*(-) = sD^* \tag{28}$$

where "D^*" denotes the common value:

$$D^*(+) = D^*(-) \tag{29}$$

and

$$f^*(+) = f^*(-) \tag{30}$$

$$b^*(+) = b^*(-) \tag{31}$$

Furthermore:

$$k_{\pm} = \pm\, [aD^*(aD^* + 2b^*)]^{1/2} \tag{32}$$

These equalities show that the decomposed case develops in a manner that is exactly parallel to the undecomposed case, as far as the homogeneous part of the solution goes. The particular part of the solution, as embodied in $C(\mu_o,\pm)$ (cf. (72) of Sec. 8.5), is now such that:

$$C(\mu_o,\pm) = \frac{\sigma_{\pm}(\mu_o)b^* + \sigma_{\mp}(\mu_o)\left[a^* + b^* \mp (\alpha/\mu_o)\right]}{\left(\frac{\alpha^2}{\mu_o^2} - k^2\right)} \tag{33}$$

Finally, we note that:

$$g_+(+) = g_-(-) = 1 + \frac{aD^*}{k} \quad , \tag{34}$$

the common value of which may be denoted by "g_+". Further, we denote by "g_-" the common value of:

$$g_+(-) = g_-(+) = 1 - \frac{aD^*}{k} \tag{35}$$

Equation (20) of Sec. 8.4 then reduces to:

$$\mp \frac{dH^*(z,\pm)}{dz} = -[a^* + b^*]H^*(z,\pm) + b^*H^*(z,\mp) + f^oH^o(z,\pm) + b^oH^o(z,\mp) \tag{36}$$

The general solution of the homogeneous part of (36) is identical in form to (16). The case of $X(0,\infty)$ is also of interest for decomposed irradiance fields. Suppose we denote by "$R_\infty(\mu_o)$" the present counterpart to R_∞ in (20). Then, parallel to (21), we have:

$$R_\infty(\mu_o) = \frac{C(\mu_o,-)}{\mu_o}\left[\frac{g_-}{g_+} - \frac{C(\mu_o,+)}{C(\mu_o,-)}\right]$$

$$= \frac{C(\mu_o,-)}{\mu_o}\left[R_\infty - \frac{C(\mu_o,+)}{C(\mu_o,-)}\right]$$

where R_∞ now uses starred distribution coefficients. The respresentation of the irradiance field in $X(0,\infty)$ is readily obtained by using (73) of Sec. 8.5 to observe that:

$$\lim_{z_1\to\infty} m_+ = 0$$

and:

$$\lim_{z_1\to\infty} m_- = N^o\,\frac{C(\mu_o,-)}{g_+} \quad ,$$

so that (71) reduces to:

$$H^*(z,+) = N^o\left[R_\infty C(\mu_o,-)e^{-kz} - C(\mu_o,+)e^{-\alpha z/\mu_o}\right] \tag{37}$$

$$H^*(z,-) = N^o\,C(\mu_o,-)\left[e^{-kz} - e^{-\alpha z/\mu_o}\right] \tag{38}$$

These equations show that eventually the ratio $H^*(z,+)/H^*(z,-)$ approaches R_∞. This ratio settles to R_∞ as soon as the effects of the collimated light, due to $N^o(0,\xi)$, have died away. It is noteworthy that the structures of (37) and (38) are identical to their counterparts in the full two-D theory (cf. also (1) and (2) of Sec. 10.3).

Many-D Models

There are generally two ways in which to gain insight into a physical theory: to work out numerical examples or special cases of the theory, and second, to generalize the theory to see it from a broader perspective. In this and the preceding section we have gained insight into the two-flow theory by pursuing its special ramifications in the various two-D and one-D models. We close this section with an activity of the second kind, that is, by finding the immediate generalization of the two-D theory to the so-called "many-D" theory.

The point of departure for the two-flow equations was the steady-state source-free version of the transfer equation (1) of Sec. 8.3. We may use this equation once again as the starting point for the present discussions. However, very little extra effort will be expended if instead we start with the time-dependent transfer equation with sources, and derive the many-D theory from that. This we shall do. Let X be an arbitrary optical medium. First we write down:

$$\frac{1}{v}\frac{N(x,\xi,t)}{\partial t} + \xi\cdot\nabla N(x,\xi,t) =$$

$$= -\alpha(x,t)N(x,\xi,t) + \int_{\Xi} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi') + N_{\eta}(x,\xi,t). \tag{39}$$

Next, we partition Ξ into n disjoint subsets, $\Xi_1, \ldots, \Xi_n$, $n \geq 2$. For example if $n = 2$ and $\Xi_1 = \Xi_+$ and $\Xi_2 = \Xi_-$, then we would return to the usual two-flow setting. If $n = 2$ and $\Xi_1 = \{\xi : \xi\cdot\mathbf{n} > 0\} = \Xi(\mathbf{n})$, and $\Xi_2 = \{\xi : \xi\cdot n < 0\} = \Xi(-\mathbf{n})$, then we would have a two-flow setting based on the partition around $\mathbf{n}$ instead of around the unit vector $\mathbf{k}$ along the z-axis. (See Fig. 8.3(a).) Thus, assuming a general partition $\{\Xi_1, \Xi_2, \ldots, \Xi_n\}$ of Ξ with respect to a unit vector $\mathbf{n}$, as in Fig. 8.3(b), we proceed to generalize (5) through (8) of Sec. 8.3 and related concepts to these. First we write: for each j, $1 \leq j \leq n$:

$$\text{"}h_j(x,t)\text{" for } \int_{\Xi_j} N(x,\xi,t)\,d\Omega(\xi) \tag{40}$$

and

$$\text{"}\mathbf{H}_j(x,t)\text{" for } \int_{\Xi_j} N(x,\xi,t)\,\xi\,d\Omega(\xi) \tag{41}$$

The vector $\mathbf{H}_j(x,t)$ is the irradiance vector induced at x at time t by the radiance distribution restricted to Ξ_j. The net irradiance induced by this vectorial flux on a surface with inner normal $\mathbf{n}$ at x and time t is:

$$\mathbf{n}\cdot\mathbf{H}_j(x,t) \tag{42}$$

and which we shall designate by "$H_j(x,\mathbf{n},t)$". Next we write

$$\text{"}D_j(x,\mathbf{n},t)\text{" for } \frac{h_j(x,t)}{H_j(x,\mathbf{n},t)} \tag{43}$$

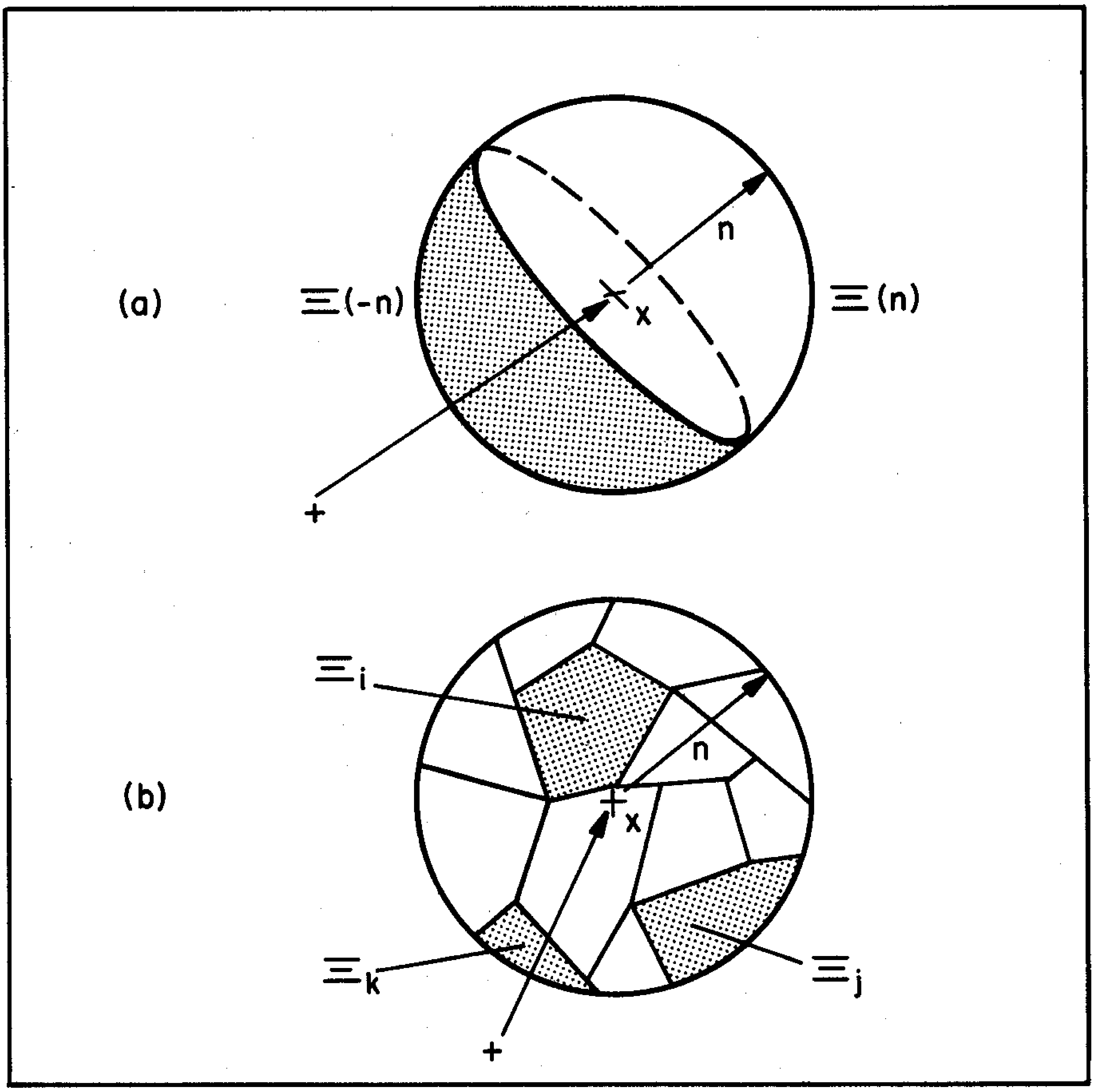

FIG. 8.3 Illustrating a two-flow decomposition of direction space Ξ, as in part (a); and a many-flow composition as in part (b). Each gives rise to a set of irradiance equations at point x in an optical medium.

The attenuating functions $\alpha_j(x,t)$, $s_j(x,t)$, $a_j(x,t)$, $j = 1, \ldots, n$ are defined analogously to (6), (13), and (14) of Sec. 8.3. Thus, e.g., we now write:

$$\text{"}\alpha_j(x,\mathbf{n},t)\text{" for } \alpha(x,t)D_j(x,\mathbf{n},t) \quad . \tag{44}$$

The four forward and backward scattering forms of the two-flow theory sublimate into the following n^2 quantities:

$$\frac{1}{H_k(x,\mathbf{n},t)} \int_{\Xi_j} \left[\int_{\Xi_k} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{45}$$

which we shall denote by "$s_{jk}(x,\mathbf{n},t)$" for $j = 1, \ldots, n$; $k = 1, \ldots, n$. Finally, integrating (39) over Ξ_j, we find:

$$\frac{1}{v}\frac{\partial}{\partial t}\left[D_j H_j\right] + \nabla \cdot \mathbf{H}_j = -\alpha_j H_j + \sum_{k=1}^{n} s_{jk} H_k + h_{\eta,j} \qquad j = 1, \ldots, n \tag{46}$$

where, for brevity, space and time coordinates have been suppressed, and where in general we have written:

$$\text{"}h_{\eta,j}(x,\mathbf{n},t)\text{"} \quad \text{for} \quad \int_{\Xi_j} N_\eta(x,\xi,t)\,d\Omega(\xi) \tag{47}$$

The system (46) is most useful when there exists a one parameter family of space-filling surfaces (in order words, Errata a one-parameter optical medium) over which the $\mathbf{H}_j$ and optical properties are constant valued. For example, spherical and cylindrical media are used occasionally in practice, and to which (46) may be applied. Then a coordinate system can usually be established in the medium, as in the plane-parallel case, so that the divergence term reduces to a single derivative of a suitable component of $\mathbf{H}_j$. For example, the reader will find it instructive to obtain $\nabla \cdot \mathbf{H}$ in spherical and cylindrical coordinate systems, and illustrate (46) for a two-flow setting in these two coordinate systems.

The general system can be cast, as it stands, into a form more nearly resembling the homogeneous terms of (19) of Sec. 8.3. To see this, write:

$$\text{"}f_j\text{"} \quad \text{for} \quad s_{jj}$$
$$\text{"}b_{jk}\text{"} \quad \text{for} \quad s_{jk} \tag{48}$$

when $j \neq k$; also write:

$$\text{"}b_j\text{"} \quad \text{for} \quad \sum_{k \neq j}^{n} s_{kj} \tag{49}$$

where "$\sum_{k\neq j}^{n}$" stands for the sum over all n indices k except j.

Furthermore, write:

$$\text{"}\alpha_j\text{"} \quad \text{for} \quad \alpha D_j \tag{50}$$

$$\text{"}a_j\text{"} \quad \text{for} \quad a D_j \tag{51}$$

$$\text{"}s_j\text{"} \quad \text{for} \quad s D_j \tag{52}$$

Then it follows that:

$$f_j + b_j = s_j \quad , \tag{53}$$

$$\alpha_j = a_j + s_j = a_j + f_j + b_j \quad , \tag{54}$$

and (46) can then be cast into the form:

$$\frac{1}{v}\frac{\partial}{\partial t}\left[D_j H_j\right] + \nabla \cdot \mathbf{H}_j = -\left[a_j + b_j\right] H_j + \sum_{k \neq j}^{n} b_{jk} H_k + h_{n,j}$$
$$j = 1 , \ldots , n \quad . \tag{55}$$

which establishes the final generalization. By letting $n \to \infty$ such that $\max \{ \Omega(\Xi_j) : j = 1 , \ldots , n \} \to 0$, (55) returns to the equation of transfer (39), and the circle is complete.

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8.7 Invariant Imbedding Concepts for Irradiance Fields

The formulations of the preceding sections of this chapter can be placed into deeper perspective when viewed from the general standpoint of the invariant imbedding concepts developed in Chapter 7. In this section we select some of the results obtained in this chapter to be so viewed. The main purpose of this activity is to indicate the conceptual and the numerical advantages gained by adopting the invariant imbedding point of view: Unsuspected symmetries and connections between the solutions of the two-flow equations spring into view when led in their general directions by the algebraic equations stored up in Sec. 7.4; furthermore the existence of quite general differential equations collected in Sec. 7.5 are now seen to hold also among the components of the irradiance field, and of the R and T factors. Furthermore, the semigroup and group-theoretic methods of Sec. 7.8 through Sec. 7.10 are awaiting their systematic translation into the irradiance context. In general, according to our observations in Sec. 8.2, *all the functional equations derived in Chapter 7 from the local or global forms of the principles of invariance also hold for the irradiance context.* Therefore, to be thorough, we could, in principle, repeat virtually all of Chapter 7 in the present section. But this is a lavish and unnecessary task for the purposes of the present work. By leaving it undone, we allow room for the few pertinent remarks made below and, more importantly, encourage students of the subject to explore such matters on their own and perhaps find new and interesting facets to develop and use. The selected examples below will indicate a few of the possible modes of approach.