$$f_j + b_j = s_j \quad , \tag{53}$$

$$\alpha_j = a_j + s_j = a_j + f_j + b_j \quad , \tag{54}$$

and (46) can then be cast into the form:

$$\frac{1}{v}\frac{\partial}{\partial t}\left[D_j H_j\right] + \nabla \cdot \mathbf{H}_j = -\left[a_j + b_j\right] H_j + \sum_{k \neq j}^{n} b_{jk} H_k + h_{\eta,j}$$
$$j = 1 , \ldots , n \; . \tag{55}$$

which establishes the final generalization. By letting $n \to \infty$ such that $\max \{ \Omega(\Xi_j) : j = 1 , \ldots , n \} \to 0$, (55) returns to the equation of transfer (39), and the circle is complete.

8.7 Invariant Imbedding Concepts for Irradiance Fields

The formulations of the preceding sections of this chapter can be placed into deeper perspective when viewed from the general standpoint of the invariant imbedding concepts developed in Chapter 7. In this section we select some of the results obtained in this chapter to be so viewed. The main purpose of this activity is to indicate the conceptual and the numerical advantages gained by adopting the invariant imbedding point of view: Unsuspected symmetries and connections between the solutions of the two-flow equations spring into view when led in their general directions by the algebraic equations stored up in Sec. 7.4; furthermore the existence of quite general differential equations collected in Sec. 7.5 are now seen to hold also among the components of the irradiance field, and of the R and T factors. Furthermore, the semigroup and group-theoretic methods of Sec. 7.8 through Sec. 7.10 are awaiting their systematic translation into the irradiance context. In general, according to our observations in Sec. 8.2, *all the functional equations derived in Chapter 7 from the local or global forms of the principles of invariance also hold for the irradiance context.* Therefore, to be thorough, we could, in principle, repeat virtually all of Chapter 7 in the present section. But this is a lavish and unnecessary task for the purposes of the present work. By leaving it undone, we allow room for the few pertinent remarks made below and, more importantly, encourage students of the subject to explore such matters on their own and perhaps find new and interesting facets to develop and use. The selected examples below will indicate a few of the possible modes of approach.

Example 1: $\mathcal{R}$ and $\mathcal{T}$ Factors in Two-D Models

Equations (24) through (43) of Sec. 8.5 represent the results of a tentative, initial excursion into the invariant imbedding domain of the reflectance and transmittance concepts associated with the two-D theory. We now develop the full structure of these $\mathcal{R}$ and $\mathcal{T}$ factors, being guided by the basic equations of the invariant imbedding relation in the irradiance case, namely (7) and (8) of Sec. 8.1. Our main goal is to represent the four $\mathcal{R}$ and $\mathcal{T}$ factors for an arbitrary subslab $X(x,z)$ of $X(a,b)$ in terms of the local scattering and absorbing properties of $X(a,b)$.

We begin with the general solution (9) of Sec. 8.5 for the irradiance field at level y in $X(a,b)$. Equations (7) and (8) of Sec. 8.1 state that the irradiances $H(y,\pm)$ are a linear combination of the upward irradiance at level z and the downward irradiance at level x, the coefficients of the combination being the complete reflectance ($\mathcal{R}$) and transmittance ($\mathcal{T}$) factors for $X(x,z)$. Evidently, m_+ and m_- in (9) of Sec. 8.5 hold the key to determining $\mathcal{R}(x,y,z)$, $\mathcal{T}(x,y,z)$, $\mathcal{R}(z,y,x)$, and $\mathcal{T}(z,y,x)$. Therefore, since (9) of Sec. 8.5 holds for all depths z in $X(a,b)$ we have:

$$H(x,-) = m_+ g_+(-)e^{k_+ x} + m_- g_-(-)e^{k_- x} \tag{1}$$

$$H(z,+) = m_+ g_+(+)e^{k_+ z} + m_- g_-(+)e^{k_- z} \tag{2}$$

These two equations can be solved for m_+, m_-. The results are:

$$m_\pm = \pm \left[H(z,+)g_\mp(-)e^{k_\mp x} - H(x,-)g_\mp(+)e^{k_\mp z} \right] \Big/ \Delta(x,z) \tag{3}$$

where we have written:

$$\text{"}\Delta(x,z)\text{" for } g_+(+)g_-(-)e^{k_+ z + k_- x} - g_+(-)g_-(+)e^{k_+ x + k_- z} \tag{4}$$

These solutions are the full symmetric forms of (19) and (28) of Sec. 8.5. The earlier forms are obtained by setting to zero the appropriate irradiances in (3).

Now, from (9) of Sec. 8.5, we can write:

$$H(y,\pm) = m_+ g_+(\pm)e^{k_+ y} + m_- g_-(\pm)e^{k_- y} \tag{5}$$

with $m_\pm$ as given in (3). Therefore, after some algebraic reductions to the forms (7) and (8) of Sec. 8.1, we find that for $a \le x \le y \le z \le b$:

$$\mathcal{R}(x,y,z) = \frac{g_+(+)g_-(+)}{\Delta(x,z)} \left[e^{k_+z + k_-y} - e^{k_+y + k_-z}\right] \tag{6}$$

$$\mathcal{R}(z,y,x) = \frac{g_+(-)g_-(-)}{\Delta(x,z)} \left[e^{k_+y + k_-x} - e^{k_+x + k_-y}\right] \tag{7}$$

$$\mathcal{T}(x,y,z) = \frac{\Delta(y,z)}{\Delta(x,z)} \tag{8}$$

$$\mathcal{T}(z,y,x) = \frac{\Delta(x,y)}{\Delta(x,z)} \tag{9}$$

These are the desired representations of the $\mathcal{R}$ and $\mathcal{T}$ factors for $X(x,z)$ in $X(a,b)$. It will be instructive for the reader to verify that the basic semigroup relations (52) and (53) of Sec. 3.7 hold in the present irradiance context, using (6) through (9). From these representations we can at once derive the standard R and T factors for $X(x,z)$. Thus, being guided by (9) through (12) of Sec. 8.1:

$$\mathcal{R}(x,x,z) = R(x,z) = \frac{g_+(+)g_-(+)}{\Delta(x,z)} \left[e^{k_+z + k_-x} - e^{k_+x + k_-z}\right] \tag{10}$$

$$\mathcal{R}(z,z,x) = R(z,x) = \frac{g_+(-)g_-(-)}{\Delta(x,z)} \left[e^{k_+z + k_-x} - e^{k_+x + k_-z}\right] \tag{11}$$

$$\mathcal{T}(x,z,z) = T(x,z) = \frac{\Delta(z,z)}{\Delta(x,z)} \tag{12}$$

$$\mathcal{T}(z,x,x) = T(z,x) = \frac{\Delta(x,x)}{\Delta(x,z)} \tag{13}$$

Furthermore from (6) through (9) we see that:

$$\mathcal{R}(x,z,z) = 0 \qquad \mathcal{R}(z,x,x) = 0 \tag{14}$$

$$\mathcal{T}(x,x,z) = 1 \qquad \mathcal{T}(z,z,x) = 1 \tag{15}$$

We can simplify (10) through (13) somewhat by making use of the homogeneity of $X(a,b)$. On the grounds of homogeneity we would expect the four $\mathcal{R}$ and $\mathcal{T}$ factors to be invariant under a vertical displacement of an amount d, as long as the vertically displaced slab remains within $X(a,b)$. To prove this rigorously, we need only first observe that:

$$\Delta(x+d,z+d) = \Delta(x,z)e^{(k_+ + k_-)d} \tag{16}$$

It follows at once from this and (6) through (9) that:

$$\mathcal{R}(x+d,y+d,z+d) = \mathcal{R}(x,y,z) \tag{17}$$

$$\mathcal{T}(x+d,y+d,z+d) = \mathcal{T}(x,y,z) \tag{18}$$

and similarly for $\mathcal{R}(z,y,x)$ and $\mathcal{T}(z,y,x)$. As a result, the standard R and T factors inherit this translation invariance:

$$R(x+d,z+d) = R(x,z) \tag{19}$$

$$T(x+d,z+d) = T(x,z) \tag{20}$$

and similarly for $R(z,x)$ and $T(z,x)$. However it is clear that the R and T factors generally possess polarity in the two-D theory, so that while $R(x,z) = R(0,z-x)$, $R(z,x) = R(z-x,0)$, we still need not have $R(0,z-x) = R(z-x,0)$, as an inspection of (10) through (13) will show.

In view of (16) we have:

$$\Delta(x,z) = \Delta(x-x,z-x) = \Delta(0,z-x)e^{+(k_+ + k_-)x} = \Delta(z-x)e^{+(k_+ + k_-)x} \tag{21}$$

Errata

In other words, we can strictly do without $\Delta(x,z)$ and retain only $\Delta(z)$, as defined in (20) of Sec. 8.5. Thus, the use of one or the other type of delta is optional.

Example 2: $\mathcal{R}$ and $\mathcal{T}$ Factors in One-D Models

By allowing the one-D assumptions (Sec. 8.6) to enter into the terms (3) through (21), some interesting and useful symmetries arise. The key symmetry is:

$$k_+ + k_- = 0 \tag{22}$$

In view of (21) and (22) we have also:

$$\Delta(x,z) = \Delta(0,z-x) = \Delta(z-x) \tag{23}$$

It follows that:

$$R(0,z-x) = R(x,z) = R(z,x) = R(z-x,0) \tag{24}$$

$$T(0,z-x) = T(x,z) = T(z,x) = T(z-x,0) \tag{25}$$

The complete $\mathcal{R}$ and $\mathcal{T}$ factors then become:

$$\mathcal{R}(x,y,z) = \frac{A}{\Delta(z-x)}\left[e^{k(z-y)} - e^{-k(z-y)}\right] \tag{26}$$

$$\mathcal{R}(z,y,x) = \frac{A}{\Delta(z-x)} \left[e^{k(y-x)} - e^{-k(y-x)}\right] \tag{27}$$

$$\mathcal{T}(x,y,z) = \frac{\Delta(z-y)}{\Delta(z-x)} \tag{28}$$

$$\mathcal{T}(z,y,x) = \frac{\Delta(y-x)}{\Delta(z-x)} \tag{29}$$

where we have written:

$$\text{"A"} \quad \text{for} \quad \left(1 - \frac{a^2D^2}{k^2}\right) \tag{30}$$

Example 3: Differential Equations for R and T Factors

The discussions of Sec. 8.6 based the derivation of the R and T factors for irradiance on knowledge of the irradiance field in X(a,b). This procedure is reversed in invariant imbedding theory: The global reflectance and transmittance properties of a medium are found first, and from these, via the imbedding relation, the internal irradiance field is determined. The R and T factors are determined by solving the differential equations they are found to obey. We shall state and discuss these differential equations in this example. The operator versions of these differential equations were derived in Sec. 7.1 and are of a fundamental kind known as *Riccati equations*. The gestalt of these equations for R and T factors is the same for the R and T operators, and as shown in Ref. [251], is independent of the geometric structure of the medium. In the irradiance context the differential equations for the R and T factors take their simplest form, the R and T factors being numerical valued functions rather than function valued functions. In such a context the heuristic manipulations of Sec. 7.3 become fully justified using ordinary calculus.

The derivations of the differential equations of the R and T factors in the two-D theory start with the local and global forms of the principles of invariance for irradiance fields. These principles are reproduced below for convenient comparison. The two main (global) principles of invariance are:

$$\text{I.} \quad H(y,+) = H(z,+)T(z,y) + H(y,-)R(y,z) \tag{31}$$

$$\text{II.} \quad H(y,-) = H(x,-)T(x,y) + H(y,+)R(y,x) \tag{32}$$

The two local forms of the principles of invariance for irradiance are:

$$-\frac{dH(y,+)}{dy} = \tau(y,+)H(y,+) + \rho(y,-)H(y,-) \tag{33}$$

$$\frac{dH(y,-)}{dy} = \tau(y,-)H(y,-) + \rho(y,+)H(y,+) \tag{34}$$

These principles are taken from (1) and (2) of Sec. 8.1 and (6) and (7) of Sec. 8.2, respectively. Adopting the assumptions of two-D theory, the local transmittance and reflectance factors become depth independent (re (8) of Sec. 8.5), so that the "y" can be dropped from the notations "$\tau(y,\pm)$", "$\rho(y,\pm)$".

The derivation procedure now follows that of (10) through (29) of Sec. 7.1 in all its essential respects. Therefore we need only state results, and in forms indigenous to the irradiance context:

$$\text{I}' \quad -\frac{\partial R(x,z)}{\partial x} = \rho(-) + [\tau(+) + \tau(-)]R(x,z) + \rho(+)R^2(x,z) \tag{35}$$

$$\text{II}' \quad \frac{\partial T(x,z)}{\partial z} = [\tau(-) + \rho(-)R(z,x)]T(x,z) \tag{36}$$

$$\text{III}' \quad \frac{\partial R(x,z)}{\partial z} = T(x,z)\rho(-)T(z,x) \tag{37}$$

$$\text{IV}' \quad -\frac{\partial T(x,z)}{\partial x} = [\tau(-) + \rho(+)R(x,z)]T(x,z) \tag{38}$$

The differences between (35) and (38) and their operator correspondents in Sec. 7.1 (starting with (18) of Sec. 7.1) immediately strike the eye: the presence of signed ρ and τ factors and the rearrangement of terms, both in the present set. Both differences are superficial and can be erased with a few strokes of the pen and some accompanying reasons. The signed ρ and τ factors reflect the two-D nature of the light field and summarize the fact that the local optical properties of $X(a,b)$, with respect to irradiance, are accordingly *anisotropic*, as discussed at some length in Sec. 8.5. If, in the developments of Sec. 7.1, we chose to explicitly consider aniostropic media, then the four operators $\rho_\pm(y)$, $\tau_\pm(y)$ would have been used throughout that discussion (see note after (4) of Sec. 7.1). However, for simplicity and for practical reasons, namely that anisotropic media are seldom encountered in practice, the developments of Sec. 7.1 took their present form. Readers may view the anisotropic versions of the differential equations for general R and T operators in Sec. 25 and Sec. 125 of Ref. [251]; furthermore, Sec. 7.7 and Sec. 7.13 contain differential equations which incorporate signed local reflectance and transmittance operators. As far as the order of the terms within I' through IV' above is concerned, we need only note that we are now working with real valued functions of real numbers rather than with operators, so that commutativity of the present multiplications is in force. As a result we have been able to rearrange the differential equations of R and T to look "more natural" to the eye.

The solutions of (35) through (38), namely $R(x,z)$ and $T(x,z)$ and their companions $R(z,x)$, $T(z,x)$ are readily obtained. However, we need only note that they are of the form (10) through (13). The advantage of the differential equations (35) through (38) is that they may be integrated even when ρ and τ *vary with depth*, so that they potentially transcend the two-D theory in versatility. The reader should now derive the completely general differential equations for $R(x,z)$ and $T(x,z)$ starting with (1) through (4) and with no assumptions on the homogeneity of $X(a,b)$. The results should be of the form:

$$\text{I}' \quad -\frac{\partial R(x,z)}{\partial x} = \rho(x,-) + [\tau(x,+) + \tau(x,-)]R(x,z) + \rho(x,+)R^2(x,z) \tag{39}$$

$$\text{II}' \quad \frac{\partial T(x,z)}{\partial z} = [\tau(z,-) + \rho(z,-)R(z,x)]T(x,z) \tag{40}$$

$$\text{III}' \quad \frac{\partial R(x,z)}{\partial z} = T(x,z)\rho(z,-)T(z,x) \tag{41}$$

$$\text{IV}' \quad -\frac{\partial T(x,z)}{\partial x} = [\tau(x,-) + \rho(x,+)R(x,z)]T(x,z) \tag{42}$$

Equations (39) and (42) are the key relations here. Using this we can find $R(x,z)$, $T(x,z)$ for $X(x,z)$, with initial conditions $R(z,z) = 0$, $T(z,z) = 1$. The reader should now develop the equations for $R(z,x)$, $T(z,x)$. (See bibliographic notes.)

Example 4: Third Order Semigroup Properties of $\mathcal{R}$ and $\mathcal{T}$ Factors

The semigroup properties of the $\mathcal{R}$ and $\mathcal{T}$ *operators*, as given in (52) and (53) of Sec. 3.7, were some of the most frequently used properties during our discussions of Chapter 7. We have already used their irradiance counterparts in Sec. 8.5 to advance the theory of boundary effects in plane-parallel media (cf. (92) and (93) of Sec. 8.5). We now assemble the full family of semigroup relations which hold among the members of $\Gamma_3(a,b)$ (re: (36) of Sec. 3.7). It will be an instructive exercise to derive these semigroup relations directly from the general semigroup relations stated in (85) through (88) of Sec. 3.7. Equations (85) through (88) of Sec. 3.7 were referred to as the *fourth order semigroup relations* since they are stated for members of $\Gamma_4(a,b)$ (re: (57) of Sec. 3.7). By allowing the depth variables w and x to coincide in the notation "$\Gamma(u,w;v,x)$", we obtain the *third order operators* of $\Gamma_3(a,b)$ and if the variables coalesce in just the right manner, the corresponding *third order semigroup relations* are forthcoming from (85) to (88) of Sec. 3.7. The requisite identifications of variables to go from $\Gamma_4(a,b)$ to $\Gamma_3(a,b)$ are given in (72) through (75) of Sec. 3.7. By means of these types of identifications and the appropriate choices of the variables in the fourth order semigroup relations, the requisite third-order counterparts can be obtained as follows.

Starting with (85) of Sec. 3.7, we set $u = a$, $v = b$. The result is:

$$\mathcal{T}(a,y;b,z) = \mathcal{T}(a,w;b,x)\mathcal{T}(w,y;x,z) + \mathcal{R}(a,x;b,w)\mathcal{R}(x,y;w,z) \ . \tag{43}$$

Now, the next step will seem like so much sorcery to the reader who has not carefully studied Examples 6 and 7 of Sec. 3.7: In the derivations of (69) and (71) of Sec. 3.7, it was discovered that the *right-end variables* of the extended $\mathcal{R}$ and $\mathcal{T}$ operators were "loose" in the sense that:

$$\mathcal{T}(u,w;v,x) = \mathcal{T}(u,w;v,x') \tag{44}$$

$$\mathcal{R}(v,w;u,x) = \mathcal{R}(v,w;u,x') \tag{45}$$

and:

$$\mathcal{T}(v,x;u,w) = \mathcal{T}(v,x;u,w') \tag{46}$$

$$\mathcal{R}(u,x;v,w) = \mathcal{R}(u,x;v,w') \tag{47}$$

for arbitrary values of x, x', and w, w', respectively. We can take advantage of this "looseness' of the right-end variables to write, in (43);

$$\mathcal{T}(a,y;b,z) = \mathcal{T}(a,y;b,y) = \mathcal{T}(a,y,b) \tag{48}$$

Furthermore, on the same basis, setting $w = a$, $x = b$, and using (84) of Sec. 3.7 to note that $\mathcal{T}(a,a;b,b)$ is an identity operator, the first summand in (43) can be exchanged for:

$$\mathcal{T}(a,a;b,b)\ \mathcal{T}(a,y;x,y) = \mathcal{T}(a,y,x) \tag{49}$$

and retaining the identifications $w = a$, $x = b$, the second summand can be exchanged for:

$$\mathcal{R}(a,x;b,x)\mathcal{R}(x,y;a,y) = \mathcal{R}(a,x,b)\mathcal{R}(x,y,a) \tag{50}$$

Assembling these results, (43) becomes:

$$\mathcal{T}(a,y,b) = \mathcal{T}(a,y,x) + \mathcal{R}(a,x,b)\mathcal{R}(x,y,a) \tag{51}$$

The physical interpretation of (51) is clearly discernible with the help of Fig. 8.4. In that figure the depths y and x are reversed from their usual lexicographic order, but that is an accident of notational choices in the present derivation. The essential physical import of (51) is that radiant flux completely transmitted from level a to level y *in the medium* $X(a,b)$ may be thought of as consisting of two parts: that completely transmitted from level a to level y *in the medium* $X(a,x)$, and that completely reflected from the lower boundary of $X(a,x)$ to level y, where the amount of flux incident at the lower boundary of $X(a,x)$ is that completely reflected from level a to level x *in the medium* $X(a,b)$. Complicated, but correct; and something the intuitions of quite experienced radiative transferists or transport theorists

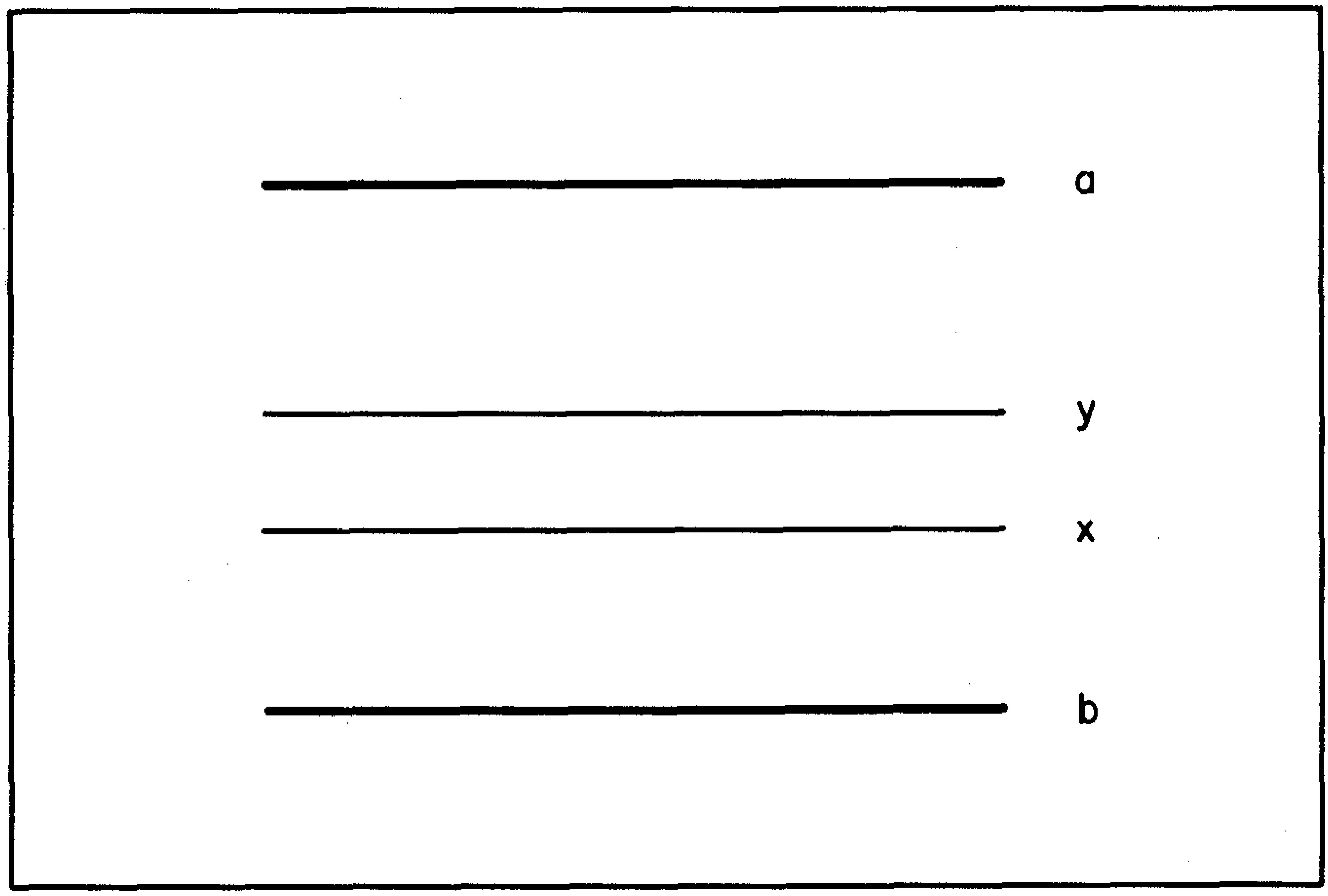

FIG. 8.4 For visualizing equation (51).

could overlook. An equation of the kind (51), but for complete reflectance operators, was found in (69) of Sec. 7.4 in constructing the star product of the invariant imbedding operators.

The reader may now deduce in a similar manner three more semigroup relations from (86) to (88) of Sec. 3.7. These, along with (51), are stated below using the customary lexicographic orders on the depth variables; as shown in Fig. 8.5.

$$\mathcal{T}(a,x,b) = \mathcal{T}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{R}(z,x,a) \quad (52)$$

$$\mathcal{T}(a,z,b) = \mathcal{T}(a,x,b)\mathcal{T}(x,z,b) \quad (53)$$

$$\mathcal{R}(a,x,b) = \mathcal{R}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{T}(z,x,a) \quad (54)$$

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,x,b)\mathcal{R}(x,z,b) \quad (55)$$

Equations (52) through (55) constitute the required complete set of *third-order semigroup relations* for members of $\Gamma_3(a,b)$. Since we have derived them without using any special assumptions on (85) through (88) of Sec. 3.7, the relations, as they stand, hold in any one-parameter optical medium in either the radiance or irradiance contexts. Three of the preceding types of equations have been derived as a matter of course during earlier discussions (69) through (72)

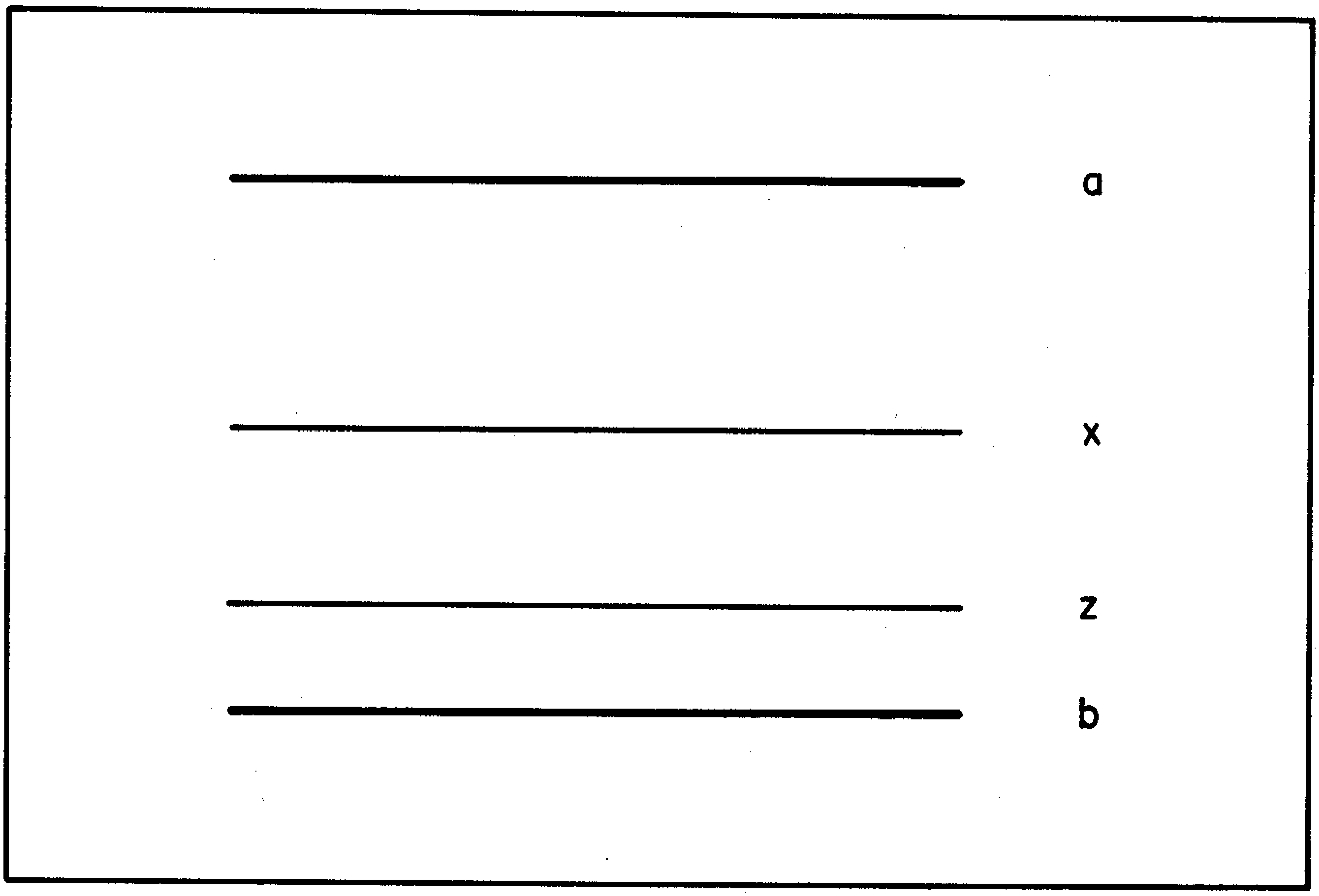

FIG. 8.5 For visualizing equations (52) through (55).

of Sec. 7.4) with (52) as the relative newcomer of the preceding group. Thus we finally arrive at a complete gathering of the third-order semigroup relations. It is clear that a companion group to that of (52) through (55) for upward incident flux follows from (85) to (88) of Sec. 3.7 by reversing the sense of the incident flows in the medium, i.e., by interchanging "a" with "b" and "x" with "z" everywhere in equations (52) through (55) but *not* in Fig. 8.5.

The reader may find it of interest to derive (52) through (55) in still another way so as to gain confidence in their validity. Thus, begin with the three invariant imbedding relations written for the setting of Fig. 8.5:

$$(N_+(z), N_-(z)) = (N_+(b), N_-(a))\mathcal{M}(a,z,b) \tag{56}$$

$$(N_+(x), N_-(x)) = (N_+(b), N_-(a))\mathcal{M}(a,x,b) \tag{57}$$

$$(N_+(x), N_-(x)) = (N_+(z), N_-(a))\mathcal{M}(a,x,z) \tag{58}$$

Then use the representation of $N_+(z)$, as given by (56), in (58). The result is:

$$(N_+(x), N_-(x)) = (N_+(b)\mathcal{T}(b,z,a) + N_-(a)\mathcal{R}(a,z,b), N_-(a))\mathcal{M}(a,x,z)) \tag{59}$$

Equating the right side of (59) with that of (57), we have, e.g.,

$$N_+(x) = N_+(b)\mathcal{T}(b,x,a) + N_-(a)\mathcal{R}(a,x,b)$$

$$= [N_+(b)\mathcal{T}(b,z,a) + N_-(a)\mathcal{R}(a,z,b)]\mathcal{T}(z,x,a) + N_-(a)\mathcal{R}(a,x,z) \ .$$

Since $N_+(b)$ and $N_-(a)$ are arbitrary:

$$\mathcal{R}(a,x,b) = \mathcal{R}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{T}(z,x,a) \quad ,$$

which is (54). The remaining three semigroup relations may be found in a similar manner.

Example 5: Systematic Analyses of Boundary Effects

In this example we amplify and systematize the discussion in Sec. 8.5 concerned with boundary effects on the irradiance field in a plane-parallel medium. The problem we shall consider is analogous to that in Example 6 of Sec. 3.9. However, our main goal now goes beyond the formulation of the boundary effects problem and takes the form of a systematic method of analysis of the problem into its component parts. The main tools we shall use are the third-order semigroup relations of the preceding example. The discussion shall be carried out in a plane-parallel medium for an irradiance field. However, both these special geometric and radiometric settings are readily exchanged for more general settings without any change in the algebraic topography of the results.

To fix ideas, we shall consider a homogeneous plane-parallel medium $X(a,b)$ with a two-D light field, with two reflecting boundaries, X_a and X_b, and a reflecting interface X_y, $a \le y \le b$. Therefore we have a composite optical medium consisting of the union of three surfaces and two slabs. See Fig. 8.6. We shall denote this space by "$X_5(a,b)$". Each of the five parts of $X_5(a,b)$ is generally endowed with a quartet of reflectance-transmittance factors: Those of X_a, X_y, X_b are as developed in Sec. 3.3 and Sec. 3.4. Those of $X(a,y)$ and $X(y,b)$ are as developed in Sec. 3.6 and Sec. 3.7. The resultant roster of interaction operators is as follows:

$$\text{For } X_u\text{:} \quad r_\pm(u), t_\pm(u), \tag{60}$$

$$\text{For } X(u,v)\text{:} \quad \begin{matrix} R(u,v), T(u,v) \\ R(v,u), T(v,u) \end{matrix} \left\{ \begin{matrix} u = a;\ v = y \\ u = y;\ v = b \end{matrix} \right\} \tag{61}$$

Thus there is a total of twelve surface interaction factors and a total of eight slab interaction factors, twenty in all. Now, for the present analysis this relatively large collection of spaces and operators is imagined to be assembled piece by piece starting with surface X_b. To this we add the slab $X(y,b)$ (considered boundaryless) to obtain what we shall refer to as "$X_2(y,b)$". We continue adding pieces until $X_5(a,b)$ is reconstructed. Thus, we shall write:

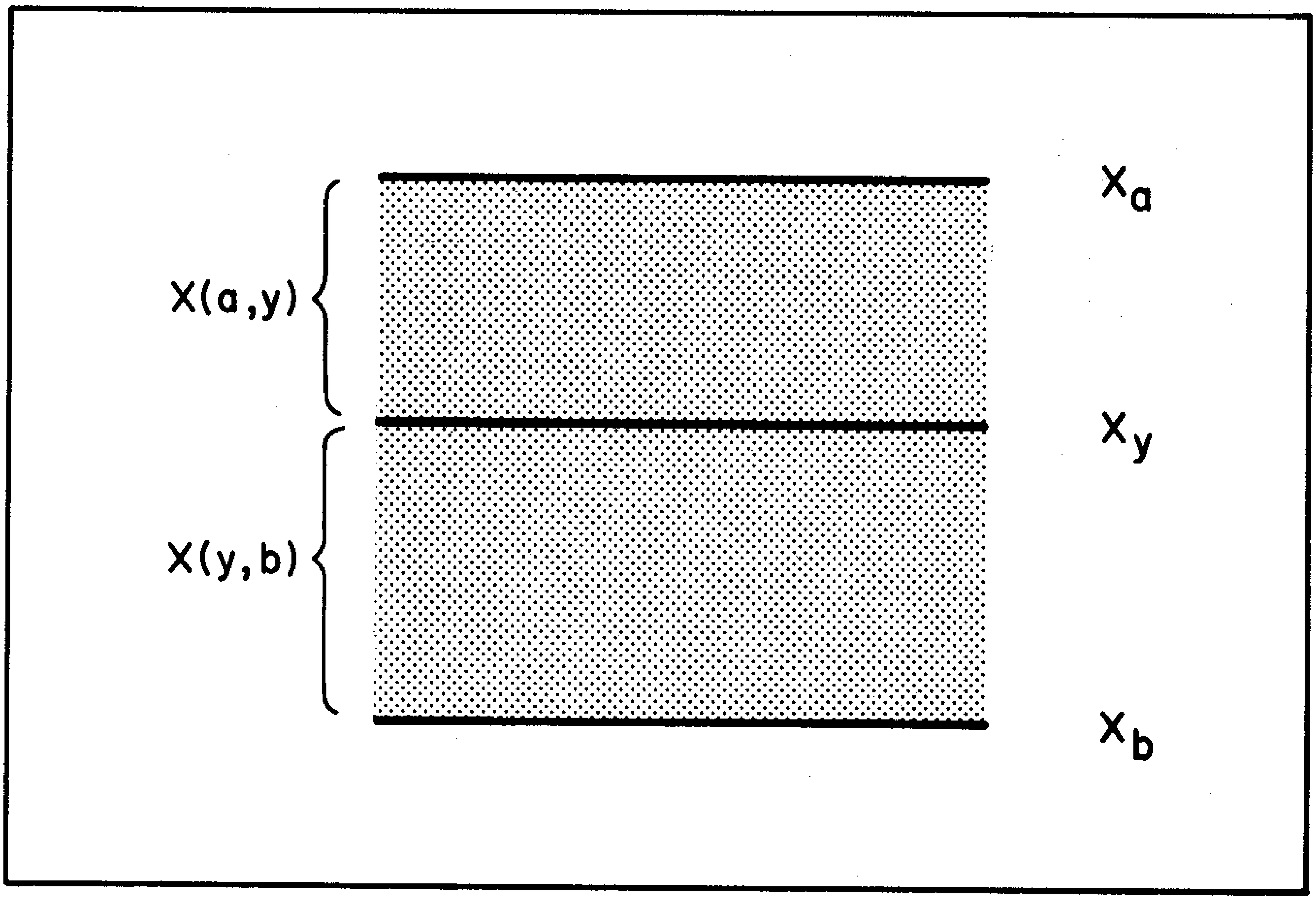

FIG. 8.6 A five-part optical medium consisting of three reflecting-transmitting surfaces (heavy lines) and two diffusing media (dotted).

$$\text{"}X_2(y,b)\text{"} \quad \text{for} \quad X(y,b) \cup X_b \tag{62}$$

$$\text{"}X_3(y,b)\text{"} \quad \text{for} \quad X_y \cup X_2(y,b) \tag{63}$$

$$\text{"}X_4(a,b)\text{"} \quad \text{for} \quad X(a,y) \cup X_3(y,b) \tag{64}$$

This mode of construction of $X_5(a,b)$ is patterned after the invariant imbedding process used in Sec. 7.13 (re: Fig. 7.25). We shall develop a discrete-space approach to the present invariant imbedding process. We are now ready for the analysis.

Stage 1 of the present invariant imbedding process (Fig. 8.7(a)) consists in finding the invariant imbedding operators for X_b. These are simply the factors $r_\pm(b)$, $t_\pm(b)$. To establish a systematic notation which will hold in all stages of the analysis we write:

$$\text{"}\mathcal{R}(b,b,b+1) \quad \text{for} \quad r_-(b) \tag{65}$$

$$\text{"}\mathcal{T}(b-1,b,b)\text{"} \quad \text{for} \quad t_-(b) \tag{66}$$

$$\text{"}\mathcal{R}(b,b,b-1)\text{"} \quad \text{for} \quad r_+(b) \tag{67}$$

$$\text{"}\mathcal{T}(b+1,b,b)\text{"} \quad \text{for} \quad t_+(b) \tag{68}$$

This notation is patterned after that of general discrete

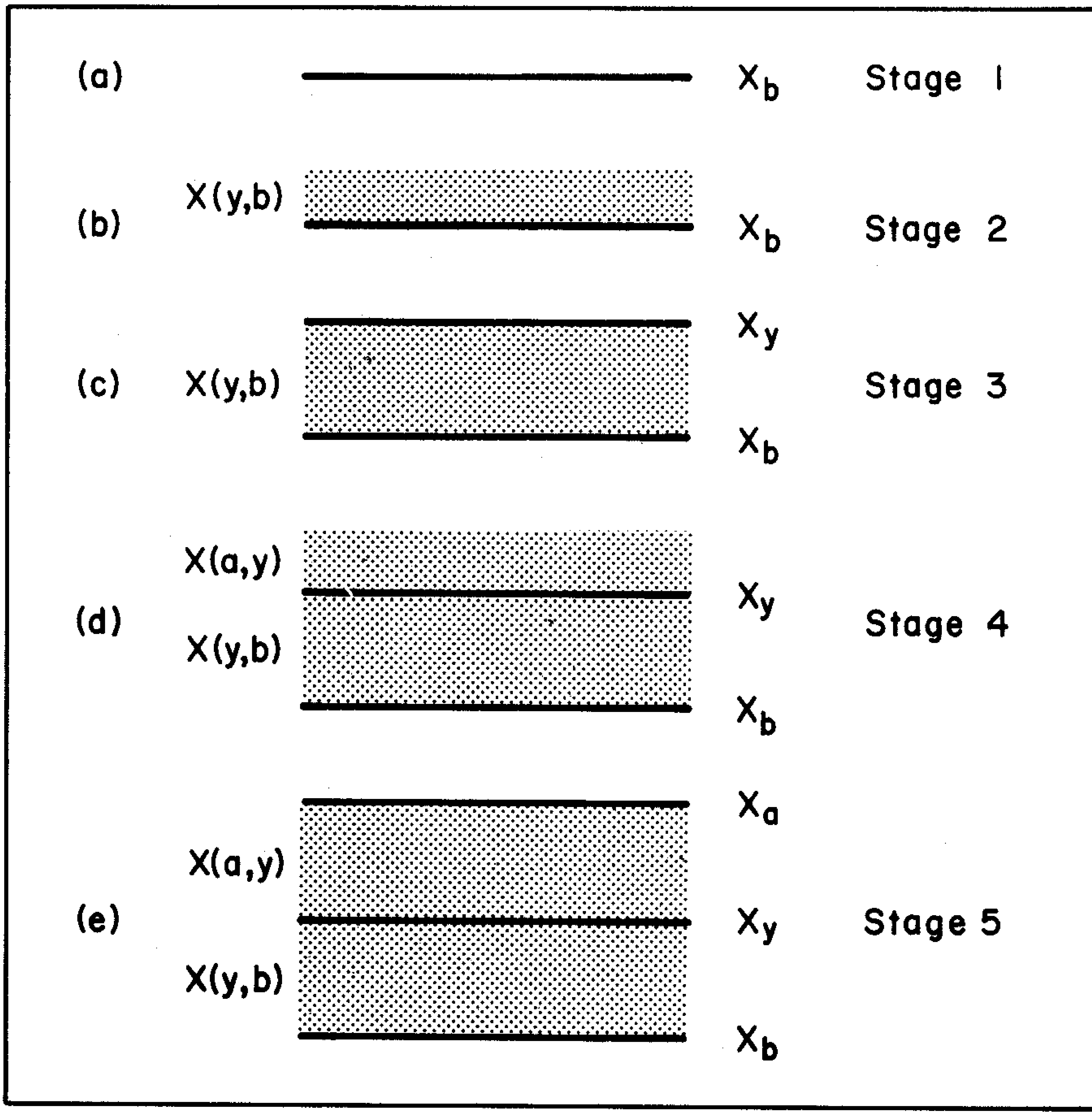

FIG. 8.7 A systematic analysis (and resynthesis) of the medium in Fig. 8.6.

space theory, (Ref. [251]). Hence, for Stage 1, the pertinent invariant imbedding relation associated with X_b is:

$$(H(b,+),H(b,-)) = (H(b+1,+),H(b-1,-))\mathcal{M}(b-1,b,b+1) \qquad (69)$$

Here $\mathcal{M}(b-1,b,b+1)$ is the 2×2 matrix made up from the four factors defined in (65) through (68). By the simple notational device of adding 1 to b and subtracting 1 from b , we can conveniently denote the incident irradiances *on* X_b *considered as an isolated medium.* This tactic will be used repeatedly below and is the signal that a discrete-space calculation is in progress.

Stage 2 of the present invariant imbedding process finds the operators for $X_2(y,b)$. The $\mathcal{M}$- operator for $X_2(y,b)$ is $\mathcal{M}(y,z,b+1)$, a 2×2 matrix. Thus, by (52) and (54), adapted to the present one-parameter medium, we have:

$$\mathcal{R}(y,z,b+1) = \mathcal{R}(y,z,b) + \mathcal{R}(y,b,b+1)\mathcal{T}(b,z,y) \qquad (70)$$

$$\mathcal{T}(y,z,b+1) = \mathcal{T}(y,z,b) + \mathcal{R}(y,b,b+1)\mathcal{R}(b,z,y) \qquad (71)$$

which hold for $a \le y \le z \le b$. These equations represent $\mathcal{R}(y,z,b+1)$ and $\mathcal{T}(y,z,b+1)$, two of the four components of $\mathcal{M}(y,z,b+1)$, in terms of the four complete $\mathcal{R}$ and $\mathcal{T}$ factors for $X(y,b)$ (known) and the operator $\mathcal{R}(y,b,b+1)$. By (55), adapted to the present setting, we have:

$$\mathcal{R}(y,b,b+1) = \mathcal{T}(y,b,b+1)\mathcal{R}(b,b,b+1) \qquad (72)$$

$\mathcal{R}(b,b,b+1)$ is known from (65). It remains to find $\mathcal{T}(y,b,b+1)$. At this point (42) of Sec. 3.7 is used to find:

$$\mathcal{T}(y,b,b+1) = T(y,b)[I - R(b,y)R(b,b+1)]^{-1} \qquad (73)$$

Here $T(y,b)$ and $R(b,y)$ are known properties of $X(y,b)$, and "$R(b,b+1)$" is another name for $\mathcal{R}(b,b,b+1)$ using the reduction equations (44) through (47) of Sec. 3.7. Therefore (70) through (73) completely analyze two of the operators of $\mathcal{M}(y,z,b+1)$ associated with $X_2(y,b)$.

The remaining two operators, $\mathcal{R}(b+1,z,y)$ and $\mathcal{T}(b+1,z,y)$ in $\mathcal{M}(y,z,b+1)$ are analyzed similarly. Thus, from (53) and (55)

$$\mathcal{R}(b+1,z,y) = \mathcal{T}(b+1,b,y)\mathcal{R}(b,z,y) \qquad (74)$$

$$\mathcal{T}(b+1,z,y) = \mathcal{T}(b+1,b,y)\mathcal{T}(b,z,y) \qquad (75)$$

which hold for $a \le y \le z < b$. These equations represent $\mathcal{R}(b+1,z,y)$ and $\mathcal{T}(b+1,z,y)$ in terms of the two complete $\mathcal{R}$ and $\mathcal{T}$ operators for $X(y,b)$ (known) and the operator $\mathcal{T}(b+1,b,y)$. By (43) of Sec. 3.7 we have:

$$\mathcal{T}(b+1,b,y) = T(b+1,b)[I - R(b,y)R(b,b+1)]^{-1}$$

Here $R(b,y)$ is a known property of $X(y,b)$ and "$T(b+1,b)$" and "$R(b,b+1)$" are other names for $t_+(b)$ and $r_-(b)$, respectively. We summarize the constructions of Stage 2 by the following equation:

$$\mathcal{M}(y,z,b+1) = \begin{bmatrix} \mathcal{T}(b+1,b,y)\mathcal{T}(b,z,y) & \mathcal{T}(b+1,b,y)\mathcal{R}(b,z,y) \\ \mathcal{R}(y,z,b)+\mathcal{R}(y,b,b+1)\mathcal{T}(b,z,y) & \mathcal{T}(y,z,b)+\mathcal{R}(y,b,b+1)\mathcal{R}(b,z,y) \end{bmatrix}$$

The preceding matrix can be analyzed into factors of the invariant imbedding type as follows:

$$\boxed{\begin{aligned} \mathcal{M}(y,z,b+1) = [C_- + \mathcal{M}(y,b,b+1)C_+]\mathcal{M}(y,z,b) \\ a \le y \le z \le b \end{aligned}} \qquad (77)$$

where C_+ and C_- are defined in (4) and (5) of Sec. 7.4. Now $I_\pm$ occurring in $C_\pm$ are simply the number 1 in the present irradiance context. Equation (77) shows how the invariant imbedding operators of two contiguous media can be algebraically combined to yield the invariant imbedding operator for their union. In this case the media are the slab $X(y,b)$ and the surface X_b.

Stage 3 of the present invariant imbedding process finds the $\mathcal{M}$-operator for $X_3(y,b)$. The $\mathcal{M}$-operator for $X_3(y,b)$ is the 2×2 matrix $\mathcal{M}(y-1,z,b+1)$. Thus, by (53), and (55), adapted to the present one-parameter medium, we have:

$$\mathcal{R}(y-1,z,b+1) = \mathcal{T}(y-1,y,b+1)\mathcal{R}(y,z,b+1) \tag{78}$$

$$\mathcal{T}(y-1,z,b+1) = \mathcal{T}(y-1,y,b+1)\mathcal{T}(y,z,b+1) \tag{79}$$

which hold for $y \leq z \leq b$. The two factors $\mathcal{R}(y,z,b+1)$ and $\mathcal{T}(y,z,b+1)$ are known from the preceding stage of the analysis. The remaining factor is obtained by means of (42) of Sec. 3.7:

$$\mathcal{T}(y-1,y,b+1) = T(y-1,y)\,[I - R(y,b+1)R(y,y-1)]^{-1} \tag{80}$$

Here "T(y-1,y)" and "R(y,y-1)" are simply other names for $t_-(y)$ and $r_+(y)$, respectively, used in discrete-space theory. Finally,

$$R(y,b+1) = \mathcal{R}(y,y,b+1) \quad , \tag{81}$$

which is known from the preceding stage of analysis. The remaining two components of $\mathcal{M}(y-1,z,b+1)$ are found by means of (52) and (54);

$$\mathcal{R}(b+1,z,y-1) = \mathcal{R}(b+1,z,y) + \mathcal{R}(b+1,y,y-1)\mathcal{T}(y,z,b+1) \tag{82}$$

$$\mathcal{T}(b+1,z,y-1) = \mathcal{T}(b+1,z,y) + \mathcal{R}(b+1,y,y-1)\mathcal{R}(y,z,b+1) \tag{83}$$

which holds for $y \leq z \leq b$. Here all factors are known from Stage 2 except for $\mathcal{R}(b+1,y,y-1)$, which can be obtained using the semigroup relation:

$$\mathcal{R}(b+1,y,y-1) = \mathcal{T}(b+1,y,y-1)\mathcal{R}(y,y,y-1)$$

Finally $\mathcal{T}(b+1,y,y-1)$ is found by means of (43) of Sec. 3.7.

Stage 3 may be summarized by the following equation:

$$\mathcal{M}(y-1,z,b+1) = \begin{bmatrix} \mathcal{T}(b+1,z,y)+\mathcal{R}(b+1,y,y-1)\mathcal{R}(y,z,b+1) & \mathcal{R}(b+1,z,y)+\mathcal{R}(b+1,y,y-1)\mathcal{T}(y,z,b+1) \\ \mathcal{T}(y-1,y,b+1)\mathcal{R}(y,z,b+1) & \mathcal{T}(y-1,y,b+1)\mathcal{T}(y,z,b+1) \end{bmatrix}$$

In other words:

$$\mathcal{M}(y-1,z,b+1) = [C_+ + \mathcal{M}(y-1,y,b+1)C_-]\mathcal{M}(y,z,b+1) \qquad y \le z \le b \tag{84}$$

The pattern that is forming is now sufficiently clear so that the reader may complete stages 4 and 5. Observe that (84), as it stands, solves the general boundary-effect problem of a medium $X(y,b)$ with interreflecting boundaries X_y and X_b.

Example 6: Invariant Imbedding Operators for Interacting Media

We consider next the problem of predicting the irradiance field within a medium $X(a,c)$ composed of two contiguous media $X(a,b)$, $X(b,c)$. The media may be of infinite depth or they may be degenerate, i.e., they may be surfaces. For example, if $X(a,b)$ is degenerate, then we write "$X(a-1,a)$" for $X(a,b)$ and construct the reflectances and transmittances in the manner described in Example 5 above. If there is an interface at level b, let it belong to $X(a,b)$. Our main purpose in this example is to present a unified algebraic approach to the problem of irradiance (or any other* radiometric) fields in sets of contiguous plane-parallel media. We have developed sufficient background for the solution of this problem in the preceding Examples 4 and 5 and in Sec. 8 5 (re: (94) of that section) to permit the broad algebraic techniques of the present example to be followed without difficulty.

Figure 8.8 depicts the composite medium $X(a,c) = X(a,b) \cup X(b,c)$. We assume that the operators (2×2 matrices) $\mathcal{M}(a,y,b)$ and $\mathcal{M}(b,y,c)$ associated with the component media $X(a,b)$, $X(b,c)$ are known. Our goal is to characterize $\mathcal{M}(a,y,c)$, the invariant imbedding operator for $X(a,c)$, in terms of the operators $\mathcal{M}(a,y,b)$ and $\mathcal{M}(b,y,c)$. The analysis of the problem reduces to the two cases, depicted in Fig. 8.8. Consider case (a). The irradiance field $(H(y,+), H(y,-))$ at depth y, $a \le y \le b$ may be viewed from two vantage points: as an irradiance field in $X(a,b)$, which is the response of the isolated medium $X(a,b)$ to the incident irradiances $H(b,+)$, $H(a,-)$; or as an irradiance field in $X(a,c)$ which is the response of $X(a,c)$ to the incident irradiances $H(c,+)$, $H(a,-)$. The first interpretation is represented as:

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*In the event that any of the present results are adapted to the *radiance* context, and media with different indices of refraction are considered, it will be understood that each radiance will be divided by the square of the index of refraction of the medium to which it pertains (cf. (4) of Sec. 7.6), so that we use N/n^2 rather than simply N throughout any formula.

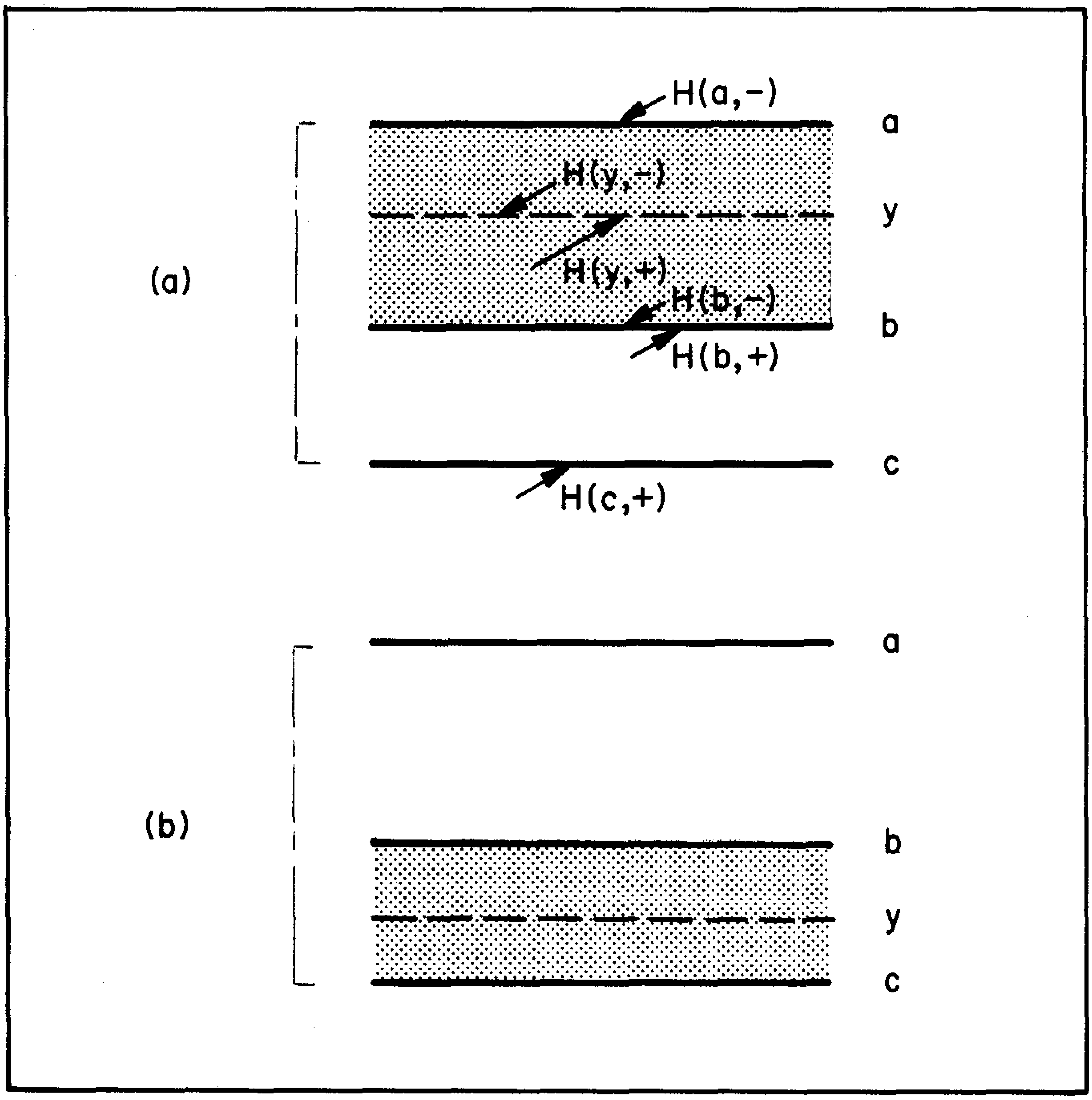

FIG. 8.8 The response of the composite medium X(a,c) can be characterized algebraically in terms of the responses of its components X(a,b) and X(b,c).

$$(H(y,+),H(y,-)) = (H(b,+),H(a,-))\mathcal{M}(a,y,b) \qquad (85)$$

The second interpretation is represented as:

$$(H(y,+),H(y,-)) = (H(c,+),H(a,-))\mathcal{M}(a,y,c) \qquad (86)$$

Finally, the irradiance field (H(b,+),H(b,-)) may be represented as:

$$(H(b,+),H(b,-)) = (H(c,+),H(a,-))\mathcal{M}(a,b,c) \qquad (87)$$

Using the contracting matrices C_+, C_- in (4) and (5) of Sec. 7.4, it follows from (87) that:

$$(H(b,+),0) = (H(c,+),H(a,-))\mathcal{M}(a,b,c)C_+$$

Further we have:

$$(0,H(a,-)) = (H(c,+),H(a,-))C_-$$

Adding these two equations, we obtain:

$$(H(b,+),H(a,-)) = (H(c,+),H(a,-))[C_- + \mathcal{M}(a,b,c)C_+] \tag{88}$$

which, operated on by $\mathcal{M}(a,y,b)$, would, according to (85), yield an alternate representation of $(H(y,+),H(y,-))$ to that given in (86). Hence, since $(H(c,+),H(a,-))$ is arbitrary,

$$\boxed{\begin{gathered}\mathcal{M}(a,y,c) = [C_- + \mathcal{M}(a,b,c)C_+]\mathcal{M}(a,y,b) \\ a \le y \le b \le c\end{gathered}} \tag{89}$$

Case (b) in Fig. 8.8 proceeds analogously, and the form of $\mathcal{M}(a,y,c)$ in this case turns out to be:

$$\boxed{\begin{gathered}\mathcal{M}(a,y,c) = [C_+ + \mathcal{M}(a,b,c)C_-]\mathcal{M}(b,y,c) \\ a \le b \le y \le c\end{gathered}} \tag{90}$$

Equation (89) is the general form of (77) and (90) that of (84). Further, (89) and (90) implicitly contain the third-order semigroup relations (52) through (55) for the settings of Fig. 8.8.

The generality of (89) and (90) resides in the possibility of either component $X(a,b)$ or $X(b,c)$ being itself a composite space made up any number of contiguous slabs and internal boundaries (see, e.g., Examples 3 and 4 of Sec. 3.4). Equations (89) and (90) serve as guides in the construction of $\mathcal{M}(a,y,c)$, knowing the corresponding operators for its component spaces. Hence (89) and (90) constitute an *inductive step* in the construction of $\mathcal{M}(a,y,c)$ in precise analogy to the customary inductive step used in the application of the principle of induction in mathematical arguments, where one proceeds from a statement $P(n)$ associated with an integer n to statement $P(n+1)$. One interesting application of (89) and (90) would be to work out the complete details of assigning distinct pairs $D_i(\pm)$ of distribution values to each slab $X(a_i,a_{i+1})$ in a sequence of n slabs comprising $X(a,b)$, so that (6) through (9) may be used in actual numerical calculations based on (89) and (90).

It remains to observe how $\mathcal{M}(a,b,c)$ is found. The reader may have noted, on the basis of the discussion in

Example 5, that equations (40) through (43) of Sec. 3.7 will serve adequately in this task. In this connection, we observe that an elegant algebraic formulation of (40) through (43) of Sec. 3.7 is possible by using the star product for $\Gamma_3(a,b)$ introduced in (75) of Sec. 7.4. For by (76) of Sec. 7.4 we have:

$$\mathcal{M}(a,b,c) = \mathcal{M}(a,b,b) * \mathcal{M}(b,b,c) \tag{91}$$

Continuing on this algebraic level, we can further resolve $\mathcal{M}(a,b,b)$ and $\mathcal{M}(b,b,c)$ by means of the M-operators (2) of Sec. 7.4. It is easy to see that:

$$\mathcal{M}(a,b,b) = \begin{bmatrix} I & R(b,a) \\ 0 & T(a,b) \end{bmatrix} = [C_+ + M(a,b)C_-] \ ; \tag{92}$$

$$\mathcal{M}(b,b,c) = \begin{bmatrix} T(c,b) & 0 \\ R(b,c) & I \end{bmatrix} = [M(b,c)C_+ + C_-] \ . \tag{93}$$

Finally, the star product * for $G_2(a,b)$, as defined in (35) of Sec. 7.4 and studied in (37) and (38) of that section, may be used analogously to (91) to find the R and T operators of the union of contiguous media. Thus, e.g., the equation:

$$M(a,c) = M(a,b) * M(b,c) \tag{94}$$

shows how to algebraically construct the standard R and T factors for $X(a,c)$, knowing those for $X(a,b)$ and $X(b,c)$.

Example 7: Differential Equations Governing $\mathcal{R}$ and $\mathcal{T}$ Factors

The preceding examples have shown the key role played by the complete $\mathcal{R}$ and $\mathcal{T}$ factors in the determination of the irradiance field in a variety of practical instances. It is of interest to observe that these $\mathcal{R}$ and $\mathcal{T}$ factors may be obtained directly by integrating the differential equation that governs them. Several such differential equations were developed in (11) through (13) of Sec. 7.5. The one we select for attention here is (38) of Sec. 7.5:

$$\boxed{\frac{d\mathcal{A}(y)}{dy} = \mathcal{A}(y)\mathcal{K}(y)} \tag{95}$$

defined for each y in the depth interval $[a,b]$ and with initial conditions:

$$\mathcal{A}(b) = [0, T(a,b)] \tag{96}$$

or:

$$\mathcal{A}(a) = [R(a,b), I] \tag{97}$$

Here we have written:

$$"\mathcal{Q}(y)" \quad \text{for} \quad (\mathcal{R}(a,y,b), \mathcal{T}(a,y,b)) \quad ,$$

and $\mathcal{K}(y)$ now has the form given in (9) of Sec. 8.2. For example, knowing $\mathcal{K}(y)$ and $R(a,b)$, one can find $\mathcal{R}(a,y,b)$, $\mathcal{T}(a,y,b)$ directly by integrating (95) from a to y. The integrations may be theoretical or numerical, where appropriate. Further observations on $\mathcal{Q}(y)$, of interest to the irradiance context, are given in (11) through (15) of Sec. 7.10.

Example 8: Method of Modules for Irradiance Fields

We now present an illustration of the method of modules, as developed in Sec. 7.8, for the case of irradiance fields. Equations (14) of Sec. 7.8 are readily put to use in irradiance computations once $\mathcal{T}(d)$ is obtained. From (8) and (21) we have at once:

$$\begin{aligned} \mathcal{T}(d) &= \lim_{z\to\infty} \mathcal{T}(0,d,z) \\ &= \lim_{z\to\infty} \frac{\Delta(d,z)}{\Delta(0,z)} \\ &= \lim_{z\to\infty} \frac{\Delta(z-d)}{\Delta(z)}\, e^{+(k_+ + k_-)d} \\ &= e^{k_- d} \end{aligned} \tag{98}$$

Equations (14) of Sec. 7.8 then take the form

$$H(jd,+) = H(0,-)e^{jk_- d}\, R_\infty(-) \tag{99}$$

$$H(jd,-) = H(0,-)e^{jk_- d} \tag{100}$$

where we have written:

$$"R_\infty(-)" \quad \text{for} \quad \lim_{z\to\infty} R(0,z) \tag{101}$$

and which, by (10), has the representation:

$$R_\infty(-) = \frac{g_-(+)}{g_-(-)} = \frac{1 + \frac{a(-)}{k_-}}{1 - \frac{a(+)}{k_-}} = \frac{k_- + a(-)}{k_- - a(+)} \qquad (102)$$

The latter equalities follow from (11) of Sec. 8.5.

Example 9: Method of Semigroups for Irradiance Fields

The method of semigroups, applied to general radiance fields in Sec. 7.9, yields formulas for $H(y,\pm)$ in $X(0,\infty)$. Thus from (10) and (12) of Sec. 7.9, we now may write:

$$\boxed{H(y,-) = H(0,-) \exp\{ (\tau(-) + \rho(+)R_\infty(-))y \}} \qquad (103)$$

and as usual:

$$H(y,+) = H(y,-)R_\infty(-) \qquad (104)$$

The present setting is $X(0,\infty)$ and we have used the two-D theory concepts $\tau(-)$, $\rho(+)$, and $R_\infty(-)$. The latter constant was defined in (101). In view of (99) (which holds for arbitrary d, so let $d = y$, thereby fixing j as 1) we see that we must have:

$$\boxed{k_- = \tau(-) + \rho(+)R_\infty(-)} \qquad (105)$$

It may be verified that this is consistent with (102). This is the desired connection between k_-, $\tau(-)$, $\rho(+)$, and $R_\infty(-)$. Equations (102) and (105) are the first views we have of an important set of general connections which hold between the theoretical and the exact observable counterparts to these concepts, and which are studied in detail in Sec. 9.2.

Example 10: Irradiance Fields Generated by Internal Sources

We devote the final example of this section to illustrating some of the relations developed in Sec. 7.3 for internal sources in the special case of irradiance field (i.e., one-dimensional) settings. As a result we gain perspective on the structure of the one-D and two-D moels for internal sources discussed in general in (44) through (66) of Sec. 8.5, and in particular in (25) of Sec. 8.6.

According to the general conversion principle, stated in the introductory remarks to this section, every functional equation developed in Sec. 7.13 may be converted to the present irradiance context. Because of this we shall devote most of the discussion to the task of bringing to light some special relations for internal-source generated irradiance fields which hold *only* in the irradiance context.

Toward this end, consider the observations in (87) through (92) of Sec. 7.13 concerned with the asymmetry of the Ψ-operator. It was observed that the invariant imbedding operators, such as $\mathcal{R}(s,y,b)$ and $\mathcal{T}(s,y,b)$ were generally distinct from their duals $\mathcal{R}^{+}(y,s,b)$ and $\mathcal{T}^{+}(y,s,b)$, respectively. An examination of the matter showed that if $\mathcal{R}(s,y,b)$ and $\mathcal{R}^{+}(y,s,b)$ were ever equal in some setting, then the computation details of the internal source problem in that setting would be effectively halved with respect to the general case. Therefore a search for a reciprocity property between $\mathcal{R}(s,y,b)$ and $\mathcal{R}^{+}(s,y,b)$ was launched. It did not require an extensive analysis to see that the requisite reciprocity property (i.e., the equality of $\mathcal{R}(s,y,b)$ and $\mathcal{R}^{+}(y,s,b)$) is generally barred by polarity of the R and T operators and general noncommutativity of the integral operators. Since commutativity of $\mathcal{R}$ and $\mathcal{T}$ factors is now available, and since the R and T factors of one-D models do not possess polarity (re: (35) and (36) of Sec. 8.5, and also (24) and (25),) we return to the matter of reciprocity of $\mathcal{R}(s,y,b)$ and $\mathcal{R}^{+}(y,s,b)$ and reexamine some of the functional relations of Sec. 7.13 in the present simpler setting. Therefore for the remainder of this example, we shall work in a separable plane-parallel medium X(a,b) in which the one-D assumptions of Sec. 8.6 are in force. First we explicitly verify that:

$$\mathcal{R}(s,y,b) = \mathcal{R}^{\dagger}(y,s,b) \tag{106}$$

$$\mathcal{T}(s,y,b) = \mathcal{T}^{\dagger}(y,s,b) \tag{107}$$

Figure 8.9 is a schematic summary of the relative location of levels s,y,b in X(a,b), pertinent to (106) and (107), Thus we have $a \le s < y \le b$. Furthermore, the entities of (106) and (107) may be seen in their correct place in a general invariant imbedding process on X(a,b) by referring to stage 2 of Fig. 7.25. Now, to test the validity of (106) we use (1) and (25) of Sec. 7.13 to write down the equivalent statement to (106):

$$T(s,y)\Psi_{-+}(y,y:a,b) = \Psi_{-+}(y,y:a,b)T(y,s) \quad .$$

By commutativity of numerical multiplication we have:

$$T(s,y)\Psi_{-+}(y,y:a,b) = T(y,s)\,\Psi_{-+}(y,y:a,b) \quad .$$

This statement can be generally true only in a one-D setting, for since $\Psi_{-+}(y,y:a,b) \neq 0$ for all $a < y < b$, we require:

$$T(s,y) = T(y,s) \quad ,$$

which holds generally only for one-D irradiance fields (re: (24) and (25)).

As a consequence of (106) and (107), and two more equations of the same kind:

$$\mathcal{R}(b,y,s) = \mathcal{R}^{\dagger}(y,b,s) \tag{108}$$

$$\mathcal{T}(b,y,s) = \mathcal{T}^{\dagger}(y,b,s) \quad , \tag{109}$$

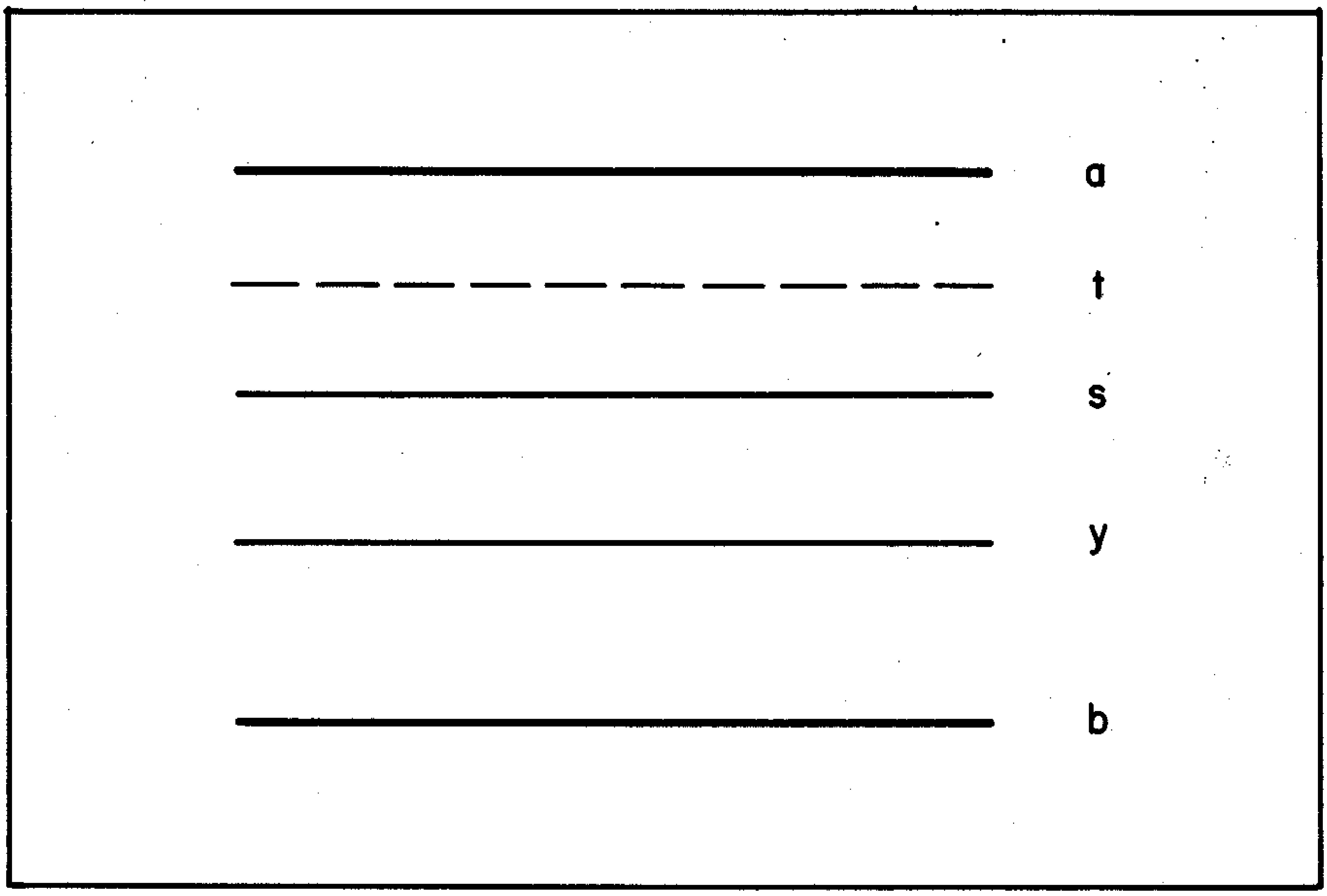

FIG. 8.9 An internal-source setting for irradiance fields. The sources are at level s, the observation level is at y. This setting falls under case 4, stage 3 of the general invariant imbedding scheme in Fig. 7.25.

we see that the number of invariant imbedding factors for X(a,b) is reduced by a factor of two.

The physical significance of equations (106) through (109) is of interest in the present discussions and we pause to make it clear. Consider (106) and Fig. 8.9. The left side of (106), namely the complete reflectance factor $\mathscr{R}(s,y,b)$, describes the amount of upward irradiance at level y in X(s,b) induced by a unit downward irradiance at level s . On the other side, the dual reflectance factor $\mathscr{R}^{\dagger}(y,s,b)$ describes the amount of upward irradiance at level s in X(s,b) induced by a unit downward irradiance at level y . The present geometric setting (plane-parallel medium) and physical assumptions for that setting (separability and isotropy of X(a,b)) combine to imply the reciprocity statement (106). The remaining three reciprocity statements are interpreted similarly.

We are now ready to illustrate some of the functional relations for internal-source generated light fields, as developed in Sec. 7.13. Equation (86) of Sec. 7.13 is one of the general results of that section and, since its associated geometric setting is precisely that of Fig. 8.9, we select it for illustration.

The first part of the illustration consists in casting (86) of Sec. 7.13 into $\mathscr{m}$-operator notation, in order to emphasize the fact that stage 3 of an invariant imbedding process (Fig. 7.25) can be carried out using the $\mathcal{M}$-operators constructed in stage 2. Furthermore, the $\mathcal{M}$-operator notation is simpler and of more intuitive value to the present discussion. The conversion to $\mathcal{M}$-operators is effected by means of (63) and (69) of Sec. 7.13:

$$\Psi(s,y:s,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(s,y,b) & \mathcal{T}(s,y,b) \end{bmatrix} = C_-\mathcal{M}(s,y,b) \quad (110)$$

$$\Psi(s,t:t,b) = \begin{bmatrix} \mathcal{T}^\dagger(s,t,b) & 0 \\ \mathcal{R}^\dagger(s,t,b) & 0 \end{bmatrix} = \mathcal{M}^\dagger(s,t,b)C_+ \quad (111)$$

$$\Psi(t,y:t,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(t,y,b) & \mathcal{T}(t,y,b) \end{bmatrix} = C_-\mathcal{M}(t,y,b) \quad (112)$$

It follows that (86) of Sec. 7.13 may be written:

$$\Psi(s,y:a,b) = C_-\mathcal{M}(s,y,b) + \int_a^s \mathcal{M}^\dagger(s,t,b)C_+\mathcal{K}(t)C_-\mathcal{M}(t,y,b)\,dt \quad (113)$$

or, directly in matrix form:

$$\Psi(s,y:a,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(s,y,b) & \mathcal{T}(s,y,b) \end{bmatrix} +$$

$$+ \int_a^s \begin{bmatrix} \mathcal{T}^\dagger(s,t,b) & 0 \\ \mathcal{R}^\dagger(s,t,b) & 0 \end{bmatrix} \begin{bmatrix} -\tau & \rho \\ -\rho & \tau \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathcal{R}(t,y,b) & \mathcal{T}(t,y,b) \end{bmatrix} dt \quad (114)$$

in which $\mathcal{K}(t)$ is now in its appropriate form for one-D irradiance theory, i.e., $\tau(y,\pm)$ and $\rho(y,\pm)$ are independent of y and $\pm$, (cf. (9) of Sec. 8.2; (11) and (12) of Sec. 8.3; and (2) through (6) of Sec. 8.6).

In order to illustrate (114) in its simplest form, we next assume that the medium $X(a,b)$ is infinitely deep. Thus, we assume $a = 0$ and $b = \infty$. The corresponding forms of the complete $\mathcal{R}$ and $\mathcal{T}$ factors are obtained from (26) and (28). From (26):

$$\mathcal{R}(s,y,\infty) = \lim_{b\to\infty}\mathcal{R}(s,y,b) = \lim_{b\to\infty}\frac{A}{\Delta(b-s)}\left[e^{k(b-y)} - e^{-k(b-y)}\right]$$
$$= R_\infty\ e^{-k(y-s)} \quad . \tag{115}$$

From (28):

$$\mathcal{T}(s,y,\infty) = \lim_{b\to\infty}\mathcal{T}(s,y,b) = \lim_{b\to\infty}\frac{\Delta(b-y)}{\Delta(b-s)} = e^{-k(y-s)} \quad . \tag{116}$$

Hence:

$$\mathcal{R}(s,y,\infty) = R_\infty \mathcal{T}(s,y,\infty) \quad . \tag{117}$$

Using (115) through (117) in (114) and recalling (106) through (109), the result is:

$$\Psi(s,y:0,\infty) = e^{-k(y-s)}\begin{bmatrix}0 & 0\\ R_\infty & 1\end{bmatrix} +$$
$$+\left[\int_0^s e^{2kt}dt\right]\ \rho e^{-k(s+y)}\begin{bmatrix}R_\infty & 1\\ R_\infty^2 & R_\infty\end{bmatrix} \tag{118}$$

There are many ways to arrange the final form of (118). For example, one such form is:

$$\boxed{\Psi(s,y:0,\infty) = e^{-k(y-s)}\begin{bmatrix}0 & 0\\ R_\infty & 1\end{bmatrix} + \frac{\rho}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right]\begin{bmatrix}R_\infty & 1\\ R_\infty^2 & R_\infty\end{bmatrix}} \tag{119}$$

From this form we can pick off the four components of $\Psi(s,y:0,\infty)$:

$$\Psi_{++}(s,y:0,\infty) = \frac{\rho R_\infty}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{120}$$

$$\Psi_{+-}(s,y:0,\infty) = \frac{\rho}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{121}$$

$$\Psi_{-+}(s,y:0,\infty) = R_\infty\ e^{-k(y-s)} + \frac{\rho R_\infty^2}{2k}\ \cdot$$
$$\cdot\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{122}$$

$$\Psi_{--}(s,y:0,\infty) = e^{-k(y-s)} + \frac{\rho R_\infty}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{123}$$

From these, in turn, we find the local Ψ-factors for one-D irradiance fields in $X(0,\infty)$:

$$\Psi_{++}(s,s:0,\infty) = \frac{\rho R_\infty}{2k}\left[1 - e^{-2ks}\right] \tag{124}$$

$$\Psi_{+-}(s,s:0,\infty) = \frac{\rho}{2k}\left[1 - e^{-2ks}\right] \tag{125}$$

$$\Psi_{-+}(s,s:0,\infty) = R_\infty\left[1 + \frac{\rho R_\infty}{2k}(1 - e^{-2ks})\right] \tag{126}$$

$$\Psi_{--}(s,s:0,\infty) = \frac{\rho R_\infty}{2k}\left[1 - e^{-2ks}\right] \tag{127}$$

The preceding set of eight equations offers an excellent opportunity to illustrate the general functional relations (31) through (34) of Sec. 3.9 and especially their complementary relations (39) through (42) of Sec. 7.13. Furthermore, the relations (6) through (15) of Sec. 7.13 are also illustrated in perhaps their simplest settings. Observe, for example, how (39) of 7.13 guided the derivation of (127) from (123), and how (14) of Sec. 7.13 was used to find (126).

When the source level s is allowed to sink lower and lower into $X(0,\infty)$, equations (124) through (127) go to the relatively simple forms:

$$\lim_{s\to\infty} \Psi_{++}(s,s:0,\infty) = \frac{\rho R_\infty}{2k} \tag{128}$$

$$\lim_{s\to\infty} \Psi_{+-}(s,s:0,\infty) = \frac{\rho}{2k} \tag{129}$$

$$\lim_{s\to\infty} \Psi_{-+}(s,s:0,\infty) = R_\infty\left[1 + \frac{\rho R_\infty}{2k}\right] \tag{130}$$

$$\lim_{s\to\infty} \Psi_{--}(s,s:0,\infty) = \frac{\rho R_\infty}{2k} \tag{131}$$

Furthermore, if both source level s and observation level y descend into $X(0,\infty)$ so that the difference $d = y-s$ remains fixed, the observed boundary effects at level 0 eventually die away and (120) through (123) yield:

$$\lim_{s\to\infty} \Psi_{++}(s,s+d:0,\infty) = \frac{\rho R_\infty}{2k} e^{-kd} \tag{132}$$

$$\lim_{s\to\infty} \Psi_{+-}(s,s+d:0,\infty) = \frac{\rho}{2k} e^{-kd} \tag{133}$$

$$\lim_{s\to\infty} \Psi_{-+}(s,s+d:0,\infty) = R_\infty e^{-kd}\left[1 + \frac{\rho R_\infty}{2k}\right] \tag{134}$$

$$\lim_{s\to\infty} \Psi_{--}(s,s+d:0,\infty) = e^{-kd}\left[1 + \frac{\rho R_\infty}{2k}\right] \tag{135}$$

An unexpected dividend accrues from the preceding array of Ψ-factor relations. Observe first that (128)-(131) agree with our intuitive ideas that the relation between the source irradiance $H^o(s,+)$ and the response field $H(s,+)$, should be the same as that between $H^o(s,-)$ and $H(s,-)$ when boundaries are far away from level s (because the medium is optically symmetric about very deep levels). By the same intuitive expectations, Ψ_{+-} and Ψ_{-+} of (126) and (130) should be numerically equal. Apparently, this does not seem to be the case. However, if we rely on the correctness of our principles and algebraic manipulations, to yield up (129) and (130) then on the basis of our intuition we are led to conclude that:

$$\frac{\rho}{2k} = R_\infty\left[1 + \frac{\rho R_\infty}{2k}\right] ,$$

or equivalently that:

$$\boxed{\frac{\rho}{2k} = \frac{R_\infty}{1 - R_\infty^2}} \tag{136}$$

where ρ is the local reflectance (i.e., backscattering) factor for the one-D theory and k and R_∞ are respectively the decay rate and reflectance factor associated with irradiance fields in $X(0,\infty)$. It follows that all the preceding results (128)-(131) may be characterized in terms of R_∞ only. The reader may establish (136) independently of the preceding argument by using the connections (105) above, with (32) of Sec. 7.3, which holds in the irradiance context also. Still further connections between k, ρ, R_∞ and related concepts are available in Chapters 9 and 10.

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8.8 A Model for Vector Irradiance Fields

The purpose of this section is to apply the vector theory of the irradiance field to an important class of scattering-absorbing optical media, namely the class of natural hydrosols consisting, e.g., of oceans, harbors, and lakes. The application is of practical value in that it yields explicit expressions for the depth-dependence of the irradiance vector in terms of its components at the surface and certain of the optical properties of these media. Furthermore, the discussion presents particularly simple interpretations of the quasi-potential and related functions, arising in vector analysis and which are pertinent to the description of natural light fields. These vector interpretations emerge naturally