8.10 Bibliographic Notes for Chapter 8

It is generally agreed that the history of the two-flow irradiance equations begins with the classic paper by Schuster [279]. The differential equations derived dealt with a pair of irradiance functions representing two counter-flowing streams of radiant energy (one outward and one inward) in a stellar atmosphere. In the hands of Schwarzschild [281], King [138], and Milne [180], Schuster's approach was developed into a relatively complete description of the light field by means of the equation of transfer for radiance. With the advent of the work of Hopf [111], the problems of radiative transfer theory took a deeper mathematical turn, and with the physical insight of Ambarzumian [1], and the industry of Chandrasekhar [43], the notions of the principles of invariance were conceived and exhaustively developed for the simplest plane-parallel settings; and radiative transfer theory as it is known today took its early definitive form.

On the other hand, there followed from Schuster's work another chain of studies which dwelled almost exclusively on his original pair of equations for irradiance; reshaping them, generalizing them, occasionally rediscovering them, and applying them to all manners of optical media from paint and paper, to the atmosphere and the sea. These took, for the most part, the form of the one-D model of Sec. 8.6. The industrial researchers and the geophysicists took alternate turns in the formulations and applications, the results being typified by papers of Schmidt [273], Benford [18], Kubelka and Munk [146], Channon, Renwick, and Storr [44], Mecke [174], Dietzius [65], Silberstein [285], Ryde and Cooper [70], and Duntley [69]. Concurrently, certain Russian authors, notably Gurevic [102], Boldyrev and Alexandrov [27], and Gershun [97], made important contributions to Schuster's theory. The latter papers are curious mixtures of the archaic forms of the equations during that period along with a few isolated brilliant innovations which only much later came into widespread use. Indeed, in the papers of Gurevic [102], and Schmidt [273], for example, may be found the rudimentary but recognizable germ of the idea behind equations (40) and (41) of Sec. 8.7. The fundamental Riccati equation (39) of Sec. 8.7 did not appear in its full form, but as a recognizable primitive variant, and with the physical significances of the coefficient functions being obscure. Stokes [291] also obtained Riccati-type equations in his early researches. The invariant imbedding formulas, of a faint but noticeable variety, can be traced back to Fresnel [94].

With the formulation of neutron diffusion problems there arose a certain amount of mutually profitable cross-fertilization of techniques between neutron diffusion and radiative transfer theories, which stems principally from the papers of Wick [319], and Chandrasekhar [42]. The ramifications of this interaction may be traced in neutron transport theory in [62]. In the Wick paper and subsequently in Chandrasekhar's work, the Schuster two-flow equations were extended to handle n-flows with particular emphasis on the form of the coefficients most suitable to numerical analysis.

rather than on their physical significance (as developed for example, in (5)-(8) of Sec. 8.3, (6)-(9) of Sec. 8.3).

Some relatively recent works based on or relatable to Schuster's theory are contained in the papers of Benford [18], [19], [20], Whitney [316], Hulbert [114], Kubelka [145], Middleton [178], a report by Sliepcevitch and associates [287], and a paper by Kottler [142]. A fairly exhaustive bibliography of the two-flow theory may be compiled from the references of the above papers.

The developments of this chapter are drawn in the main from [221], in which the two-flow equations (19) of Sec. 8.3 were first rigorously derived from the equation of transfer and with particular emphasis on the structure of the functions $f(z,\pm)$, $b(z,\pm)$, $a(z,\pm)$, $\alpha(z,\pm)$, $s(z,\pm)$. Reference [221] is also the source of the two-D model and related concepts. The invariant imbedding relation and the principles of invariance for irradiance were developed in [243], along with a Green's function construction of the R and T factors. The latter construction is simply a special case of the method of the interaction principle, and reproduces in miniature the constructions of Chapter XIV of Ref. [251]. Sec. 8.8 is based in the main on [227]. The divergence law of the light field (15) of Sec. 8.8, along with its most general form is given in [220]. A theory of the irradiance vector $\mathbf{H}(x)$, and the corresponding divergence law in vacua is developed in [187] and [188]. Gershun [98] was first to explicitly recognize the importance of the photometric counterpart to the irradiance vector $\mathbf{H}(x)$, and Milne [180] noted the divergence law's occurrence in astrophysical optics. This same law may be found in Chandrasekhar [43]. The canonical form for irradiance, as given in (40) of Sec. 8.9, was developed in [223].

The theory of internal sources given in Sec. 8.5 has been considerably extended in subsequent invariant imbedding studies of linear hydrodynamics.[1-2] These new techniques (derived in the hydrodynamic context) are directly applicable to the Schuster two flow equations with source terms.

The determination of the four transfer functions $R(x,z)$, $T(x,z)$, $R(z,x)$, $T(z,x)$ (cf. (39)-(42) of Sec. 8.7) can be made *simultaneously* via integration of so called *Riccati quartets* of differential equations, as developed recently in linear hydrodynamics.[3]

1. Preisendorfer, R. W., *Forcing Long Surface Waves Through Two-Port Basins. I. Circuit Variables*, NOAA-JTRE-156, HIG-76-11 Hawaii Institute of Geophysics, November 1976.

2. Preisendorfer, R. W., *Forcing Long Surface Waves Through Two-Port Basins. II. Two-Flow Variables*, NOAA-JTRE-157, HIG-76-12 Hawaii Institute of Geophysics, November 1976.

3. Preisendorfer, R. W., *Multimode Long Surface Waves in Two-Port Basins*, NOAA-JTRE-125, HIG-75-4 Hawaii Institute of Geophysics, January 1975.