brevity, the optical medium, i.e., the quintuple, is usually referred to simply as "X".

Finally, we make clear the meaning of the term "optical property" as it is used in radiative transfer theory, by means of the following:

Definition 6: An *optical property* of an optical medium X is any interaction operator s, including its components s_{ij}, if any, arising from the use of the interaction principle in X.

Several comments can be made on the preceding definition which will help tie together the ideas of this and the following sections. First, we deduce from definitions 1 and 6 that the inherent optical properties are precisely those which arise when the incident radiometric quantities used in the interaction principle are *radiance distributions*; all other types of incident radiometric quantities by definition give rise to apparent optical properties. Second, when S in the interaction principle of Chapter 3 is a one, two, or three dimensional subset of X, then the associated optical property is a global property. If S is a subset of X consisting of one point (a zero dimensional subset of X) then the associated property is a local property. Finally, it is helpful to distinguish between optical properties which are associated with diffuse or n-ary (in other words *decomposed*) radiometric functions and undecomposed functions (re: (19) through (22) of Sec. 5.1 and Sec. 8.4). The undecomposed radiometric functions are those that are *directly observed* in nature with the standard radiant flux meters and their variant instrumental forms; and to emphasize this feature, we shall in the present work also refer to such radiometric quantities and their associated optical properties as *directly observable*.

9.2 Directly Observable Quantities for Light Fields in Natural Hydrosols

In this section we introduce a set of directly observable, apparent optical properties which have been found to be of extraordinary power in the radiometric description of natural hydrosols. Because of the importance of these properties we shall introduce them and sketch their logical and historical background in some detail.

Introduction

The increasing accuracy of experimental determinations of the optical absorption and scattering properties of oceans, lakes, and harbors has necessitated exact knowledge of the possible interrelations between these properties. The classical one-D model of the two-flow analysis of the light field has been an important source of such relations (Sec. 8.6). While it continues to be a source of useful simple models for engineering calculations, it has become inadequate to collate the data of basic experimental research

The general two-flow theory of Sec. 8.3 was formulated to meet these new experimental needs. The two-flow formulation is based on the exact equation of transfer for the light field in scattering-absorbing media. The main results accumulated below give useful relations among the magnitudes of the downwelling and upwelling irradiances, their depth rates of change, and the scattering and absorption properties of natural hydrosols. The formulations are applicable in general to arbitrarily stratified plane-parallel media.

To gain an adequate insight into the formulations below, it is necessary to recall some of the historical background of the classical one-D irradiance theory, and hence radiative transfer theory itself. Radiative transfer theory arose, in part, in the attempts of astrophysicists to understand the problem of the passage of light through stellar and planetary atmospheres. The theory subsequently was found to be of use in terrestrial settings and consequently was directed by geophysicists to the problem of the passage of light through natural optical media such as the atmosphere of the earth, and its waters such as oceans and lakes. While both fields of astrophysics and geophysics share a common interest in the theory of optical media that scatter and absorb light, their experimental procedures are vastly different: The geophysicist has easy access to his optical media and can amass first hand data from even the remotest places of a natural aerosol or hydrosol. This fact has far-reaching effects on the form of the attendant theory of radiative transfer: The astrophysicist has no such direct access and is forced to devise radically different mathematical models, and limit his explorations to theoretical studies of such models and their observable consequences.

Thus when the geophysicists adopted the more useful of these astrophysical models, they inherited, for better or worse, certain simplifying assumptions built into them, assumptions which had been explicitly and deliberately introduced to make them analytically tractable and of some practical worth. In the course of time, as the study of the light fields in natural aerosols and especially natural hydrosols became progressively more exact and exhaustive, these adopted models, which were extremely valuable in planning and interpreting the early geophysical experiments, became progressively inadequate, especially in their predictive and correlative powers. However, the formulas connected with these models had proved so convenient, and their simplicity so appealing, that their impending loss set off a search for more exact replacements which were to retain, whenever possible, the utility and simplicity of the classical formulas. The results of such a search are summarized in the discussion that follows. The model considered is the modern version of the two-flow theory as developed in Sec. 8.3. In what follows we will use the general formulation of the two-flow equations, and require precise knowledge of the assumptions adopted in the classical formulations. The requisite groundwork has been covered in adequate detail in Chapter 8, and the following discussion will draw freely from the results of that chapter.

While the motivation and main emphasis of this study is in connection with that branch of radiative transfer known as hydrologic optics, it would perhaps be of interest to observe that the results summarized below can actually be applied to the experimental study of the transfer of radiation through any arbitrarily stratified plane-parallel medium with arbitrary incident lighting and reflecting boundary conditions.

Classical Two-Flow Theory: The Theoretical K-Functions

In the present discussion we will prepare the way for the desired general formulations by returning to the classical one-D model of the two-flow theory, as it has been developed and used in hydrologic optics, and isolating those formulas which have been of importance in checking and collating experimental data. The key notion in all that follows is that of the diffuse absorption coefficient k and its generalizations.

As we saw in (36) of Sec. 8.6, the classical one-D theory for decomposed irradiance fields in natural hydrosols is embodied in the pair of equations,

$$\frac{dH^*(z,-)}{dz} = -\,[a^* + b^*]\,H^*(z,-) + b^*H^*(z,+) + f^oH^o(z,-)$$

$$-\frac{dH^*(z,+)}{dz} = -\,[a^* + b^*]\,H^*(z,+) + b^*H^*(z,-) + b^oH^o(z,-) \qquad (1)$$

Here $H^*(z,-)$ and $H^*(z,+)$ are the irradiances at depth z induced, respectively, by downwelling diffuse flux (-) and upwelling diffuse flux (+). The term, "diffuse flux," as defined in (22) of Sec. 5.1, refers to flux which has been scattered one or more times. $H^o(z,-)$ is the downwelling residual or reduced irradiance at depth z and refers to flux which has not been scattered. The upward residual irradiance $H^o(z,+)$ is zero; a^*, b^* are the volume absorption and backward scattering coefficients for the diffuse flux; f^o and b^o are the forward and backward scattering coefficients for the downwelling residual flux. The quantities a^*, b^*, f^o, and b^o are assumed constant and known, $H^o(z,-)$ is assumed known for each depth z, and is given explicitly in (22) of Sec. 8.4.

Diffuse Absorption Coefficient k

The parameter which plays the greatest single role in the classical one-D theory is the so-called diffuse absorption coefficient k, given in the form (re: (32) of Sec. 8.6).

$$k = +\,[a^*(a^* + 2b^*)]^{1/2} = 2[a(a + b^*)]^{1/2} \qquad (2)$$

The alternate expression follows from the assumption (usually made in classical applications) that the diffuse flux is described by a uniform radiance distribution in the upper and lower hemispheres; a is the volume absorption coefficient, and is related to a* , under the above assumption, by $a^* = 2a$ (re: (27) of Sec. 8.6). From a mathematical point of view, k is a natural consequence of the solution procedure for (1), which yields the general solutions:

$$H^*(z,-) = m_+g_-e^{kz} + m_-g_+e^{-kz} - A_-(z)H^o(0,-)$$

and (3)

$$H^*(z,+) = m_+g_+e^{kz} + m_-g_-e^{-kz} - A_+(z)H^o(0,-) \quad .$$

Here:

$$g_- = 1 - \frac{a^*}{k} \quad , \quad g_+ = 1 + \frac{a^*}{k} \quad ,$$

and m_+ , m_- are constants of integration determined by the values $H^*(0,-)$, $H^*(z_1,+)$, and $H^o(0,-)$ where z_1 is the depth of the medium. Finally, $A_\pm$ is a determinable function (re: (67) of Sec. 8.5) whose specific form is not of interest at present.

In order to fully understand the significance of the experimental K-functions discussed below, we underscore the fact that k , as represented in (2) and as used in the system (3), is a purely mathematical construct: It is devoid of any physical meaning as it emerges from the solution procedure. It has generally no basis in any realizable physical operation. Only in the case of an optically infinitely deep medium irradiated by a uniform radiance distribution may a physical interpretation be assigned to k . For, in this case, the system (3) is such that we may deduce the following representations of the undecomposed irradiances (compare with (22) and (23) of Sec. 8.6):

$$H(z,-) = H^o(z,-) + H^*(z,-) = m_-g_+e^{-kz} = H(0,-)e^{-kz}$$

(4)

$$H(z,+) = H^o(z,-) + H^*(z,+) = m_-g_-e^{-kz} = H(0,+)e^{-kz}$$

and

$$H^o(z,+) = 0$$

In order to obtain the system (4) directly from (3), it was assumed that $H^o(z,-)$, the reduced downwelling irradiance, is associated with an angularly uniform radiance distribution, and behaves with depth in the following manner:

$$H^o(z,-) = H^o(0,-)e^{-\alpha^o z} \quad ,$$

where $\alpha^o = \alpha D^o$, and $D^o = 2$. Furthermore, $f^o = f^*$, $b^o = b^*$, $a^o = a^*$. Under this assumption, along with $H^o(z,+) = 0$, it turns out $A_+(z) = 0$ and that $A_-(z) = e^{-\alpha^o z}$.

The final step to (4) was to deduce that $m_+ = 0$, and to recall that the directly observable irradiance H(z,-) is the sum of H (z,-) and H*(z,+). It is now clear that (4) could also be arrived at using the one-D model for the undecomposed irradiance field (re: (22) and (23) of Sec. 8.6).* In the above case, and only that case, is it possible to interpret K as the decay rate of the irradiance for both streams; and the system (4) suggests the following operational definition for k :

$$k = \frac{-1}{H(z,-)} \frac{dH(z,-)}{dz} = \frac{-1}{H(z,+)} \frac{dH(z,+)}{dz} \quad . \tag{5}$$

Even with this interpretation of k , its formal nature is inescapable; it is a rare medium, indeed, which exhibits a light field of *exactly* the structure described by the system (4).

The term "diffuse absorption coefficient," for k stems from the following three observations: first, and least important is the fact that k is defined in terms of coefficients a* , b* which are associated with the diffuse flux making up the light field. Second and more important is the fact that in the extreme case where the medium exhibits only absorption, we have formally, k = a*. And finally, if the medium exhibits no absorption, k = 0 (see also (13) of Sec. 10.8).

The R-Infinity Formulas

In addition to the determination of the depth dependence of the values H(z,-) and H(z,+), the experimental determination of the ratio H(z,+)/H(z,-) is of interest because knowledge of this ratio aids in the solution of underwater visibility and photography problems and certain aspects of marine biology problems.

The classical two-flow model yields several expressions for this ratio, the most important being:

$$R_\infty = H(z,+)/H(z,-) = g_-/g_+ = (k - a^*)/(k + a^*) \tag{6}$$

which follows directly from (4), and which was derived earlier in (21) of Sec. 8.6. By using the definition of k , and some simple algebraic rearrangements, (6) may be cast into either of the following two equivalent forms:

$$R_\infty = \frac{a^* + b^* - k}{b^*} \quad , \tag{7}$$

*This is the more direct of the two methods. However, the alternate route, patterned after the classical approach, is retained to reinforce the point made in this section, namely the inadequacy of the classical approach to form a theory of *directly observable* radiometric concepts in natural optical media. (See appendix, sheet 1 of [210] for original derivation details.)

$$R_\infty = \frac{b^*}{a^* + b^* + k} \quad . \tag{8}$$

The primary virtue of these formulas lies in their simple, explicit inclusion of b^*, whose magnitude may then be estimated if the remaining quantities are known.

Finally, by using the identity

$$\alpha^* = a^* + s^* = a^* + f^* + b^* \quad ,$$

we have:

$$R_\infty = \frac{\alpha^* - f^* - k}{b^*} \quad , \tag{9}$$

and alternately:

$$R_\infty = \frac{b^*}{\alpha^* - f^* + k} \quad . \tag{10}$$

In the above formulas, α^* is the volume attentuation coefficient for the diffuse flux; α^* is related to the basic volume attenuation coefficient α, under the one-D assumptions, by the formula: $\alpha^* = 2\alpha$; α is the sum of the volume absorption coefficient a, and the volume total scattering coefficient s. If α is known, (9) and (10) may then be used to estimate f^*.

The Inequalities

One final set of relations which have consistently proved their usefulness in checking the experimentally obtained values of optical constants of turbid media are the inequalities:

$$a^* \leq k < \alpha \quad . \tag{11}$$

The left inequality follows from (6) by observing that R_∞, k, and a^* can never be negative. It also may be obtained from (2) by omitting the nonnegative number "$2b^*$" from the binomial.

The right-hand inequality is more difficult to establish using simple models. It can be established only under certain conditions on the lighting and inherent optical properties of the medium (see, e.g., (9) of Sec. 6.6). Such conditions will come to light in the discussions of k_∞ in (43) of Sec. 10.5. It is on this basis that the right-hand inequality is understood to hold.* The exact version of (11)

*In media that exhibit no scattering, there can be no diffuse flux. The definition of k is so rigid, however, that a formal contradiction of the inequality $k < \alpha$ can be obtained by setting $s = 0$. For then $b^* = 0$, and $k = a^* = 2a = 2\alpha$.

that holds in real media will be derived below (cf. (26) and (27)).

Observations on Inadequacies of Classical Theory

The increasing inadequacy of the two-flow model is perhaps most succinctly summarized in the preceding discussion of the basic inequalities of the theory. But this is a mathematical matter which should carry less weight than some more striking inadequacies on the experimental front: the observed nonconstancy of R_∞, the lack of an operational definition of k, except in extreme hypothetical cases, and the fact that the basic system (1) refers to nonobservable quantities, namely $H^*(z,\pm)$. However, most of the trouble stems from the fact that the angular structure of *both* the diffuse and reduced radiance distribution have, for all depths, been assigned a fixed uniform structure in both the upper and lower hemispheres. From this follows the constancy of all the absorption and scattering coefficient functions, and that of R_∞, which is in direct conflict with experimental observations.

The question then arises: Is there some procedure which retains the conceptual simplicity of the classical two-flow analysis and at the same time incorporates an accurate representation of the effects of the depth dependence of the observable radiance distributions? The answer is yes. The details of the solution are presented below.

Exact Two-Flow Theory: Experimental K Functions and R Functions

We turn now to the systematic development of the directly observable counterparts to the concepts occurring in the one-D and two-D models for irradiance fields. First of all, the distribution functions for the downwelling and upwelling streams are defined by writing:

$$\text{"}D(z,-)\text{"} \quad \text{for} \quad \frac{h(z,-)}{H(z,-)} \tag{12}$$

and

$$\text{"}D(z,+)\text{"} \quad \text{for} \quad \frac{h(z,+)}{H(z,+)} \tag{13}$$

The quantities $h(z,-)$ and $h(z,+)$ are the scalar irradiance induced, respectively, by the down and upwelling streams at depth z (cf. (11) of Sec. 2.7). They may be measured by suitably shielded spherical collectors, whereas, the ordinary irradiances $H(z,-)$ and $H(z,+)$ are measured by flat plate collectors. The ratio of a hemispherical scalar irradiance to an irradiance provides a simple means of characterizing the directional distribution of flux in each stream, as we saw in some detail in Sec. 8.5. But their usefulness extends considerably beyond this service, as we shall see below.

In what follows, $H(z,-)$ and $H(z,+)$ are the undecomposed downwelling and upwelling irradiances: the irradiances actually measured by horizontal flat plate collectors. The exact equations governing these source-free media are given by (19) of Sec. 8.3 and are:

$$\frac{dH(z,-)}{dz} = -[a(z,-)+b(z,-)]H(z,-)+b(z,+)H(z,+)$$

$$-\frac{dH(z,+)}{dz} = -[a(z,+)+b(z,+)]H(z,+)+b(z,-)H(z,-) \tag{14}$$

The other functions appearing in (14) are the absorption, $a(z,\pm)$, and backward scattering, $b(z,\pm)$ functions for the respective streams; their exact definitions and properties are given in (8) and (13) of Sec. 8.3. For brevity, source terms have been omitted from the system (14); that is, terms which describe self-luminous or transpectral sources of radiant flux distributed throughout the medium. Their inclusion does not modify the essential forms of the following results. However, as noted in Sec. 1.2 and in the discussion of definitions 4 and 5 in Sec. 9.1, true sources are relatively scarce in natural hydrosols, and so may be omitted from most basic discussions.

In (5) the diffuse absorption coefficient was expressed as the logarithmic derivative of the irradiances of each flow. It was shown that this simple operational characterization was possible only in a very special set of circumstances in the classical two-flow theory. It turns out, however, that such an operation is the natural way to characterize the decay (and growth) rate of each stream in real media. Each stream in general has its own depth rate of change. These rates of change are not completely independent, as we shall see below, and they are generally different in magnitude. Thus we are led to write:

$$\text{"}K(z,-)\text{"} \quad \text{for} \quad \frac{-1}{H(z,-)}\frac{dH(z,-)}{dz} \,,$$

$$\text{"}K(z,+)\text{"} \quad \text{for} \quad \frac{-1}{H(z,+)}\frac{dH(z,+)}{dz} \,. \tag{15}$$

We call $K(z,\pm)$ the *K-function* (or absorption function) for downward (-) or upward (+) irradiance. Just as the K-functions for each stream are found to change with depth, so does the irradiance ratio, where we write:

$$\text{"}R(z,-)\text{"} \quad \text{for} \quad \frac{H(z,+)}{H(z,-)} \,, \tag{16}$$

and hence its reciprocal also changes with depth z, where we write:

$$\text{"}R(z,+)\text{"} \quad \text{for} \quad \frac{H(z,-)}{H(z,+)} \,. \tag{17}$$

We call $R(z,\pm)$ the *reflectance function* for downward (-) or upward (+) irradiance. The term $R(z,-)$ in (16) is the exact experimental counterpart to R_∞, and may be thought of as the reflectance of the hypothetical plane surface at depth z, with respect to downwelling flux. A similar interpretation exists for (17).

From (14) and the preceding definitions:

$$\boxed{K(z,-) = a(z,-) + b(z,-) - b(z,+)R(z,-)} \quad , \qquad (18)$$

$$\boxed{-K(z,+) = a(z,+) + b(z,+) - b(z,-)R(z,+)} \quad , \qquad (19)$$

which show how the experimental K-functions are linked to the absorption and scattering properties of the medium. From these, we immediately obtain the exact counterparts to (7) and (8):

$$\boxed{R(z,-) = \frac{a(z,-) + b(z,-) - K(z,-)}{b(z,+)}} \qquad (20)$$

$$\boxed{R(z,-) = \frac{b(z,-)}{a(z,+) + b(z,+) + K(z,+)}} \qquad (21)$$

While the overall resemblance to (7) and (8) is quite evident, care should be taken to distinguish the relatively subtle roles now played by the functions associated with each flow.

The counterparts to (9) and (10) are obtained by making use of the general identity ((18) of Sec. 8.3):

$$\alpha(z,\pm) = a(z,\pm) + s(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm) \quad ,$$

Thus, in general:

$$\boxed{R(z,-) = \frac{\alpha(z,-) - f(z,-) - K(z,-)}{b(z,+)}} \qquad (22)$$

$$\boxed{R(z,-) = \frac{b(z,-)}{\alpha(z,+) - f(z,+) + K(z,+)}} \quad . \qquad (23)$$

The Basic Reflectance Relation

The general counterpart to (6) is singled out for special attention because it is the most useful representation of relfectance functions in practice. It relates the six direct observables: the two K-functions, the two distribution functions, the volume absorption function, and the reflectance function. It therefore may replace (6) by providing an exact formula to check the consistency of the experimentally determined values of these functions. Furthermore, from a theoretical point of view, the general counterpart to (6) is closely related to the divergence relation for the light field (re: (18) of Sec. 9.8) in fact directly derivable from it, as shown below. Alternate derivations, of course, can be made directly from the system (14). Now, for the derivation at hand, the divergence relation for the light field in stratified media may be written in the form:

$$\frac{d\overline{H}(z,+)}{dz} = a(z)h(z) \quad , \qquad (24)$$

where $\overline{H}(z,+) = H(z,+) - H(z,-)$, and where $h(z)$ is the scalar irradiance at depth z, and $a(z)$ is the value of the volume absorption function at depth z.

Rewriting this with the help of (9) of Sec. 2.7, as:

$$\frac{dH(z,+)}{dz} - \frac{dH(z,-)}{dz} = a(z)[h(z,+) + h(z,-)] \quad ,$$

and dividing each side by, say $H(z,-)$, we have:

$$\frac{H(z,+)}{H(z,-)} \frac{1}{H(z,+)} \frac{dH(z,+)}{dz} - \frac{1}{H(z,-)} \frac{dH(z,-)}{dz} =$$

$$= a(z)\left[\frac{h(z,+)}{H(z,+)} \frac{H(z,+)}{H(z,-)} + \frac{h(z,-)}{H(z,-)}\right]$$

Applying the appropriate definitions, this may be rewritten:

$$- R(z,-)K(z,-) + K(z,-) = a(z)[D(z,+)R(z,-) + D(z,-)]$$ Errata

$$= a(z,+)R(z,-) + a(z,-) \quad .$$

Solving for $R(z,-)$:

$$\boxed{R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)}} \qquad (25)$$

which is the desired exact experimental counterpart to (6).

The Exact Inequalities

To obtain the exact counterparts to the classical inequalities (11), we encounter a reversal of difficulty: the counterpart to $k < \alpha$ is relatively simple to establish, and its validity is completely general; the counterpart to $a^* \leq k$ requires additional assumptions, but of a kind which are a consequence of the generality of the present formulations rather than their shortcomings.

The first member of (14) may be written as:

$$\frac{dH(z,-)}{dz} = -\alpha(z,-)H(z,-) + \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi)$$

so that:

$$K(z,-) = \alpha(z,-) - \frac{1}{H(z,-)} \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi) \quad .$$

This representation may be obtained by using the definition of $K(z,-)$, recalling (18) of Sec. 8.3, and the derivation leading to (9) of Sec. 8.3. Since the subtracted member of the right side is never negative, we have immediately:

$$K(z,-) \leq \alpha(z,-)$$

for all z. Finally, whenever $0 \leq K(z,+)$, we have (see note below) from (25):

$$a(z,-) \leq K(z,-) \quad ,$$

which establishes the desired inequalities:

$$\boxed{a(z,-) \leq K(z,-) \leq \alpha(z,-)} \quad , \tag{26}$$

or equivalently:

$$a(z) \leq \frac{K(z,-)}{D(z,-)} \leq \alpha(z) \quad .$$

and alternately

$$0 \leq \frac{K(z,-)}{D(z,-)} - a(z) \leq s(z)$$

A corresponding set of inequalities for the upwelling stream may be obtained in a similar way:

$$\boxed{a(z,+) \leq -K(z,+) \leq \alpha(z,+)} \tag{27}$$

or equivalently:

$$a(z) \leq -\frac{K(z,+)}{D(z,+)} \leq \alpha(z) \quad .$$

and alternately:

$$0 \leq -\frac{K(z,+)}{D(z,+)} - a(z) \leq s(z)$$

The right-hand side of (27), as that of (26), holds in general; the left side of (27) holds whenever $K(z,-) \leq 0$, a condition completely symmetric to the condition, $0 \leq K(z,+)$ used to establish the left side of (26).

The Significance of the Condition $0 \leq K(z,+)$

The significance of the condition, $0 \leq K(z,+)$, used to establish the left side of (26), is quite important; and the adoption of this condition raises some interesting questions. First of all, we observe that if the condition holds, then the denominator of (25) is positive. Since $R(z,-)$ is positive, this then requires the numerator of (25) to be positive, from which the desired inequality follows. But one may ask: is this condition ever violated? In other words, can we ever have: $K(z,+) < 0$? Before answering this, we recall that $K(z,+) < 0$ means that $K(z,+)$ is negative, and physically this means that the function $H(\cdot,+)$ is *increasing* with *increasing* depth at z. A similar interpretation exists for the condition $K(z,-) < 0$. The preceding question may then be put very concretely as follows: As one measures upwelling and downwelling irradiances in a real stratified optical medium, is it ever possible to observe an *increase* in these irradiances as depth is increased? The answer is yes. There are two possible mechanisms which generally allow such a phenomenon to be observed.

The first mechanism is that associated with self-luminous sources within the medium. For example, light-giving organisms distributed in a horizontal layer of water clearly make it possible for increases of irradiance to be observed as the irradiance collectors approach the layer and pass downward through the layer. These layers can occur at quite large depths. Other examples are given by various physical emission processes, fluorescence, i.e., e.g., scattering with change in wavelength, et cetera. The latter mechanisms, as noted several times, ordinarily play a subordinate role in many natural hydrosols,* but in the atmosphere, they can be important. If the emission terms $h_\eta(z,\pm)$ are included in the two-flow equations, this phenomenon can be represented

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*When biological processes in natural hydrosols are of interest in hydrologic optics studies, fluorescence can play import roles in the associated radiative transfer process. Errata In this case transpectral scattering theory (cf. Sec. 19 of [251] is the appropriate theory to use.

explicitly, and a suitable parallel theory can be built up around such a phenomenon.

The second mechanism is that of simple scattering processes: the redirection of radiant flux without change in wavelength. This results in an effective storage of radiant energy within scattering media. It seems plausible that if an increase of irradiance is induced by this mechanism in natural water, the increase should noticeably occur at depths near the surface of the medium. For in these regions of small depth the diffuse light field is still building up in magnitude, and just the right kind of inhomogeneities may possibly contribute to the effect. Generally, in optically deep scattering media, the downwelling diffuse radiant flux is zero at the surface, increases with increasing depth, reaches a maximum at some depth, and then falls off in a more or less exponential way ever afterward. But the diffuse irradiance $H^*(z,-)$ is not directly observed. Superimposed on it is the reduced irradiance $H^\circ(z,-)$, which clearly must decrease continuously with depth, starting right from the upper surface. Thus, whether or not the directly observable irradiance $H(z,\pm) = H^\circ(z,\pm) + H^*(z,\pm)$ exhibits any increase with increasing depth clearly depends upon the magnitudes and relative rates of change of each of its components.

A theoretical discussion of the conditions which govern the growth of the light field in stratified media is out of place here. We merely note in passing that many such conditions can be extracted from expressions like (14), (18), or (19) given above; furthermore, various approximate models of the light field such as the two-D theory discussed in Chapter 8 give very explicit, if only approximate, criteria for the growth of the light field. Some of these possibilities will be explored in Chapter 10. Finally, the possibility of the growth of *radiance* values has been predicted by a simple model for radiance distributions in natural hydrosols. This prediction has been verified by experiment. The model is based on the canonical form of the equation of transfer; see in particular (12) of Sec. 4.5.

Relative Magnitudes of H and K Functions

For source-free stratified media, we can make several general observations about the relative magnitudes of the observable H-values and K-values. These have been of help in checking and collating experimental data. First of all, from the integrated divergence relation (24), we deduce that:

$$\bar{H}(z_2,+) - \bar{H}(z_1,+) = \int_{z_1}^{z_2} a(z)h(z)dz \geq 0 \quad ,$$

so that:

$$\boxed{\bar{H}(z_2,-) \le \bar{H}(z_1,-)} \tag{28}$$

which demonstrates that the *net downward* irradiance function $\bar{H}(\cdot,-)$ $(= H(\cdot,-) - H(\cdot,+))$ never increases with depth. Here z_1 and z_2 are any two depths, z_2 being the greater. If the medium is, in particular, finitely deep with a bottom surface whose reflectance r is such that $0 \le r \le 1$, (re: *discussion of solutions*, (6) and (7) of Sec. 3.1) then (28) immediately implies that, for all depths z,

$$\boxed{R(z,-) = \frac{H(z,+)}{H(z,-)} \le 1} \tag{29}$$

If the medium is optically infinitely deep, we have a similar result. For in this case, we have for all depths z:

$$\bar{H}(z,-) = \int_z^\infty a(z')h(z')dz' \ge 0$$

so that

$$H(z,+) \le H(z,-) \quad ,$$

from which (29) follows once more.

We can derive a correspondingly general inequality that must hold between $K(z,-)$ and $K(z,+)$. From (25):

$$K(z,-) - K(z,+)R(z,-) = a(z,-) + a(z,+)R(z,-) \ge 0 \quad ;$$

whence:

$$K(z,+)R(z,-) \le K(z,-) \tag{30}$$

or equivalently:

$$\boxed{\frac{dH(z,-)}{dz} \le \frac{dH(z,+)}{dz}} \quad .$$

This relation throws some light on the question raised above. Relation (30) shows that if $K(z,-)$ is negative, then necessarily $K(z,+)$ is negative, too. Conversely, if $K(z,+)$ is positive, then so must $K(z,-)$ be positive. Finally, (30) hints at real situations in which $K(z,+)$ may well be negative while $K(z,-)$ is positive. Ideal examples of each of these three situations are easily found; however, occurrences in real media have not yet been sought.

Characteristic Equation for $K(z,\pm)$

The classical two-flow theory gives a convenient expression for k in terms of absorption and scattering coefficients as in (2). There is a remarkable corresponding formula which characterizes $K(z,-)$ and $K(z,+)$ in addition to (18) and (19). This exact counterpart to (2) is obtained by eliminating $R(z,-)$ from (20) and (21). The result is:

$$1 = \frac{b(z,-)}{K(z,-) - a(z,-)} - \frac{b(z,+)}{K(z,+) + a(z,+)} \tag{31}$$

That this is the general counterpart to (54) of Sec. 8.5 may be verified for example by setting, as such a verification requires, $b(z,-) = b(z,+) = b^*$, $a(z,-) = a(z,+) = a^*$, and $K(z,-) = K(z,+) = k$. When this is done, (31) reduces to (2).

The Depth Rate of Change of $R(z,-)$

Since the experimental counterpart to R_∞ generally varies with depth, it is of interest to characterize the variation in terms of the experimental K-functions. The desired formula follows immediately from the definition (16) of $R(z,-)$:

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad . \tag{32}$$

From this we see that the constancy of $R(\cdot,-)$ is equivalent to the equality of $K(\cdot,-)$ and $K(\cdot,+)$. In other words, $R(\cdot,-)$ is constant over any interval (z_1,z_2) of depths when and only when $K(z,-) = K(z,+)$ for every depth z in the interval (z_1,z_2).

Connections Among the K Functions

In this paragraph we will briefly discuss the connection between the k of the classical theory (as in (2)) and the exact K-functions introduced in (5) above. First of all we observe the simple connection that exists between the scalar irradiances $h(z,\pm)$ and the irradiances $H(z,\pm)$ within the framework of the classical theory. Recall that both the diffuse and reduced radiance distributions in both the upper and lower hemispheres are assumed uniform; therefore, for all depths z,

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} = 2 \quad .$$

This means that $h(z,\pm)$ and $H(z,\pm)$ differ multiplicatively only by a fixed factor 2. Thus the operational definition (5) for k lets us conclude:

$$k = -\frac{1}{H(z,\pm)}\frac{dH(z,\pm)}{dz} = -\frac{1}{h(z,\pm)}\frac{dh(z,\pm)}{dz}$$

In other words, the classical theory says that k may be estimated equally well from measurements of scalar irradiances or ordinary irradiances. As demonstrated above (see (12) and (13)) distinctions between h and H are often necessary not only in theory but in careful experimental practice. Consequently, when it becomes necessary to discuss the growth or decay of, say, $h(\cdot,-)$ with depth, its logarithmic derivative is generally considered distinct from that of $H(\cdot,-)$. A similar statement is true for $h(\cdot,+)$. Thus we are led to consider operations of the kind:

$$-\frac{1}{h(z,\pm)}\frac{dh(z,\pm)}{dz} ,$$

and to distinguish these from the operations

$$-\frac{1}{H(z,\pm)} \cdot \frac{dH(z,\pm)}{dz}$$

we write:

$$\text{"}k(z,\pm)\text{"} \quad \text{for} \quad -\frac{1}{h(z,\pm)}\frac{dh(z,\pm)}{dz} \qquad (33)$$

As a mnemonic, we observe that in the exact theory for real media, the little k's go with little h's and big K's go with big H's. In real media the connection between these is:

$$\boxed{k(z,\pm) = K(z,\pm) - \frac{1}{D(z,\pm)}\frac{dD(z,\pm)}{dz}} \qquad (34)$$

We may now state the connection we set out to establish: $k(\cdot,\pm)$ and $K(\cdot,\pm)$ are equal over some interval (z_1,z_2) of depths when and only when the distributon function $D(\cdot,\pm)$ is constant over that interval. This is precisely the situation that holds for optically infinitely deep media in the classical theory so that we have in that setting: $k(\cdot,\pm) = K(\cdot,\pm)$. Furthermore, in this case, $R(\cdot,-) = R_\infty$, and thus is independent of depth. From (32) we may then conclude that $K(\cdot,-) = K(\cdot,+)$. The net conclusion is that for each depth z, the four quantities: $k(z,+)$, $k(z,-)$, $K(z,+)$, $K(z,-)$, which are generally four distinct quantities in real media, are constrained in the one-D model of the two-flow theory to be identical, their common value being k, as given by (2). In this way we justify the generally interchangeable use of k and K in any discussion which has the constancy of the distribution functions in the background.

K-Function for Radiance

The K-functions discussed throughout this section are associated with an *exact* formulation of the two-flow analysis of the light field. They are the little k's for the little h's, and the big K's for the big H's. In more detailed experimental studies of the light field, namely those that document the radiance distribution values $N(z,\theta,\phi)$ at each depth z in all directions (θ,ϕ), a corresponding K-function has been found extremely useful in theoretical work (re: (20) of Sec. 4.5 and Secs. 10.5 and 10.6) and in graphical and tabular representations of these distributions. It is defined by writing:

$$"K(z,\theta,\phi)" \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} \tag{35}$$

No confusion should arise from the continued use of the letter "K": (35) will always explicitly exhibit three variables or places for them when clarity is threatened, the other K's only one. We note in passing that this function, analogously to the other K-functions discussed above, has several interesting theoretical consequences in addition to its immediate experimental uses. However, a discussion of these matters is deferred until Chapter 10.

General K Functions

To round out the discussion of the experimental K-functions, we note that all of the K-functions defined above fall into a specific class, each member of which is defined by an operation of the kind:

$$-\frac{1}{A}\frac{dA}{dz} \quad ,$$

where A could be any of the functions: $H(\cdot,\pm)$, $h(\cdot,\pm)$, $N(\cdot,\theta,\phi)$. Some further possibilities for A are, $h(\cdot)$, $\overline{H}(\cdot,+)$. Furthermore we observe, by means of the divergence relation (21) or more generally by (15) of Sec. 8.8, that the basic volume absorption function may be defined as the result of the operation,

$$\frac{1}{h}\frac{d\overline{H}}{dz} \quad ,$$

on the two types of irradiances shown. Finally, the K-function (1) of Sec. 4.5 should be noted. On the basis of these examples, it appears that the most general notion of an experimental attentuation function (i.e., a general K-function) is definable by an operation of the kind,

$$\frac{1}{A}\frac{dB}{dz} \quad , \quad \text{or} \quad \frac{\nabla B}{A} \tag{36}$$

on any two *observable* radiometric quantities A and B.

Integral Representations of the K Functions

The K-function for radiance is basic in the same way that radiance itself is basic; that is, as a fountainhead of representations of the various radiometric concepts. Thus, it is easily shown that:

$$K(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\xi)K(z,\xi)\xi \cdot \mathbf{n}d\Omega(\xi)}{\int_{\Xi_\pm} N(z,\xi)\xi \cdot \mathbf{n}d\Omega(\xi)} \tag{37}$$

$$k(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\xi)K(z,\xi)d\Omega(\xi)}{\int_{\Xi_\pm} N(z,\xi)d\Omega(\xi)} \tag{38}$$

$$k(z) = \frac{\int_{\Xi} N(z,\xi)K(z,\xi)d\Omega(\xi)}{\int_{\Xi} N(z,\xi)d\Omega(\xi)} \tag{39}$$

where $k(z)$ is the negative logarithmic depth derivative of scalar irradiance $h(z)$. Equations (37) through (39) are indicative of the type of integral representations of the various K-functions defined in (36).

Integral Representations of the Irradiance and Radiance Fields

It follows at once from the definitions of the various K-functions introduced above that the directly observable irradiances $H(z,\pm)$ can be given the following integral representations:

$$H(z,\pm) = H(x,\pm) \exp\left\{ - \int_x^z K(y,\pm)\, dy \right\} \tag{40}$$

where x,y,z are three depths in stratified plane-parallel media $X(a,b)$ such that $a \leq x \leq y \leq z \leq b$. Similarly:

$$N(z,\xi) = N(x,\xi) \exp\left\{ - \int_x^z K(y,\xi)\, dy \right\} \tag{41}$$

Since $K(z,\pm)$ and $K(z,\xi)$ are thus observed to play the general roles of absorption functions analogously to a and k, we can alternately refer to them as *absorption functions* for H or N, as the case may be. (See (29) of Sec. 9.3.)

As an example of the use of (40), let us determine the K-function belonging to a spherically symmetric light field about a point source, imbedded in a natural or laboratory hydrosol. The two-flow equations governing radiative transfer across spherical surfaces of radius r and concentric with the source are governed by (46) of Sec. 8.6 in which now $\nabla \cdot \mathbf{H}(z,\pm)$ takes the form:

$$\nabla \cdot \mathbf{H}(z,\pm) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 H(r,\pm))$$

where $H(r,+)$ is the centripetal flux and $H(r,-)$ is the centrifugal flux at radius r. Hence for a steady spherically symmetric light field, the negative logarithmic derivative of $H(r,-)$ with respect to r is:

$$K(r,-) = a(r,-) + b(r,-) - b(r,+)R(r,-) + \frac{2}{r} \quad . \quad (42)$$

A similar formula holds for the centripetal flux by suitably changing signs in the arguments (cf. (19)).

If r and s are any two radii with $r \leq s$, then (40) and (42) yield the formula:

$$\frac{H(s,-)}{H(r,-)} = \left(\frac{r}{s}\right)^2 T_a(r,s)T_b(r,s) \qquad (43)$$

where we have written:

$$"T_a(r,s)" \quad \text{for} \quad \exp\left\{ - \int_r^s a(u)D(u,-)\,du \right\}$$

$$"T_b(r,s)" \quad \text{for} \quad \exp\left\{ - \int_r^s [b(u,-) - b(u,+)R(u,-)]\,du \right\}$$

Here T_a and T_b are special transmittances built up from absorption and backscatter coefficients, respectively. For media with relatively small $b(u,\pm)$ it follows that $T_b \doteq 1$, and so a practical rule of thumb for K in spherically symmetric fields is:

$$K(r,-) \simeq a(r,-) + \frac{2}{r} = a(r)D(r,-) + \frac{2}{r}$$

and (43) becomes

$$H(s,-) = H(r,-)\left(\frac{r}{s}\right)^2 T_a(r,s)$$

$$= H(r,-)\left(\frac{r}{s}\right)^2 \exp\left\{-\int_r^s a(u)D(u,-)\,du\right\} .$$

A simple, but approximate, operational procedure for measuring volume absorption function in spherically symmetric fields is forthcoming from this. Since

$$\ell n\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} = \int_r^s a(u)D(u,-)\,du \quad ,$$

on holding r fixed and varying only s :

$$a(s) = \frac{1}{D(s,-)} \cdot \frac{d}{ds}\,\ell n\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} \qquad (44)$$

where $D(s,-)$ is the value of the distribution function $D(\cdot,-)$ at radial distance s from the source, and $a(s)$ the required value of the volume absorption function at the same radial distance s . This method of determining the volume absorption function supplements those discussed in Sec. 13.8. Finally, the preceding example shows how, with only minor modifications, all the exact two-flow theory formulas for stratified plane-parallel fields can be used to obtain their correspondents in stratified spherical fields--i.e., spherically symmetric fields of irradiance.

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9.3 The Covariation of the K Function for Irradiance and Distribution Functions

The purpose of this section is to establish the theorem that at arbitrary fixed depths z the attenuation function value $K(z,-)$ and the distribution function value $D(z,-)$ vary *directly* (but not necessarily linearly) one with the other, in all steady state stratified real plane-parallel media whose volume scattering functions are predominantly forward scattering. In this way we establish a useful criterion for the behavior of $K(z,-)$ in terms of the intuitively simpler concept $D(z,-)$. The theorem is expected to find its greatest use in natural hydrosols. By way of background to these results we now discuss in some detail the physical significance of $K(z,-)$ and $D(z,-)$.

Some Elementary Physical and Geometrical Features of K(z,-) and D(z,-)

It is a well-known fact in hydrologic optics that the amount of light in a natural hydrosol such as an ocean or deep lake decreases essentially in an exponential manner with depth from the surface of the hydrosol. The simplest