$$H(s,-) = H(r,-)\left(\frac{r}{s}\right)^2 T_a(r,s)$$

$$= H(r,-)\left(\frac{r}{s}\right)^2 \exp\left\{-\int_r^s a(u)D(u,-)du\right\} \quad .$$

A simple, but approximate, operational procedure for measuring volume absorption function in spherically symmetric fields is forthcoming from this. Since

$$\ln\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} = \int_r^s a(u)D(u,-)du \quad ,$$

on holding r fixed and varying only s :

$$a(s) = \frac{1}{D(s,-)} \cdot \frac{d}{ds} \ln\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} \tag{44}$$

where $D(s,-)$ is the value of the distribution function $D(\cdot,-)$ at radial distance s from the source, and $a(s)$ the required value of the volume absorption function at the same radial distance s . This method of determining the volume absorption function supplements those discussed in Sec. 13.8. Finally, the preceding example shows how, with only minor modifications, all the exact two-flow theory formulas for stratified plane-parallel fields can be used to obtain their correspondents in stratified spherical fields--i.e., spherically symmetric fields of irradiance.

9.3 The Covariation of the K Function for Irradiance and Distribution Functions

The purpose of this section is to establish the theorem that at arbitrary fixed depths z the attenuation function value $K(z,-)$ and the distribution function value $D(z,-)$ vary *directly* (but not necessarily linearly) one with the other, in all steady state stratified real plane-parallel media whose volume scattering functions are predominantly forward scattering. In this way we establish a useful criterion for the behavior of $K(z,-)$ in terms of the intuitively simpler concept $D(z,-)$. The theorem is expected to find its greatest use in natural hydrosols. By way of background to these results we now discuss in some detail the physical significance of $K(z,-)$ and $D(z,-)$.

Some Elementary Physical and Geometrical Features of $K(z,-)$ and $D(z,-)$

It is a well-known fact in hydrologic optics that the amount of light in a natural hydrosol such as an ocean or deep lake decreases essentially in an exponential manner with depth from the surface of the hydrosol. The simplest models of the light field exhibit this fact (cf. (22) of Sec. 8.6). This fact may be expressed succinctly as:

$$H_z = H_o e^{-kz} \tag{1}$$

where H_z is the amount of radiant flux of a given wavelength falling downward on a unit horizontal area (i.e., the *irradiance*) at depth z, and K is a constant determined by measuring the slope of a semilog plot of measured values of H_z versus depth z. The quantity K has dimensions: per unit length, and is usually called the *attentuation coefficient* for irradiance, for the medium under study. Like H_z the quantity K depends implicitly on a specific wavelength λ of the radiant energy penetrating the hydrosol. For many engineering questions, questions of underwater photography, television and visibility, and for many purposes of marine biologist, the story of H_z and K may appropriately end with equation (1). However, for the more demanding purposes of geophysicists charged with the tasks of determining the fundamental optical constants of natural hydrosols, and for those who must make sense out of the experimental data leading to the numerical determination of the fundamental constants, the story of H_z and K has only begun to be told with equation (1).

Confronting these latter investigators in their quest for precisely measured radiometric quantities which must be interrelated by consistent rules of calculation, is a wealth of intricate and nonlinear detail in the depth behavior of H_z. The simple exponential behavior of H_z as summarized in equation (1) must now as a matter of experimental expediency be discarded and in its stead be made to appear a more detailed formula exhibiting the same general outlines of (1), but containing now all the potential variations which may be uncovered in a careful documentation of the light field in real hydrosols. As we saw in (40) of Sec. 9.2, this more detailed formula can take the form:

$$H(z,-) = H(0,-) \exp\left\{-\int_0^z K(z',-)dz'\right\} \quad . \tag{2}$$

Equation (2) is the physicists' generalization of the mathematician's simple model of the light field expressed in equation (1). Let us examine (2) in detail and thereby uncover its similarities and dissimiliarities with (1). First, $H(z,-)$ represents the *measured* irradiance at depth $z \geq 0$ in the medium produced by downwelling radiant flux on a unit horizontal area. Hence $H(0,-)$ is the downwelling irradiance of such flux on such a surface at depth $z = 0$ measured just below the air-water film. Second, suppose we plot the general equation (2) on semilog paper with depth z as abscissa and $H(z,-)$ as ordinate. Equation (2) gives the value of $H(z,-)$ at a general depth z: Hence equation (2) may then also be used to give the value of $H(z,+\Delta z,-)$, i.e., the downwelling irradiance at a depth $z+\Delta z$, where Δz is any finite positive increment in depth. The appropriate formula for this is:

$$H(z + \Delta z,-) = H(0,-) \exp \left\{ - \int_0^{z + \Delta z} K(z',-)dz' \right\} \quad .$$

Now, as in elementary calculus, we may approximate the slope of the semilog plot of $H(z,-)$ at depth z by letting z grow by an increment Δz, by finding the new value $H(z + \Delta z,-)$ and then by performing the operation:

$$\frac{\ell nH(z + \Delta z,-) - \ell nH(z,-)}{\Delta z} \simeq \left[\begin{matrix}\text{slope of } \ell nH(z,-) \\ \text{at depth } z\end{matrix}\right] \tag{3}$$

The smaller the magnitude of Δz, the more accurate is the estimate of the slope of the curve by this operation. Let us now perform the operation on the values of $H(z,-)$ and $H(z + \Delta z,-)$, as given by (1) and (2). From (1):

$$\ell nH(z,-) = \ell nH(0,-) - \int_0^{z} K(z',-)dz' \tag{4}$$

and from (2):

$$\ell nH(z + \Delta z,-) = \ell nH(0,-) - \int_0^{z + \Delta z} K(z',-)dz' \quad . \tag{5}$$

Inserting these values in (3) we have:

$$\left[\begin{matrix}\text{slope of } \ell nH(z,-) \\ \text{at depth } z\end{matrix}\right] \simeq \frac{\int_0^{z} K(z',-)dz' - \int_0^{z + \Delta z} K(z',-)dz'}{\Delta z} \quad . \tag{6}$$

We now recall two elementary facts from integral calculus, the first is:

$$\int_0^{a + \Delta} f\,dz = \int_0^{a} f\,dz + \int_a^{a + \Delta} d\,dz \quad , \text{ Errata} \tag{7}$$

which shows how the range $(0,a + \Delta)$ of integration may be broken into two parts: $(0,a)$ and $(a,a + \Delta)$ for any function integrable over $(0,a + \Delta)$; and the second fact that:

$$\int_a^{a + \Delta} f\,dz \simeq \Delta\, f(a) \quad , \tag{8}$$

whenever Δ is small and f is continuous at a. Applying these two facts to (6), where $K(\cdot,-)$ now takes the place of f in (7) and (8), we have, on application of (7) to (6):

$$\begin{bmatrix}\text{slope of } \ln H(z,-) \\ \text{at depth } z\end{bmatrix} \simeq \frac{\int_z^{z+\Delta z} K(z',-)\,dz'}{\Delta z} \qquad (9)$$

Then, applying fact (8) to (9) and letting $\Delta z \to 0$ we have:

$$- \begin{bmatrix}\text{slope of } \ln H(z,-) \\ \text{at depth } z\end{bmatrix} = K(z,-) \quad . \qquad (10)$$

Equation (10) tells us that $K(z,-)$ is the negative of the slope of the semilog plot of $H(z,-)$ versus depth z.

Another way of obtaining (10) is to directly differentiate (2), the result being:

$$\frac{dH(z,-)}{dz} = - K(z,-)H(z,-) \quad ,$$

and then solve for $K(z,-)$:

$$K(z,-) = - \frac{1}{H(z,-)} \frac{dH(z,-)}{dz}$$

$$= - \frac{d \ln H(z,-)}{dz} \quad . \qquad (11)$$

This latter method is more elegant than the preceding pedestrian method, and thereby brings out more succinctly the geometric meaning of $K(z,-)$.

We may perform the same operation on H_z as given in equation (1). Hence, either by the z-method or the simple derivative scheme shown above, (1) yields:

$$K = - \begin{bmatrix}\text{slope of } \ln(h_z) \\ \text{at depth } z\end{bmatrix} = - \frac{1}{H_z} \frac{dH_z}{dz} \quad . \qquad (12)$$

Thus both K and $K(z,-)$ are the negative slopes of the semilog plots of H_z versus depth z. In this way they are similar. But the point where they differ is in the fact that K is independent of depth and that $K(\cdot,-)$ is not independent of depth. And in this difference lies precisely the difference between mathematical fiction and physical reality. Is there a simple explanation for this gap in terms of the accumulated concepts of hydrologic optics? We now consider in detail an explanation of this difference in terms of the currently accepted concepts of radiative transfer theory, as applied to hydrologic optics.

A careful examination of the experimental evidence leading to H_z determinations shows that there are actually two mechanisms in source-free media which may give rise to the gap between the simple classical K and the modern $K(z,-)$. We have up to this point slanted the discussion to bring out only one of these mechanisms, the one which we may term the *physical* (or dynamical) mechanism of the variation of $K(\cdot,-)$. Thus, in the preceding discussion we centered attention on the *depth behavior* of H(z,-), a behavior which is discerned only as an irradiance probe moves downward into a natural hydrosol and continuously records the magnitude of H(z,-) as z continuously increases. Now if the external lighting conditions on the upper boundary of the source-free medium are arbitrary but fixed in time, that is stationary, and if the hydrosol's fundamental inherent optical properties (namely the volume attenuation α and volume scattering function σ) are arbitrary but stationary, then the depth dependence of K(•,-) is an indicator of the natural interaction of the penetrating photons with the material of the medium; it provides a running account of the depth-rate of attenuation of the number of the downwelling photons streaming past the level z. This dynamical aspect of the variations of K(•,-) will be studied in an elementary but detailed manner in Secs. 10.3 and 10.4. For our present purposes we may thus consider the physical (or dynamical) mechanism behind the depth behavior of K(•,-) to be fairly well understood. There remains, however, the problem of the *geometric mechanism* which also gives rise to variations of K(•,-) to which we now turn, and which forms the central problem under study in the present section.

If, instead of continuously moving the irradiance probe vertically downward in a natural hydrosol, we hold the depth z fixed and let H(z,-) be recorded as a function of time, we would expect in general, because of the continuously changing external lighting conditions above any natural hydrosol, a time dependence of H(z,-) for the fixed depth z. For example, if the probe is set at a depth of five meters at 0600 hours local time in a certain lake, we would expect H(z,-) generally to increase as the sun rises to reach a maximum around 1200 hours, and then to descend at 1800 to a reading comparable to the 0600 reading. On this basically regular diurnal variation of H(z,-) there is superimposed a relatively more rapid variation in H(z,-), induced for example by the movement of clouds or cloud layers between the sun and the hydrosol's surface. Even though the time rates of these latter changes in the external lighting conditions are thousands of times greater than those associated with the more stately diurnal changes, they nevertheless are far too small to cause any true transients in the natural light field. For this reason all these changes in the light field are of a *quasi-stationary* character.

Now, suppose a probe were to be sent quickly downward to accurately record the stationary light field all the while the sun is covered by a cloud and then, when the cloud has just passed away from the sun and allows it to shine with full strength on the hydrosol, another quick but precise probe uncovers the irradiance's depth profile during this sunny condition. If this were done then we would discern upon careful examination of the two semilog plots of H(z,-) that H(z,-) at each given depth on the overcast plot would differ from the value at that same depth on the sunny plot.

Actually, experimenters need not go out of their way to uncover this phenomenon; it crops up with exasperating inevitability in any painstaking (and thus time-consuming) mapping of the irradiance depth-profile in natural hydrosols. For in a particularly deep hydrosol the determination of the vertical depth-profile may take on the order of a half hour; and when the probe ascends past one of the relatively shallower depths in a check re-run, the sun may have moved as much as five to ten degrees, cloud covers may have changed, thus producing (with most modern equipment) easily measured changes in the structure of the light field in the interim.

When plots of $H(z,-)$ are made of each of these runs, the logarithmic slope $K(z,-)$ on each plot may be noticeably nonconstant with depth; furthermore, and this is the crux of the matter at hand, the $K(z,-)$ value at a fixed depth z for the downward run may differ from that for the upward run. The experimenter would, at this juncture, if he considers this difference in $K(z,-)$ from one plot to another with care, soon realize that the difference may be the result of a superposition of two basically different physical mechanisms: On the one hand there is of course the possibility that the *inherent* optical properties of the medium may have changed during the interval between the times that the probe has visited the given depth z; on the other hand there is the possibility that the external lighting conditions have changed during this period and that this change has in some way become manifest in the difference in the K-values for the given depth z.

If the investigator had made provisions to record during the same period, and over the same depth interval, the radiance distributions within the medium and the *inherent* optical properties of the medium, then he may be able to quantitatively, at least in principle, ascertain, by means of the representations (18) and (19) of Sec. 9.2 and the known structures of $a(z,\pm)$, $b(z,\pm)$, those parts of the differences in the $K(z,-)$ values which are traceable to the changes in the inherent optical properties. Thus, once again we are in possession of sufficient knowledge to understand and cope with the physical aspects of the behavior of $K(z,-)$. There remains, however, that component of the change of $K(z,-)$ at a given depth z which is traceable to the change of the external lighting conditions.

In order to relate the way in which $K(z,-)$ changes with external lighting conditions we must have some means of specifying in a precise manner the concept of "lighting condition." Clearly, in choosing a precise characterization of this concept the absolute amount of the incident radiant flux is of no essential importance. Of critical importance, however, is the *relative* amounts of radiant flux which arrive on the upper boundary of the medium, or on some internal horizontal plane, from the infinite number of possible directions in the hemisphere of incidence. One obvious, and incidentally the most complete, characterization would be by means of the radiance distribution $N(z,\cdot)$ at depth z. While this means may be of considerable use in other contexts, it requires of the experiments a prodigious auxiliary effort to provide the necessary measuring and recording apparatus to obtain this large number of readings. In the interests of experimental expediency, what is needed is a characterization of the relative values of $N(z,\cdot)$ without having to measure each of the infinitude of values $N(z,\xi)$, $\xi \in \Xi$ where Ξ is the set of all unit vectors in euclidean space. Since relative values are of primary interest, the ideal characterization would then involve not more than two readings of some simple kind.

In searching for two simple radiometric measurements which would be capable of characterizing the relative values of the downwelling radiance distribution, it would be of some convenience to the experimenter if he could make use of his existing data, namely the values of $H(z,-)$. To see what possible choices remain if $H(z,-)$ is adoped as one of the two radiometric measurements which will characterize the *relative* magnitudes of $N(z,\cdot)$, let us express $H(z,-)$ in terms of these values. By definition, we have:

$$H(z,-) = \int_{\Xi_-} N(z,\xi)\xi \cdot \mathbf{n}\, d\Omega(\xi) \quad ,$$

where $\mathbf{n}$ is now the unit inward normal to the hydrosol. In practice, $N(z,\cdot)$ is usually determined by a suitably chosen *finite* set $\{\xi_1, \ldots, \xi_n\}$ of downward directions, and $H(z,-)$ is then computed by the rule (re: (6) through (17) of Sec. 2.5):

$$H(z,-) = \sum_{i=1}^{n} N(z,\xi_i)\xi_i \cdot \mathbf{n}\, \Delta\Omega_i \quad ,$$

where $\Delta\Omega_i$ is the solid angle associated with the direction ξ_i. The quantities $\Delta\Omega_i$ are subject to no specific restrictions except that they be small enough so that $N(z,\cdot)$ is fairly uniform over the associated direction sets and that of course $\Sigma\,\Delta\Omega_i = 2\pi$. Now the *relative* magnitudes of the quantitites $N(z,\xi_i)$, $i = 1, \ldots, n$ may be obtained by choosing any one of them, say $N(z,\xi_1)$, and forming the quotients $N(z,\xi_i)/N(z,\xi_1)$, which we shall denote by: "$g(z,\xi_i)$". Then the quotient:

$$\frac{N(z,\xi_1)}{H(z,-)} = \frac{\Delta\Omega_1}{\sum_{i=1}^{n} g(z,\xi_i)\xi_i \cdot \mathbf{n}\, \Delta\Omega i}$$

would serve as a measure of the way in which the $g(z,\xi_i)$ are distributed over the downwelling hemisphere Ξ_-. However, such a measure falls short of being satisfactory for several reasons: First, we have isolated a particular value $N(z,\xi_i)$, and therefore have distinguished it with artificial importance; actually any one of the $n - 1$ other values would serve just as well. Secondly, in order to measure $N(z,\xi_1)$ we would require the services of a specially designed radiance meter, or bring into use for extremely restricted purposes the actual radiance distribution measuring apparatus--which might as well then be used to determine a working sample of $N(z,\cdot)$. Finally, we would prefer to measure an amount of flux comparable in magnitude to $H(z,-)$; for $N(z,\xi_1)$ would generally be a far smaller number than $H(z,-)$

which must then be divided into or divided by $N(z,\xi_1)$, thereby setting the stage for the disruption of numerical accuracy in the data reduction tasks that follow.

The cumulative effect of these observations is to lead to the choice of the sum

$$h(z,-) = \sum_{i=1}^{n} N(z,\xi_i)\Delta\Omega_i \quad ,$$

as the most logical choice for the second radiometric measurement. Its integral representation is:

$$h(z,-) = \int_{\Xi_-} N(z,\xi)\,d\Omega(\xi)$$

In this way we are led to consider the ratio

$$D(z,-) = \frac{h(z,-)}{H(z,-)} \tag{13}$$

which we have encountered before ((15) of Sec. 8.3 and (1) through (7) of Sec. 8.5) and which we have termed the *distribution function*. In terms of the finite-summation representation of $h(z,-)$ and $H(z,-)$, $D(z,-)$ becomes:

$$D(z,-) = \frac{\sum_{i=1}^{n} N(z,\xi_i)\Delta\Omega_i}{\sum_{i=1}^{n} N(z,\xi_i)\xi_i\cdot\mathbf{n}\,\Delta\Omega_i} \tag{14}$$

or, in terms of their integral representations:

$$D(z,-) = \frac{\int_{\Xi_-} N(z,\xi)\,d\Omega}{\int_{\Xi_-} N(z,\xi)\,\xi\cdot\mathbf{n}\,d\Omega} \quad . \tag{15}$$

The quantity $h(z,-)$ may be measured by simple devices, (see [305] and Chapter 13 below). We have seen in Chapter 8 how $D(z,-)$ serves to characterize the distribution of the irradiating flux. Thus, let $n = 1$ in (14), i.e., let the flux come from any single solid angle $\Delta\Omega$ in the general direction ξ; the distribution function then is:

$$D(z,-) = \frac{1}{\xi\cdot\mathbf{n}} \quad . \tag{16}$$

Hence if the irradiation is incident vertically at depth z, $\xi = \mathbf{n}$ and $D(z,-) = 1$. In general, the *more obliquely incident the pencil of radiation*, the smaller the dot product $\xi\cdot\mathbf{n}$, and *the larger is the distribution function*. This conclusion, although obtained for a special case, is nevertheless valid for any number of pencils.

In what follows we will show in what manner the concepts $K(z,-)$ and $D(z,-)$ are qualitatively interdependent. This will be done by noting the behavior of the intuitively more simple and predictable quantity $D(z,-)$ which, as we have repeatedly seen, serves as a convenient characterization of the lighting conditions. In this way we may gain insight into the dependence of the more complex quantity $K(z,-)$ on given lighting conditions.

The General Law Governing K(z,-) and D(z,-)

We now have sufficient background established so as to derive with some clarity the general law which governs the exact interrelation of $K(z,-)$ and $D(z,-)$. We start with the canonical representation of $K(z,-)$ derived from the equation of transfer in (40) of Sec. 8.9. This representation of $K(z,-)$ now takes the form:

$$K(z,-) = \alpha(z,-) - \frac{\int_{\Xi_-} N_*(z,\xi)\,d\Omega}{H(z,-)} \quad . \tag{17}$$

To obtain (17) from (40) of Sec. 8.9, we set $\Xi_0 = \Xi_-$ and set $\mathbf{n} = -\mathbf{k}$ (see Fig. 8.11). An alternate derivation may be obtained by following the steps leading to (26) of Sec. 9.2. To keep the present discussion self contained, we observe that the various terms in (17) are defined as follows:

(i) $\alpha(z,-) = \alpha(z)D(z,-)$, where, of course, $\alpha(z)$ is the value of the volume attenuation function at depth z.

(ii) $N_*(z,\xi)$ is the value of the path function at depth z for the direction ξ. Its analytical representation is:

$$N_*(z,\xi) = \int_{\Xi} N(z,\xi)\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad , \tag{18}$$

where $\sigma(z;\xi';\xi)$ is the value of the volume scattering function at depth z for the incident ξ' and the scattered direction ξ.

In order to extract from (17) the desired explicit connection between $K(z,-)$ and $D(z,-)$, we must reformulate the second term of (17) in such a way as to cause $D(z,-)$ to appear explicitly in that term. Toward this end we first recall that $D(z,-)$ is be definition the ratio of $h(z,-)$ to $H(z,-)$. Now the appearance of the integral in the second term of (17) has all the external earmarks of an $h(z,-)$ type quantity; this is suggested by observing that $N_*(z,\cdot)$ is a radiance function and it is integrated, like $N(z,\cdot)$, over Ξ_-, the set of downward directions. Thus in close analogy with $h(z,-)$, we write:

$$"h_*(z,-)" \quad \text{for} \quad \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi) \tag{19}$$

Physically, $h_*(z,-)$ is the downwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth z.

The analogy need not end with (19); in fact we can extend it quite naturally to include the $*$-counterparts to $h(z,+)$, and to scalar irradiance $h(z)$ itself. Thus in analogy to $h(z,-)$ we write (as in (11) of Sec. 2.7):

$$"h(z,+)" \quad \text{for} \quad \int_{\Xi_+} N(z,\xi)\,d\Omega(\xi) \quad . \tag{20}$$

which is the upwelling scalar irradiance at depth z (and which forms the basis for defining $D(z,+) = h(z,+)/H(z,+)$ for the upwelling stream); we further write:

$$"h_*(z,+)" \quad \text{for} \quad \int_{\Xi_+} N_*(z,\xi)\,d\Omega \tag{21}$$

which represents the upwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth z. Finally, the scalar irradiance $h(z)$ at depth z, being defined by writing (as in (3) of Sec. 2.7):

$$"h(z)" \quad \text{for} \quad \int_{\Xi} N(z,\xi)\,d\Omega \quad , \tag{22}$$

has its scattered analogy in the form $h_*(z)$ where we have written:

$$"h_*(z)" \quad \text{for} \quad \int_{\Xi} N_*(z,\xi)\,d\Omega \quad . \tag{23}$$

Thus corresponding to:

$$h(z) = h(z,-) + h(z,+) \quad , \tag{24}$$

which is based on (9) of Sec. 2.7, we have:

$$h_*(z) = h_*(z,-) + h_*(z,+) \quad . \tag{25}$$

An extremely useful and unexpectedly simple relation subsists between $h(z)$ and its scattered counterpart $h_*(z)$, namely that the ratio of $h_*(z)$ to $h(z)$ is precisely the value of the volume total scattering function s:

$$s(z) = \frac{h_*(z)}{h(z)} \quad . \tag{26}$$

The derivation of this relation along with some suggestions for its use in practical direct determinations of the values of s are given in (7) of Sec. 13.7.

We have now assembled all the required concepts needed for a complete formulation and discussion of the general law governing $K(z,-)$ and $D(z,-)$. Starting with (17) and using definition (19) we may write:

$$K(z,-) = \alpha(z,-) - \frac{h_*(z,-)}{H(z,-)} \quad . \tag{27}$$

By means of the definition of $D(z,-)$ this may be written:

$$K(z,-) = \alpha(z,-) - \frac{h_*(z,-)}{h(z,-)} D(z,-) \quad ,$$

and, finally, using the definition of $\alpha(z,-)$, we may recast this into the presently desired form of the law governing $K(z,-)$ and $D(z,-)$:

$$\boxed{K(z,-) = \left[\alpha(z) - \frac{h_*(z,-)}{h(z,-)}\right] D(z,-)} \tag{28}$$

The Absorption-Like Character of K(z,-)

Before presenting the general proof of the covariation of $K(z,-)$ and $D(z,-)$ which will be based on some observations of the structure of (28), we pause to discuss the general radiative transfer nature of the function $K(\cdot,-)$. We will show by general arguments and also by means of a simple example that $K(z,-)$ is essentially an "absorption coefficient," i.e., it serves as an analytical bookkeeping device for the depth rate of *absorption* of the stream of downwelling photons as the stream passes a general depth z.

The heart of equation (28) resides in the difference of the two bracketed terms. The first term is a value of the volume attenuation function which shows that $K(z,-)$ first of all takes cognizance of the simultaneous loss of photons by means of both scattering and absorbing mechanisms. Thus, as a stream of photons crosses an hypothetical surface at depth z, the stream suffers a loss by having some of the photons scattered in all directions about the point of crossing and also by having some of its photons of (implicitly stated) wavelength λ converted into photons of (generally) longer wavelength or into nonradiant energy. Now the second term, involving $h_*(z,-)$ and being *subtracted* from $\alpha(z)$, in effect returns to the downwelling stream all photons which have been scattered in the "forward direction"--the direction of motion of the downwelling stream. The net loss to the downwelling stream is then represented by this difference; it represents the amount of radiant flux per unit area that either has been scattered back into the upwelling stream, or which has suffered true absorption. The first of these alternate possible activities (scattering back into the stream) points up the dynamic interpretation of the magnitude of K(z,-), and in this respect, it is quite like the local transmittance function $\tau(z,-)$ discussed in connection with (7) of Sec. 8.3. To see the predominantly absorption-like character of K(z,-), we now consider two extreme examples of optical media: One that is purely absorbing, the other purely scattering; we will show that in each of these two extremes the values of K(•,-) tend either immediately or eventually to become directly proportional to the given values a(z) of the volume absorption function. These arguments will be general versions of those leading to the demonstration of the absorption-like character of k in (5) of Sec. 9.2.

First, let s(z) = 0 for all z, and let a(z) be arbitrary. We then have the extreme case of a purely absorbing medium. By hypothesis, it follows that $N_*(z,\xi) = 0$ for each z and for all ξ. Furthermore, since $\alpha(z) = a(z) + s(z)$ it follows that $\alpha(z) = a(z)$. With these observations, equation (28) reduces immediately to the simple form:

$$\boxed{K(z,-) = a(z)D(z,-)} \quad . \tag{29}$$

Here D(z,-) generally depends on depth along with a(z). However if depth z and a(z) are held fixed and the external lighting conditions are varied so that D(z,-) is changed, it is quite clear that K(z,-) varies directly and *linearly* with these changes in D(z,-). This is the first and simplest instance of the covariation of K(z,-) and D(z,-). The essentially absorption-like character may be seen by holding D(z,-) fixed. Then K(z,-) varies directly with the value a(z) of the absorption function.

Next, let a(z) = 0 for all z in an optically infinitely deep medium in which s(z) = s > 0 for all z. We then have the other extreme case of a purely scattering medium. In such a medium, according to (15) of Sec. 8.8, the divergence $\nabla \cdot \mathbf{H}(z)$ of the net-irradiance vector vanishes at each depth z. In particular, in a plane-parallel medium such as that around which the present discussion is centered, this divergence relation takes the simple form:

$$\frac{d\overline{H}(z,-)}{dz} = 0$$

where $\overline{H}(z,-) = H(z,-) - H(z,+)$, H(z,+) being the upwelling irradiance at depth z. It follows that, for every z,

$$\overline{H}(z,-) = c$$

where c is a constant. Since the medium is in steady state and no photons which enter it are ever lost by absorption, we expect that the time rate of emergence H(0,+) of photons per unit are at the surface just equals the time rate of incidence per unit area H(0,-). Hence

$$H(z,-) = H(z,+)$$

for all z. Now the greater the depth z in the medium, the more alike optically are the regions above and below the level z, and the more alike are their optical responses to the equal irradiances impinging on the common internal boundary at depth z. (See, e.g., the discussion of (128)-(131) of Sec. 8.7.) Thus we are led, by considerations of increasing symmetry above and below horizontal planes at great depths, to conclude that $h(z,-) \to h(z,+)$ as $z \to \infty$. Hence at great depths in the present medium, $D(z,-) \to D(z,+)$, so that the angular structure of the upwelling and downwelling radiance distributions become equal (cf. (2) of Sec. 8.5), and in fact (by symmetry) uniform. An immediate consequence of these observations is the fact that:

$$\frac{h_*(z,-)}{h(z,-)} \to \frac{h_*(z,-) + h_*(z,+)}{h(z,-) + h(z,+)} = \frac{h_*(z)}{h(z)} = s$$

in which (26) was used for the last equality. Hence, since equation (28) for the present case of $a(z) = 0$ takes the form:

$$K(z,-) = \left[s - \frac{h_*(z,-)}{h(z,-)} \right] D(z,-) \quad ,$$

we conclude that:

$$K(z,-) \to 0 (= a(z)) \quad , \tag{30}$$

as $z \to \infty$. Thus, in the both extreme types of media, where one is purely absorbing and the other purely scattering, the diffuse attenuation function was shown to tend toward values which are directly proportional to the values of the volume absorption function for these media. It is in this sense that we understand the absorption-like character of $K(z,-)$. Another instance (36) supporting this interpretation of $K(z,-)$ will be encountered as a matter of course in the concluding observations of this section. There we shall present a practical rule of thumb based on the covariation of $K(z,-)$ and $D(z,-)$.

Forward Scattering Media

A necessary prerequisite to the establishing of the general statement of the covariation rule between $K(z,-)$ and $D(z,-)$ is the introduction of the notion of a *forward scattering medium.* Briefly, a forward scattering medium is one for which the volume scattering function has a predominant forward scattering lobe as compared to its backward scattering lobe. We shall assume that the medium is isotropic.

For a precise definition, let $\mathbf{k}$ be the unit outward normal to the plane-parallel medium, and let ξ be an arbitrary element of Ξ ; then we write:

$$"s_{-\pm}(z,\xi)" \quad \text{for} \quad \int_{\Xi_\pm} \sigma(z;\xi;\xi')d\Omega(\xi') \quad , \tag{31}$$

whenever $\xi \in \Xi_-$ and:

$$"s_{+\pm}(z;\xi)" \quad \text{for} \quad \int_{\Xi_\pm} \sigma(z;\xi;\xi')d\Omega(\xi') \qquad (32)$$

whenever $\xi \in \Xi_+$. Some general properties of these s-functions are easily deduced. For example, if the medium is isotropic, then $s_{+-}(z;\xi) = s_{-+}(z;\xi')$, provided that $\xi \cdot \mathbf{k} = -\xi' \cdot \mathbf{k}$. Further, for these ξ, ξ': $s_{++}(z;\xi) = s_{--}(z'\xi')$. For example, if $\xi \cdot \mathbf{k} = 0$, then $s_{--}(z;\xi) = s(z)/2$ since the axis of the scattering lobe would then lie in the horizontal plane at depth z and the region of integration would be over precisely half the scattering lobe (see Fig. 9.1). Furthermore, for every $\xi \in \Xi$

$$s_{--}(z;\xi) + s_{-+}(z;-\xi) = s(z) \quad .$$

The connection between $s_{+\pm}(z;\xi)$ and the forward and backward scattering functions for collimated irradiance is quite close and should be noted (see (43) and (44) of Sec. 8.4). The value $s_{--}(z;\xi)$ has the interpretation of a *forward scattering* function for the direction ξ while $\sigma_{-+}(z;\xi)$ has that of a backward scattering function. Physically $s_{--}(z;\xi)$ gives the fraction of flux of a beam of downward direction ξ that is scattered in all downward (forward, with respect to ξ) directions (see Fig. 9.1). If now we write: "Θ" for arc cos $(-\xi \cdot \mathbf{k})$ we may write "$s_{--}(z,\xi)$" as "$s_{--}(z,\Theta)$" and we finally may define a *forward scattering medium as one for which $s_{--}(z,\Theta)$ decreases monotonically with increasing Θ in the range $0 \leq \Theta \leq \pi/2$.* Clearly, (32) implies

$$s_{++}(z,\xi) + s_{+-}(z,\xi) = s(z)$$

for all $\xi \in \Xi_-$, and analogously to $s_{-\pm}(z,\xi)$ we define $s_{+\pm}(z,\Xi)$ as the *forward (+) or backward (-) scattering* function for the upward direction ξ in Ξ_+. Since $s(z)$ is independent of ξ (the medium has been assumed isotropic), we have alternate means of characterizing a forward scattering medium now using $s_{-+}(z,\Theta)$, which of necessity is monotonically increasing with Θ in any forward scattering medium.

The Covariation Rule for K(z,-) and D(z,-)

We may now state the covariation rule:

Let X be an arbitrarily stratified plane-parallel forward scattering medium with given fixed inherent optical properties. If $z \geq 0$ is any fixed depth in X, then $K(z,-)$ and $D(z,-)$ increase, remain constant, or decrease together. Thus if, in particular, over a certain time period $K(z,-)$ and $D(z,-)$ exhibit increments in their values of magnitude $\Delta K(z,-)$ and $\Delta D(z,-)$, then these increments must be simultaneously positive, zero, or negative.

The complete proof of the rule is tedious because it requires an analysis of the total light field throughout the

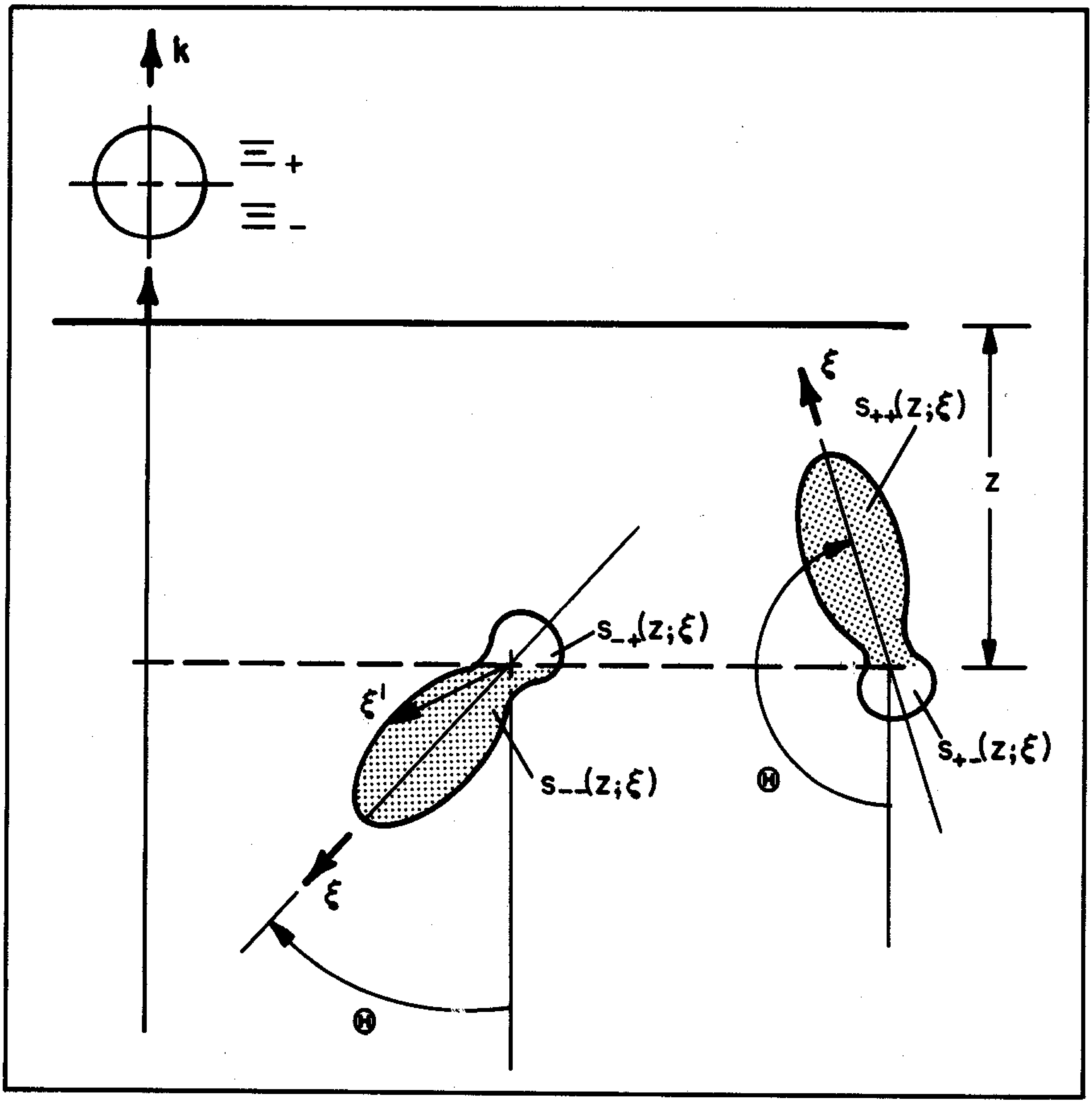

FIG. 9.1 Defining forward and backward scattering functions for the proof of the covariance of $K(z,-)$ and $D(z,-)$.

medium. However, the following argument forms the essential outline of a proof and makes the rule plausible. Suppose that, under the conditions of the hypothesis above, $\Delta D(z,-)$ is positive and $\Delta D(z,+)$ is zero. This implies an increase in the magnitude of $D(z,-)$ which, by our previous discussion of the properties of $D(z,-)$, implies that the preponderance of the radiation has now shifted to make a greater angle with the unit inward normal to an hypothetical surface at depth z. Now the term $h_*(z,-)/h(z,-)$ in (28) is simply a weighted mean of the volume scattering function values $\sigma(z;\cdot;\cdot)$ wherein the weighting factors are the downwelling radiance components. As observed above, the increase in $D(z,-)$ requires that the radiance components carrying the preponderance of the downwelling flux to be now associated with *larger* Θ values in the arguments of forward scattering function $s_{--}(z,\Theta)$; since the medium is forward scattering, this weighted mean then generally experiences a *decrease* in magnitude; the net result being an *increase* in the bracketed quantity in (28). The total change of $K(z,-)$ is the combination of the increase of the bracketed quantity and the increase in the factor $D(z,-)$; that is, $K(z,-)$ experiences an *increase* in magnitude. Summarizing: An increase in $D(z,-)$ is attended by an increase in $K(z,-)$ all other things remaining fixed in the forward scattering medium. A similar argument may be applied to the assumption that $D(z,-)$ exhibits a decrease. With these two facts established it is then a necessary consequence of continuity in all physical situations that $\Delta K = 0$ whenever $\Delta D = 0$.

Illustrations of the Rule

Example 1. Most natural hydrosols are forward scattering media; in fact $s_{--}(z,\Theta)$, when $\Theta = 0$, occasionally is on the order to ten to twenty times the magnitude of $s_{--}(z,\pi/2)$ over the visible spectrum. The values $\sigma(z;\xi;\xi)$ and $\sigma(z;\xi;\xi_1)$, where $\xi \cdot \xi_1 = 0$, often subtend a ratio of forward to side scattering on the order of 100 to 1, over the visible spectrum. Even more dramatic ratios > 100:1 are indicated in Figs. 1.72 and 1.73. The rule may safely be extended even to natural aerosols, these media being predominantly forward scattering; even the borderline case of Rayleigh atmospheres wherein $s_{--}(z,\Theta) = s(z)/2$ for Θ in $[0,\pi/2]$, are subject to the rule, since the square bracketed quantity in (28) does not generally change magnitude, with a change in $D(z,-)$.

Example 2. As a specific illustration, suppose the sky above a lake is completely overcast and that the downwelling distribution and diffuse attenuation functions at some relatively shallow depth z have values D_0 and K_0, respectively, under this overcast condition. Suddenly, the near-zenith sun breaks through the clouds. The resulting value D_1, of the distribution function, is expected to be less than D_0: $D_1 < D_0$, which follows from the fact that the predominant portion of the radiation now comes from generally less oblique directions. It follows that $\Delta D < 0$, that is the increment of $D(z,-)$ is negative, and thus the covariation rule requires that $K(z,-)$ is negative, so that the new value K_1 of the diffuse attentuation function is less than K_0: $K_1 < K_0$.

Example 3. As a final illustration, suppose that we fix attention on a relatively shallow depth in a natural hydrosol which is irradiated by a clear sunny sky for an entire afternoon. As the sun descends, $D(z,-)$ clearly increases because the direction of the predominant portion of the irradiating flux supplied by the sun increases its angle with the vertical (i.e., $1/|\xi \cdot \mathbf{k}|$ increases). The covariation rule would then require that $K(z,-)$ exhibit a corresponding increase.

The Contravariation of K(z,+) and D(z,+)

Up to this point in the discussion we have excluded any detailed mention of the upwelling irradiance $H(z,+)$. However, all that has been discussed for the downwelling stream of radiant energy may be applied *mutatis mutandis* to the upwelling stream, i.e., by replacing minus signs by plus signs in a systematic manner, etc. Therefore, the discussion of the interdependence of $K(z,+)$ and $D(z,+)$ may be conducted relatively quickly by pointing up only the basic differences between the two cases.

Now the first of these main differences between the upwelling and downwelling streams $H(z,+)$ and $H(z,-)$ lies in their magnitudes over the visible portion of the spectrum; the ratio of their magnitudes, $H(z,+)/H(z,-)$ known as the reflectance of the medium at depth z:

$$R(z,-) = \frac{H(z,+)}{H(z,-)}$$

is very nearly and almost universally in the neighborhood of 0.02. Thus $H(z,-)$ is on the order of 50 times the size of $H(z,+)$. Secondly, a fairly constant tie exists between the two streams by virtue of the ratio and sum of their distribution functions. It is found that almost universally over the visible portion of the spectrum (see, e.g., Table 1 of Sec. 8.5).

$$\frac{D(z,+)}{D(z,-)} = 2 \quad , \tag{33}$$

$$D(z,-) + D(z,+) = 4 \quad , \tag{34}$$

expressed to the nearest integer, which then requires that $D(z,-) = 4/3$, and $D(z,+) = 8/3$. The + stream counterpart to (17) is:

$$- K(z,+) = \alpha(z,+) - \frac{\int_{\Xi_+} N_*(z,\xi)\,d\Omega}{H(z,+)} \quad ,$$

which may be reduced to a corresponding expression to (28):

$$\boxed{- K(z,+) = \left[\alpha(z) - \frac{h_*(z,+)}{h(z,+)}\right] D(z,+)} \quad . \tag{35}$$

In much the same manner as $K(z,-)$, the dynamical and geometric mechanisms giving rise to the depth and temporal changes of $K(z,+)$ may be discussed in complete detail. The only precautionary observation that should be made here is that the dynamical mechanisms governing $K(z,+)$ should be examined as depth z *decreases*, this being the natural direction of flow of the upward stream. Finally, owing to the negative sign in front of K(z,+), the signs of the increments in K(z,+) and D(z,+) are *opposite*. Thus there is what we may term as a *contravariation* in the magnitudes of K(z,+) and D(z,+). This is the final distinction that must be made between the two streams, for the present.

Whence does this striking difference in the relative variations of the magnitudes of K(z,+) and D(z,+) arise? In what light should this difference be viewed? The answer to the first question is that the difference arises in the definitions of K(z,+) and K(z,-); each is defined by writing:

$$"K(z,\pm)" \quad \text{for} \quad -\frac{1}{H(z,\pm)}\frac{dH(z,\pm)}{dz} \quad .$$

Now a plane-parallel medium representing a natural hydrosol, by its very physical nature and usual coordinate system, normally invites the choice of the downward direction as the direction of increasing z values. Thus the spatial evolution of quantities associated with the downwelling stream are treated in a natural way, i.e., so that the natural unfolding of radiant energy in a downward direction takes place in the direction of the natural unfolding of the coordinate system, i.e., along with increasing z coordinates. The upwelling stream on the other hand naturally evolves spatially in the direction of decreasing z values, hence the contravariation, or topsy turvy interdependence of K(z,+) and D(z,+). This contravariation therefore is not an essential phenomenon and so can be erased and converted to a covariation if we reinterpret the derivative dH/dz as dH/d(-z) when considering the upward-flowing case. In answer to the second question, all that can reasonably be done is to view this state of affairs as an inessential perversity of standard coordinate systems, and to understand that it is the inevitable result of an attempt to depict an inherently three-dimensional process by an artificial two-dimensional symbolism designed by a basically one-dimensional thought process.

A Covariation Rule of Thumb

The general law (28) governing the interdependence of K(z,-) and D(z,-), while of extreme of importance in establishing the exact relationship between these two quantities, is somewhat unwieldy for use in quick estimates of the relative magnitudes of their increments. We conclude the present section with the derivation of a simple rule of thumb, based on experimental evidence, which relates in a linear manner the relative magnitudes of K(z,-) and D(z,-), and also their increments. We begin with the exact expression for R(z,-) in terms of K(z,±) and a(z,±) as given in (25) of Sec. 9.2:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)} \quad .$$

Here $a(z,\pm) = a(z)D(z,\pm)$. Now it is an experimental fact that the difference between the magnitudes of $K(z,+)$ and $K(z,-)$ over the visible spectrum is very small for many *practical* purposes, being on the order of five percent of the magnitude of $K(z,-)$. (This may be seen in Sec. 10.4.) Hence we may replace $K(z,+)$ by $K(z,-)$ in the preceding formula and solve for $K(z,-)$. The result is:

$$\boxed{K(z,-) = a(z)\left[\frac{R(z,-)D(z,+) + D(z,-)}{1 - R(z,-)}\right]} \quad . \qquad (36)$$

Equation (36) may be taken, as it stands, as a rule of thumb connecting $K(z,-)$ with $a(z)$. This equation serves to underscore the conclusion reached earlier in this work that K(z.-) is basically an absorption-like optical property of a medium.

To arrive at the desired rule of thumb we now make use of the experimental-numerical relations (33) and (34) between $D(z,+)$ and $D(z,-)$ and also of the fact that $R(z,-)$ is of the order 0.02. The result is:

$$K(z,-) = a(z)\left[\frac{R(z,-)\dfrac{D(z,+)}{D(z,-)} + 1}{1 - R(z,-)}\right] D(z,-)$$

$$= a(z)\left[\frac{\dfrac{1}{50} \times 2 + 1}{1 - \dfrac{1}{50}}\right] D(z,-)$$

$$= \frac{52}{49}\, a(z)D(z,-) = 1.06\; a(z)D(z,-) \quad .$$

Hence:

$$\boxed{\Delta K(z,-) = 1.06\; a(z)\Delta D(z,-)} \quad , \qquad (37)$$

which is the desired rule of thumb relating the covariation of $K(z,-)$ and $D(z,-)$.

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9.4 General Analytic Representation of the Observable Reflectance Function

The concept studied in this section is the observable reflectance function $R(\cdot,-)$ whose value at a depth z in an arbitrarily stratified plane-parallel optical medium is given by:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad ,$$

where, as usual, the quantities $H(z,\pm)$ are the observed upwelling (+) and downwelling (-) irradiances at depth z in the medium (re: (16) of Sec. 9.2). Several representations of the function $R(\cdot,-)$ are established which will, (a)