(c) The distribution functions $D(z,\pm)$ are *practically constant* with depth.

(d) The ratio of $\alpha(z)$ to $a(z)$ and hence the ratio $s(z)/\alpha(z)$, where $\alpha(z) = a(z) + s(z)$, appears to be *practically constant* with depth.

10.3 Formulation of the Shallow-Depth Model for K and R Functions

On the basis of the experimental evidence summarized in Sec. 10.2, in particular statements (c) and (d), we may adopt the two-D model of the light field as developed in Sec. 8.5. The equations forming the basis of this model have been explored in detail throughout Chapter 8. Therefore it simply remains to solve the equations of this model for the particular context at hand. We shall adopt the two-D model for decomposed light fields as given in Sec. 8.5. In addition to the assumptions leading to (67) of Sec. 8.5, we specifically emphasize that the optical medium is:

(i) Optically infinite deep.

(ii) Separable. (The ratio of $s(z)/\alpha(z)$ is invariant with depth--see experimental statement (d) of Sec. 10.2.)

(iii) Irradiated by a collimated radiance distribution of magnitude N^o incident at an angle θ_o from the normal to its upper boundary.

Formulas for $H(z,\pm)$

It follows from the two-D theory, in particular (71)-(73) of Sec. 8.5, that under the present conditions the expressions for $H(z,\pm)$ $(= H^o(z,\pm) + H^*(z,\pm))$ are:

$$H(z,-) = N^o \left[C(\mu_o,-)e^{-k_\infty z} + [\mu_o - C(\mu_o,-) \cdot e^{-\alpha z/\mu_o}\right] \quad (1)$$

$$H(z,+) = N^o \left[C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}\right] \quad (2)$$

The quantities $C(\mu_o,\pm)$, k_∞, and R_∞ and their component parts are defined in detail in Sec. 8.5. It is of interest to compare (1) and (2) with (37) and (38) of Sec. 8.6. For convenience we repeat basic formulas of Sec. 8.5; (72) of Sec. 8.5 is of the form:

$$C(\mu_o,\pm) = \frac{\sigma_\pm(\mu_o)b^*(\mp) + \sigma_\pm(\mu_o)\left[a^*(\mp) + b^*(\mp) \mp \left(\frac{\alpha}{\mu_o}\right)\right]}{\left(k_+ + \frac{\alpha}{\mu_o}\right)\left(k_- + \frac{\alpha}{\mu_o}\right)} \quad (3)$$

where $\mu_o = \cos\theta_o$ (cf. (70) of Sec. 8.5). Furthermore:

$$k_{\pm} = \frac{1}{2}\left\{\left[a^*(+) + b^*(+) - a^*(-) - b^*(-)\right] \pm \left[(a^*(+) + b^*(+) + a^*(-) + b^*(-))^2 - 4b^*(-)b^*(+)\right]^{1/2}\right\} \qquad (4)$$

and we have chosen to adopt the notation " k_∞ ," to point up the infinite depth of the medium. Hence:

$$k_\infty = -k_- \quad , \quad k_\infty \leq \alpha \quad , \qquad (5)$$

Finally from (102) of Sec. 8.7:

$$R_\infty = \frac{k_\infty - a^*(-)}{k_\infty + a^*(+)} \quad . \qquad (6)$$

Because of assumption (iii), the $H(z,\pm)$ depend implicitly on the quantity μ_o and to explicitly note this we would write "$H(z,\pm;\mu_o)$." If the medium is irradiated by an arbitrary radiance distribution $N^o(\mu,\phi)$ then the associated irradiances are found by an appropriate integration of the normalized forms of (1) and (2) (set $N^o = 1$ in (1), (2)):

$$H(z,\pm) = \int_{\phi=o}^{2\pi}\int_{\mu=o}^{1} H(z,\pm;\mu)N^o(\mu,\phi)\,d\mu d\phi \quad .$$

However, the present results can be deduced by a direct examination of (1) and (2) for an arbitrary μ_o without having to consider the general μ_o - effects as summarized in the preceding equation.

Formulas for $K(z,\pm)$

Using the definitions of the K - functions and the formulas for $H(z,\pm)$ given in (1) and (2), we have

$$K(z,-) = \frac{k_\infty C(\mu_o,-)e^{-k_\infty z} + \frac{\alpha}{\mu_o}\left(\mu_o - C(\mu_o,-)\right) e^{-\alpha z/\mu_o}}{C(\mu_o,-)e^{-k_\infty z} + \left(\mu_o - C(\mu_o,-)\right) e^{-\alpha z/\mu_o}} \qquad (7)$$

or

$$\boxed{K(z,-) = \frac{k_\infty - \frac{\alpha}{\mu_o}A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}} \quad , \qquad (8)$$

where we have written:

$$\text{"}A(\mu_o,-)\text{"} \quad \text{for} \quad \frac{C(\mu_o,-) - \mu_o}{C(\mu_o,-)} \tag{9}$$

Furthermore:

$$K(z,+) = \frac{k_\infty C(\mu_o,-)R_\infty e^{-k_\infty z} - \frac{\alpha}{\mu_o} C(\mu_o,+)e^{-\alpha z/\mu_o}}{C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}} \tag{10}$$

or

$$\boxed{K(z,+) = \frac{k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{1 - A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}} \tag{11}$$

where we have written:

$$\text{"}A(\mu_o,+)\text{"} \quad \text{for} \quad \frac{1}{R_\infty} \frac{C(\mu_o,+)}{C(\mu_o,-)} \quad . \tag{12}$$

Formulas (8) and (11) may be used to predict the depth dependence of $K(z,\pm)$. We deduce immediately from these equations the general fact that:

$$\lim_{z\to\infty} K(z,\pm) = k_\infty \quad . \tag{13}$$

Furthermore the K - functions approach this limit in a monotonic manner, as can be seen by taking their derivatives with respect to z :

$$\frac{dK(z,\pm)}{dz} = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)^2 A(\mu_o,\pm)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{\left[1 - A(\mu_o,\pm)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}\right]^2} \quad . \tag{14}$$

The monotonic behavior of $K(z,\pm)$ follows from the observation that the derivatives $dK(z,\pm)/dz$ are of *constant sign* for all z and for arbitrary choice of μ_o. In particular, the model predicts that:

$$\frac{dK(z,\pm)}{dz} > 0 \quad \text{if} \quad A(\mu_o,\pm) > 0 \left.\vphantom{\frac{a}{b}}\right\} \begin{array}{l} K \text{ increasing} \\ H \text{ concave downward} \end{array} \tag{15}$$

and:

$$\frac{dK(z,\pm)}{dz} = 0 \quad \text{if} \quad A(\mu_o,\pm) = 0 \left.\right\} \begin{array}{l} K \text{ constant} \\ H \text{ linear} \end{array} \tag{16}$$

and:

$$\frac{dK(z,\pm)}{dz} < 0 \quad \text{if} \quad A(\mu_o,\pm) < 0 \left.\right\} \begin{array}{l} K \text{ decreasing} \\ H \text{ concave upward} \end{array} \tag{17}$$

It thus appears that the model qualitatively reproduces the experimentally observed depth behavior of $K(z,\pm)$ as summarized in (a) of Sec. 10.2. The question of whether $K(z,+)$ and $K(z,-)$ increase or decrease with increasing depth is settled by evaluating the quantities $A(\mu_o,+)$ and $A(\mu_o,-)$, respectively, and applying the criteria (15)-(17). Clearly the increase or decrease of the K-functions is governed, according to the present model, by the nature of the external lighting conditions (summarized by the parameter μ_o) and the salient optical properties of the medium used in two-D models (summarized by α, k_∞, $A(\mu_o,\pm)$). Some specific illustrations of this fact are given below. For the present, we *turn to the consideration of* $R(z,-)$.

Formula for R(z,-)

Using the definition of $R(z,-)$ and the formulas for $H(z,\pm)$ given in (1) and (2), we have:

$$R(z,-) = \frac{C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}}{C(\mu_o,-)e^{-k_\infty z} + (\mu_o - C(\mu_o,-)e^{-\alpha z/\mu_o}} , \tag{18}$$

or

$$\boxed{R(z,-) = R_\infty \frac{1 - A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}} . \tag{19}$$

Formula (19) may be used to predict the depth dependence of $R(z,-)$. We deduce immediately that, in general,

$$\lim_{z\to\infty} R(z,-) = R_\infty ,$$

and that $R(z,-)$ approaches this limit in a monotonic manner, as can be seen by taking the derivatives of (19) with respect to z :

$$\frac{dR(z,-)}{dz} = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)\left(R_\infty - R(0,-)\right)\mu_o C(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}{\left[C(\mu_o,-) + (\mu_o - C(\mu_o,-))e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}\right]^2} \tag{20}$$

or

$$\frac{dR(z,-)}{dz} = \frac{R_\infty\left(\frac{\alpha}{\mu_o} - k_\infty\right)[A(\mu_o,+) - A(\mu_o,-)]e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}{\left[1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}\right]^2} \tag{21}$$

The monotonic behavior of $R(z,-)$ follows from the observation that $dR(z,-)/dz$ is of *constant sign* for all z, and for arbitrary fixed choice of μ_o. It appears that the model can qualitatively reproduce the experimentally observed depth behavior of $R(z,-)$ as summarized in (b) of Sec. 10.2 above.

In particular, the model predicts that:

$$\frac{dR(z,-)}{dz} > 0 \quad \text{if} \quad A(\mu_o,+) > A(\mu_o,-) \quad \Big\} \; R \text{ increasing} \tag{22}$$

$$\frac{dR(z,-)}{dz} = 0 \quad \text{if} \quad A(\mu_o,+) = A(\mu_o,-) \quad \Big\} \; R \text{ constant} \tag{23}$$

$$\frac{dR(z,-)}{dz} < 0 \quad \text{if} \quad A(\mu_o,+) < A(\mu_o,-) \quad \Big\} \; R \text{ decreasing} \tag{24}$$

The increase or decrease of $R(z,-)$ with increasing depth is therefore governed by the relative magnitudes of $A(\mu_o,\pm)$, according to the criteria (22)-(24).

Comparisons of Experimental Data with Calculations Based on the Model

Figure 10.5 shows a graphical comparison of the experimentally determined K-function values (the crosses) with the calculated values of these functions (solid curves) using the formulas (8) and (11) deduced from the model. Table 1 gives a numerical comparison of the values. The agreement between experimental data and theory appears to be good.

Figure 10.6 shows a graphical comparison of the experimentally determined R-function values (crosses) with the calculated values of these functions (solid curve) using the formula (19) deduced from the model. Table 1 includes a numerical comparison of the values. The agreement between the computed and measured values is excellent in this case.

A word about the calculation procedure may be in order. The following values of the optical properties were used: $k_\infty = 0.178$/meter, $\alpha = 0.430$/meter. The quantities $A(\mu_o,\pm)$, for curve-fitting purposes, may be considered as constants of

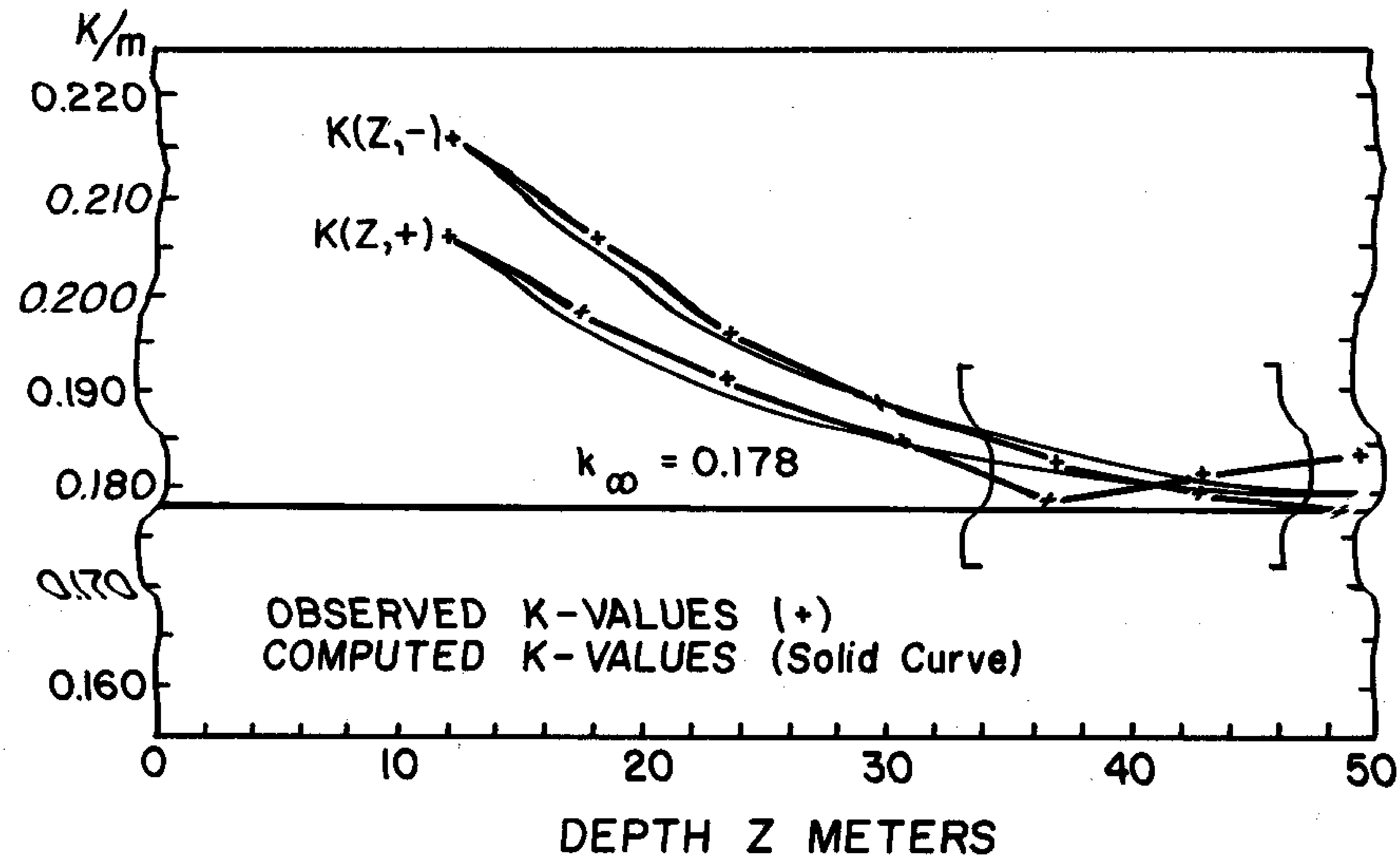

FIG. 10.5 Comparison between experimental and theoretical K - function values, as given by equations (8) and (11).

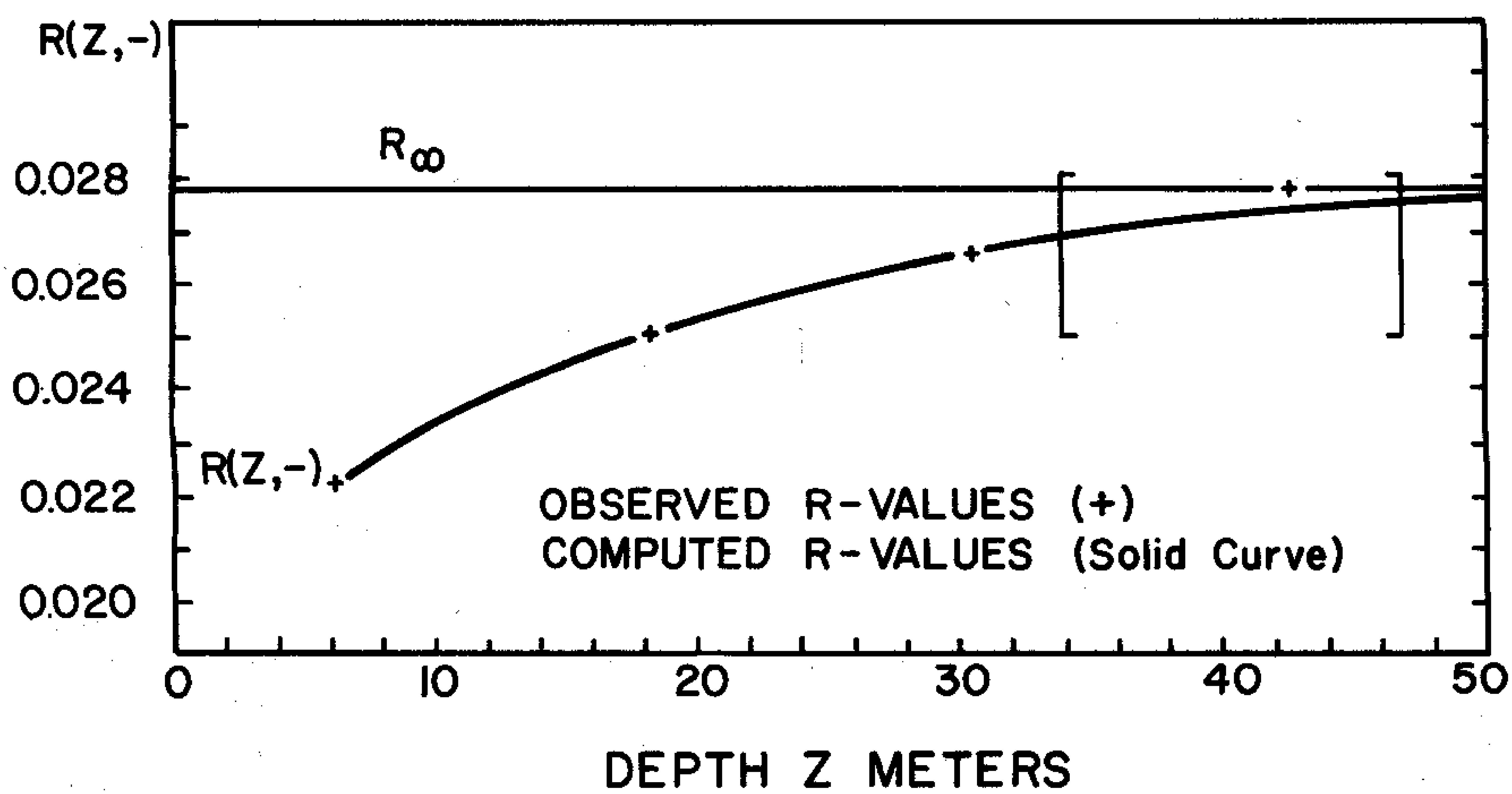

FIG. 10.6 Comparison between experimental and theoretical reflectance function values, as given by (19).

integration. Their values were therefore determined in the present by using the following boundary conditions:

$$K(12.2,-) = 0.216/\text{meter} \quad ,$$

$$K(12.2,+) = 0.206/\text{meter} \quad .$$

TABLE 1

Comparison of Calculated and Measured K and R Functions

z meters	K(z,-)		K(z,+)		R(z,-)	
	Data	Calculated	Data	Calculated	Data	Calculated
6.10	-	-	-	-	0.0221	0.0221
12.20	0.216	0.216	0.206	0.206	-	-
18.30	0.206	0.204	0.198	0.196	0.0250	0.0249
24.41	0.196	0.195	0.191	0.189	-	-
30.52	0.189	0.188	0.185	0.185	0.0266	0.0266
36.64	0.183	0.184	0.179	0.182	-	-
42.76	0.180	0.182	0.182	0.180	0.0279	0.0274
48.88	0.178	1.180	0.184	0.179	-	-
54.99	-	-	-	-	0.0258	0.0277

Note: $\mu_o = 1.582$ $A(\mu_o,+) = -1.337$ $A(\mu_o,-) = -2.141$

$R_\infty = 0.0278$ $k_\infty = 0.178/\text{meter}$ $\alpha = 0.430/\text{meter}$

$\lambda = 480 \pm 64$ mμ

A value $\mu_o = 1.583$ was found by computing the slope at $z = 12.2$ meters of the experimental $K(z,-)$ curve. This is an effective μ_o in the sense that it simulates the *noncollimated* external lighting conditions and interreflection effects at the boundary. Recall that assumption (ii) of the model (stated before (8) of Sec. 8.5) makes it strictly applicable only to media with nonreflecting boundaries irradiated by a collimated radiance distribution. In this way the lengthy integration process of the kind described above (following (6), and which is strictly necessary) was conveniently bypassed.

When the values of the constants $A(\mu_o,\pm)$ and μ_o, so found at the single depth $z = 12.2$ meters, were substituted in (8), (11), and (19), these formulas predicted a set of values of $K(z,\pm)$ and $R(z,-)$ for all other depths. These predicted values are shown in Table 1.

Hypotheses on the Fine Structure of Light Fields in Natural Hydrosols

We have seen (Sec. 10.2) that there is experimental evidence of a regular nonlinear trend in the logarithmic derivatives and the ratios of the upwelling and downwelling irradiances in near-homogeneous natural hydrosols. On the basis of this evidence, and the ability of the present mathematical model of the light field for homogeneous natural hydrosols to quantitatively reproduce these effects, we conclude that these nonlinearities are effects which may be expected to be observed in all homogeneous natural hydrosols. We are thus led to tentatively propose the following hypotheses about the fine structure of the light field in all homogeneous source-free natural hydrosols. The part of the hypotheses concerned with the limiting behavior of the K and R functions will be proved in detail in Sec. 10.7 but is included here for completeness.

I. *The ratio of the upwelling to the downwelling irradiance, i.e., the observable reflectance function $R(z,-)$, is monotonic increasing or decreasing with increasing depth z; $R(z,-)$ always approaches a limit R_∞, which depends only on the inherent optical properties of the hydrosol and which is independent of the external lighting conditions at the upper boundary of the medium.*

II. *The logarithmic derivatives $K(z,\pm)$ i.e., the K functions for the upwelling and downwelling irradiances are monotonic increasing or decreasing with increasing depth z; $K(z,\pm)$ always approaches a common limit k_∞ which depends only on the inherent optical properties of the hydrosol and which is independent of the external lighting conditions at the upper boundary of the medium.*

On the basis of the experimental evidence cited above, and on an examination of the mathematical model of the observed phenomena, we can propose an additional hypothesis which goes on to state more specifically the depth behavior of the K- and R-functions.

III. *In all homogeneous source-free natural hydrosols:*

(a) $K(z,-) > K(z,+)$ for all $z \geq 0$.

(b) $dK(z,\pm)/dz < 0$ for all $z \geq 0$.

We immediately deduce that:

$$\frac{dR(z,-)}{dz} > 0 \quad , \tag{25}$$

which follows from (a) and general relation ((32) of Sec. 9.2)

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad . \tag{26}$$

Hypothesis III is more specific than hypotheses I and II: *(b) implies hypothesis II, and Equation (25) asserts that the* reflectance function $R(z,-)$ monotonically *increases* with increasing depth, thus (a) implies a specific alternate in I.

The hypothesis cited in III is actually but one of a score of possibilities. It has, however, a relatively high probability of occurring. The sense of this "probability" will be made clear below in Sec. 10.4.