10.4 Catalog of K Configurations for Shallow Depths

In order to facilitate further theoretical studies of hypothesis III in Sec. 10.3 and to anticipate alternate possibilities, we shall develop in this section a catalog of all possible K-function configurations, as predicted by the two-D model.

The catalog of K-configurations in Figs. 10.7-10.12 is a graphical listing of all ways in which K(z,-) and K(z,+) may approach their common limit k_∞ in various homogeneous source-free natural hydrosols. It would be of interest to try to reproduce each of the possible configuration under controlled laboratory conditions.

A *K-configuration* is defined as an ordered quadruple of the four quantities: k_∞, K(0,+), K(0,-), and 0. The catalog

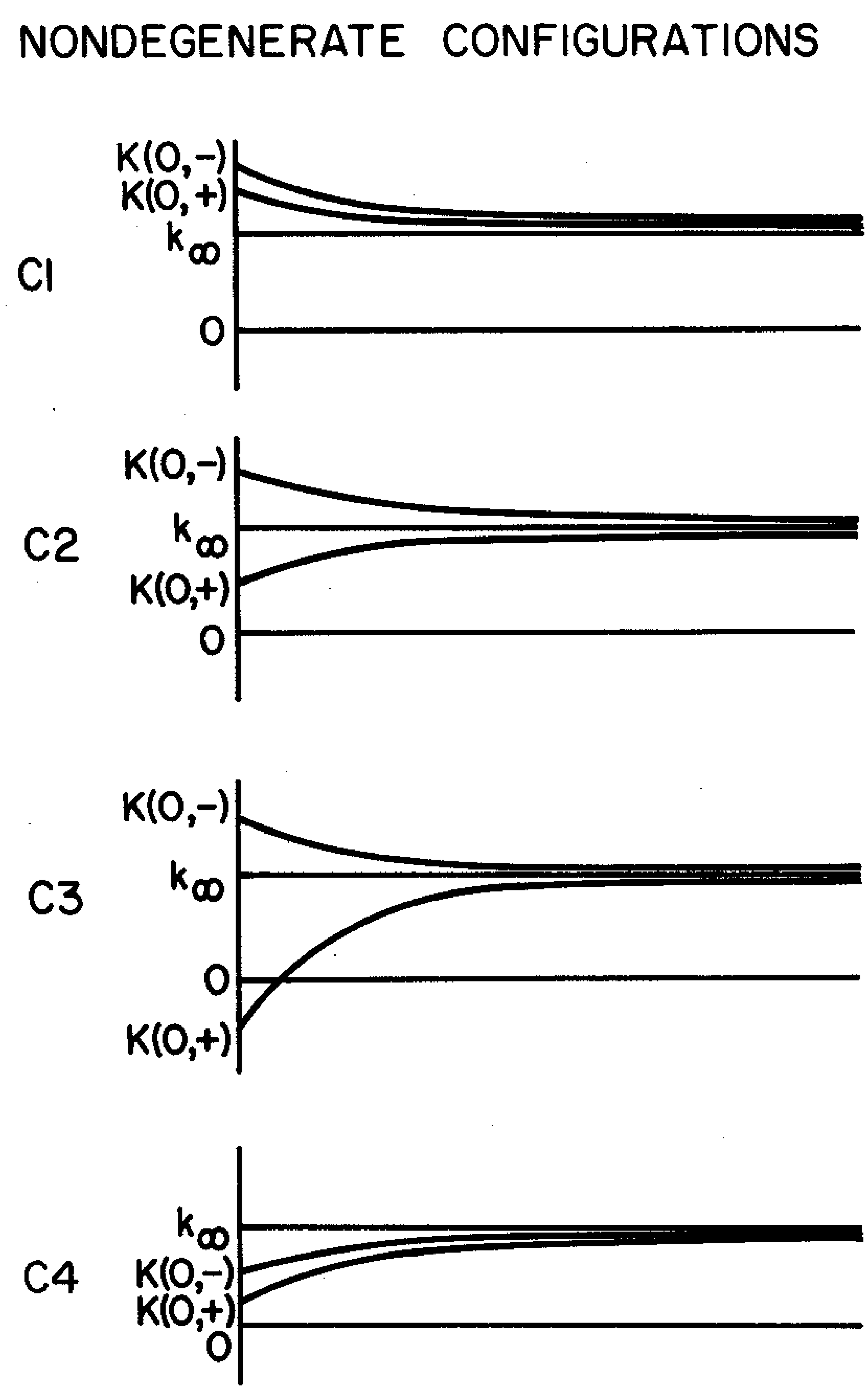

FIG. 10.7 Figures 10.7-10.12 catalog the totality of distinct depth-dependent configurations possible between the three K-functions K(z,±), k(z) in homogeneous stratified plane-parallel optical media, as deduced from the two-D theory for irradiance fields. See text for further details.

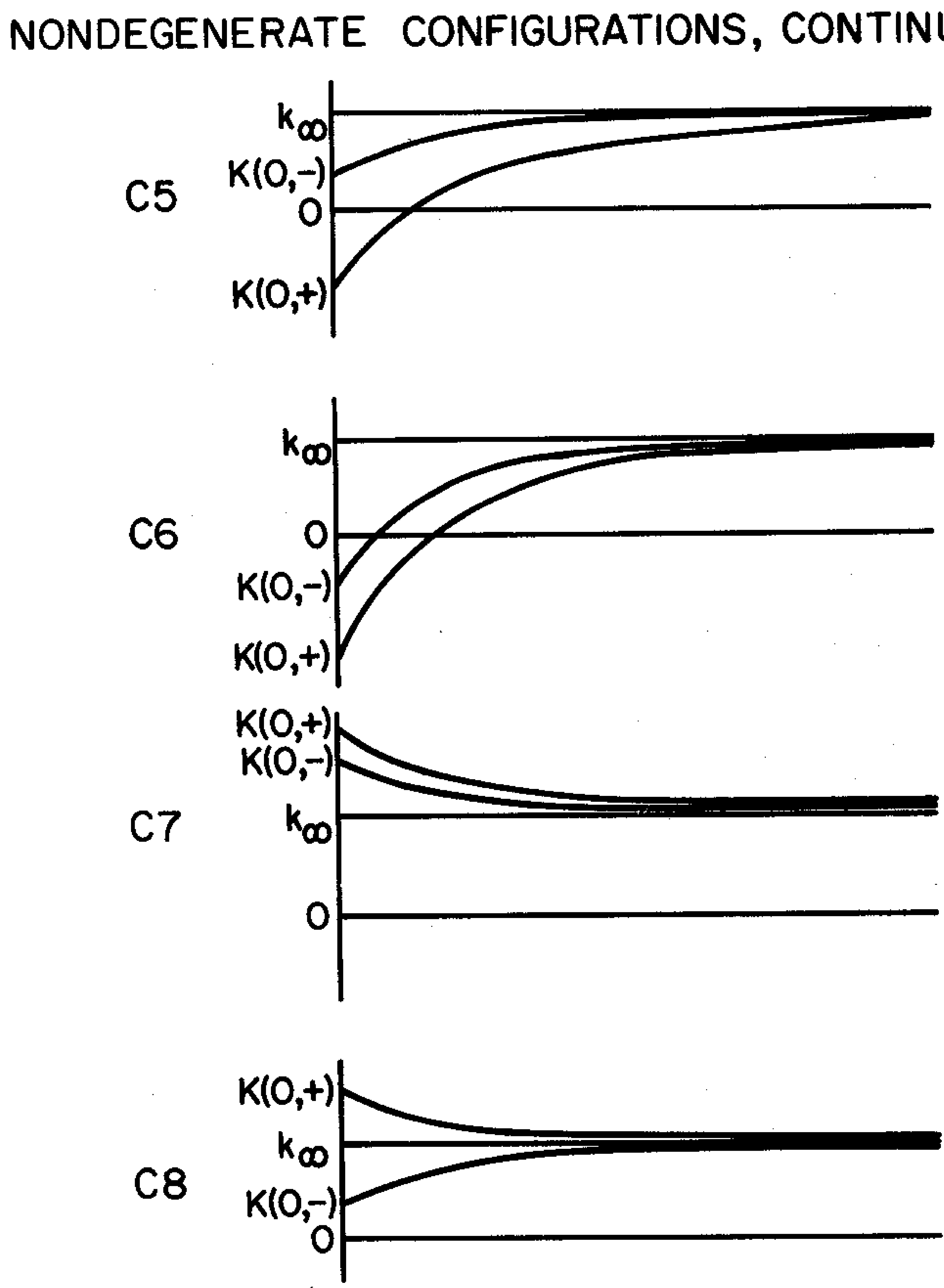

FIG. 10.8

consists of 25 configurations. These configurations are divided into three main classes:

1. Nondegenerate Configurations (9 members)
2. Degenerate Configurations
   (a) First Kind (8 members)
   (b) Second Kind (3 members)
3. Forbidden Configurations (5 members)

The *nondegenerate configuration* is defined as one in which:

$$0 \neq k_\infty \neq K(0,-) \neq K(0,+) \neq k_\infty \quad . \tag{1}$$

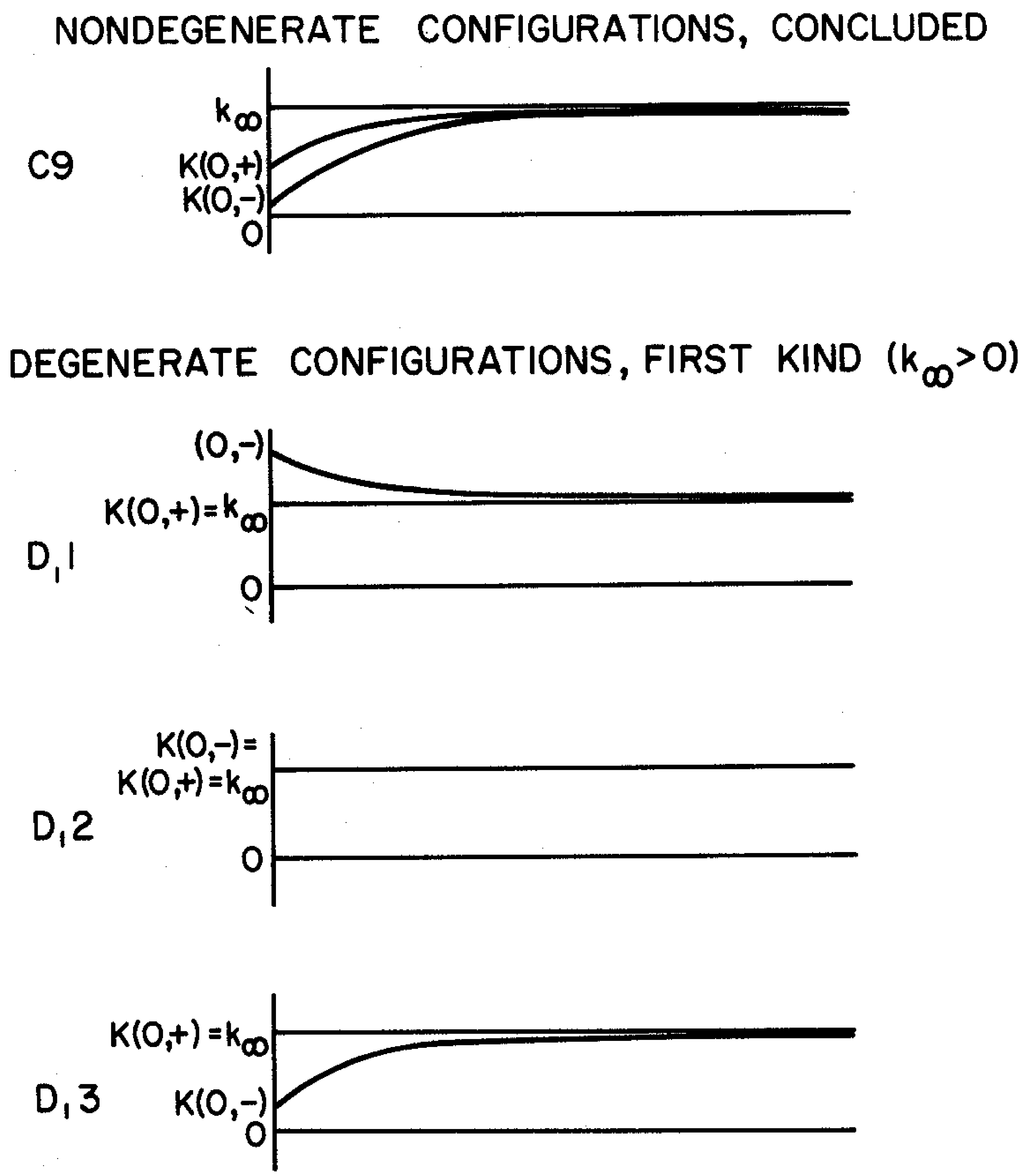

FIG. 10.9

A *degenerate configuration* is defined as one in which *at least* one of the inequalities in (1) is replaced by an equality.

A *forbidden configuration* is one in which the following basic inequality of the general theory ((30) of Sec. 9.2) is violated:

$$K(0,+)R(0,-) \leq K(0,-) \quad . \tag{2}$$

Observe that the K-configurations are defined in terms of the values of the K-functions for z = 0. This is possible because the monotonic behavior of the K and R functions, as established in 10.3, fixes their relative behavior at all depths once their initial values are known. For example, in C1 of Fig. 10.7:

$$K(0,-) > K(0,+) > k_\infty > 0 \quad .$$

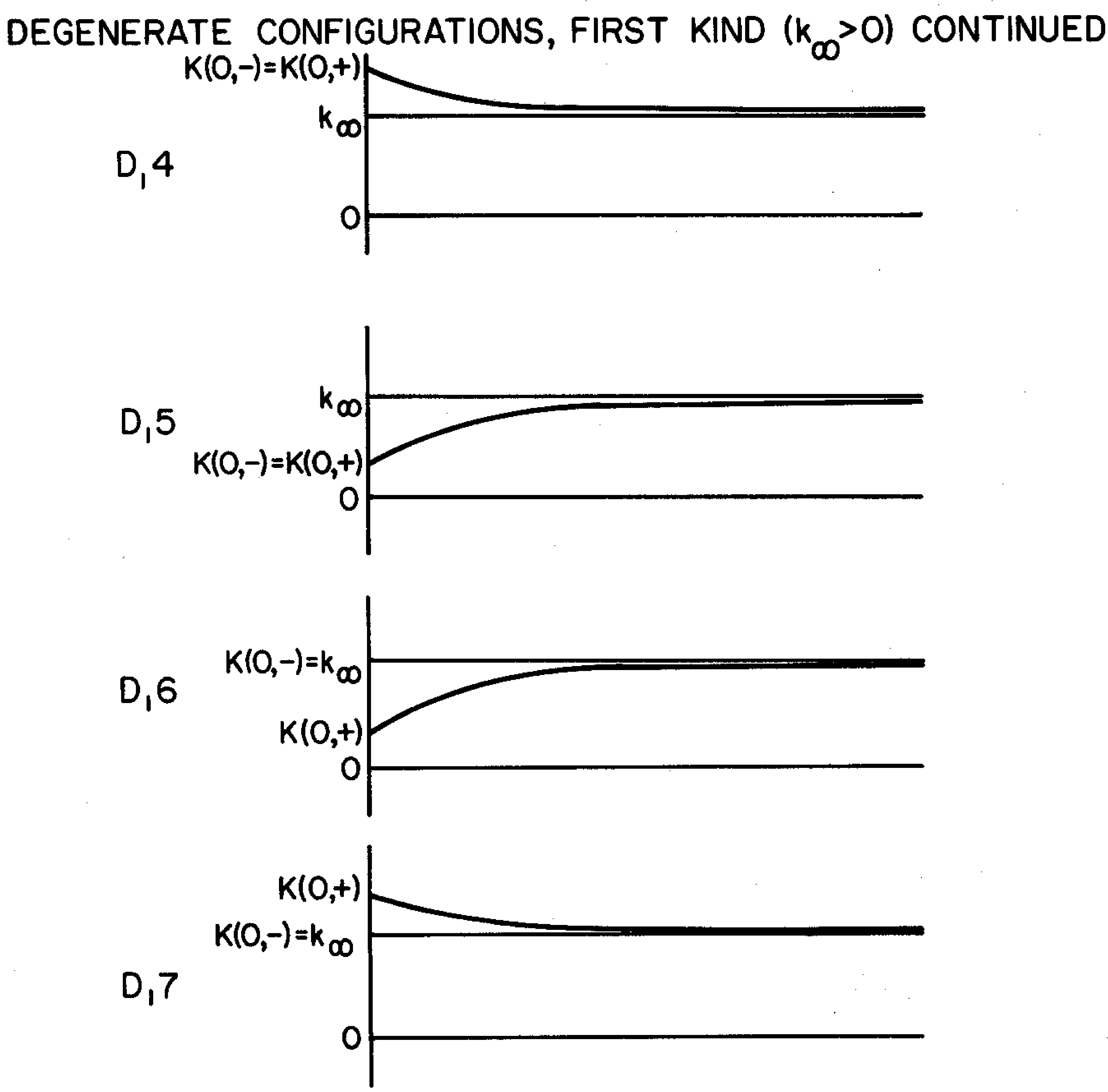

FIG. 10.10

Then since $K(z,\pm)$ must always decrease or always increase toward its limit we must have, in the present example, a *decrease* of both $K(z,+)$ and $K(z,-)$ toward k_∞. Furthermore, since $R(z,-)$ also exhibits a fixed monotonic behavior for all z, C1 must have--by virtue of (26) of Sec. 10.3-- $K(z,-) > K(z,+)$ for all z. Similar arguments show that all the configurations are well-defined in terms of the initial values of the K-functions. Knowing the initial magnitudes of $K(0,+)$ and $K(0,-)$ therefore fixes each configuration for all z. The general relation (26) of Sec. 10.3 may be used to determine whether $R(z,-)$ increases or decreases for a particular K-configuration.

We now give evidence that a configuration in which $K(z,-) > K(z,+)$ is preferred to one with $K(z,-) < K(z,+)$. We begin by noting that the most unlikely configuration is $D_2 3$ which is associated with nonabsorbing (purely scattering)

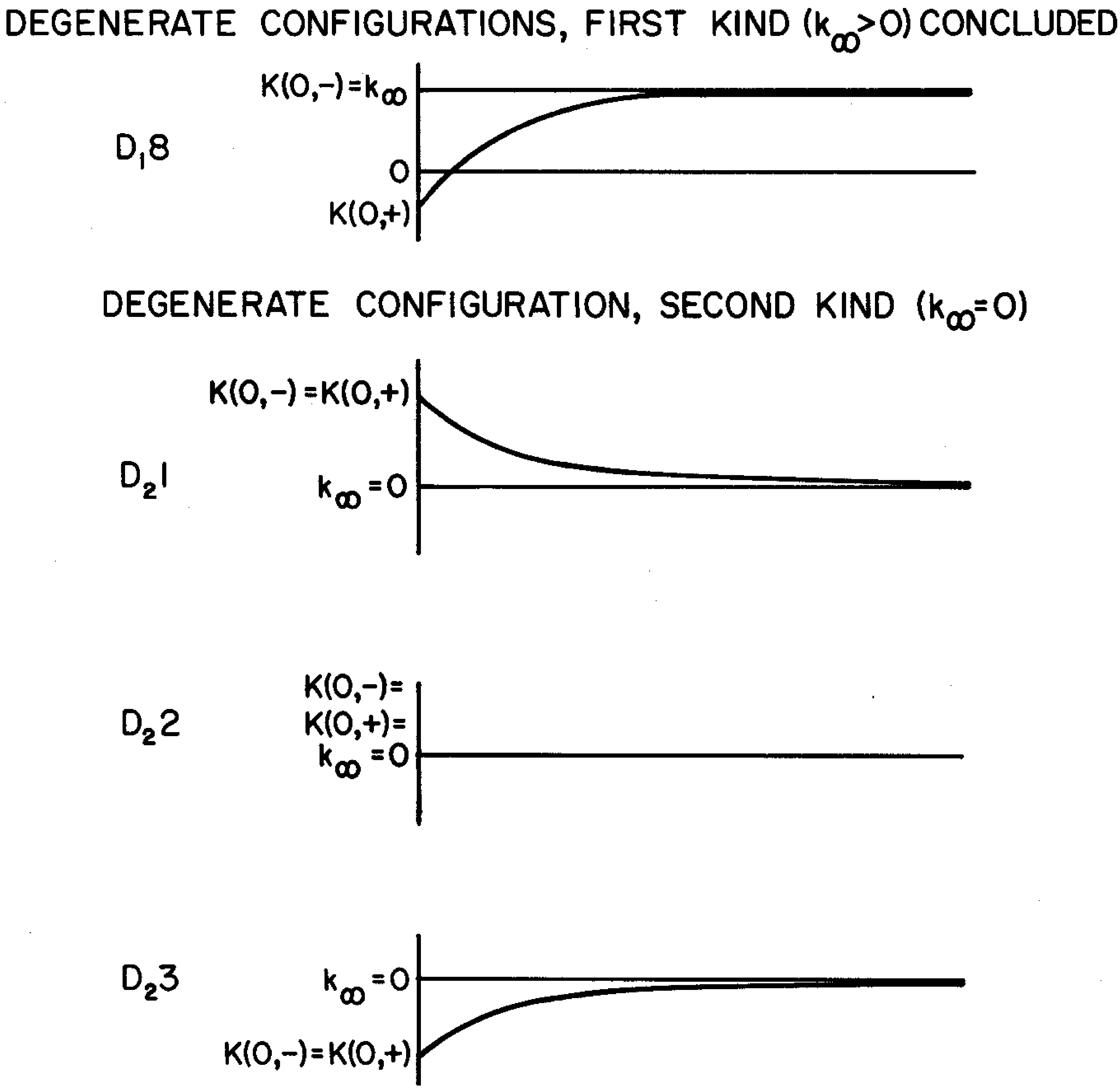

FIG. 10.11

media with $\sigma_+(\mu_0) \neq 0$ for all μ_0. In such media, we have by (18) of Sec. 8.8,

$$\frac{d\bar{H}(z,+)}{dz} = a(z)h(z) = 0 \quad .$$

Hence, for some constant C,

$$\bar{H}(z,+) = H(z,+) - H(z,-) = C$$

for all $z \geq 0$. The preceding formula implies that

$$K(z,+) = K(z,-)$$

for all $z \geq 0$, whence by (26)

$$R(z,-) = R_\infty \leq 1$$

for all $z \geq 0$. Now from the fixed value of $\bar{H}(z,+)$ and (1)

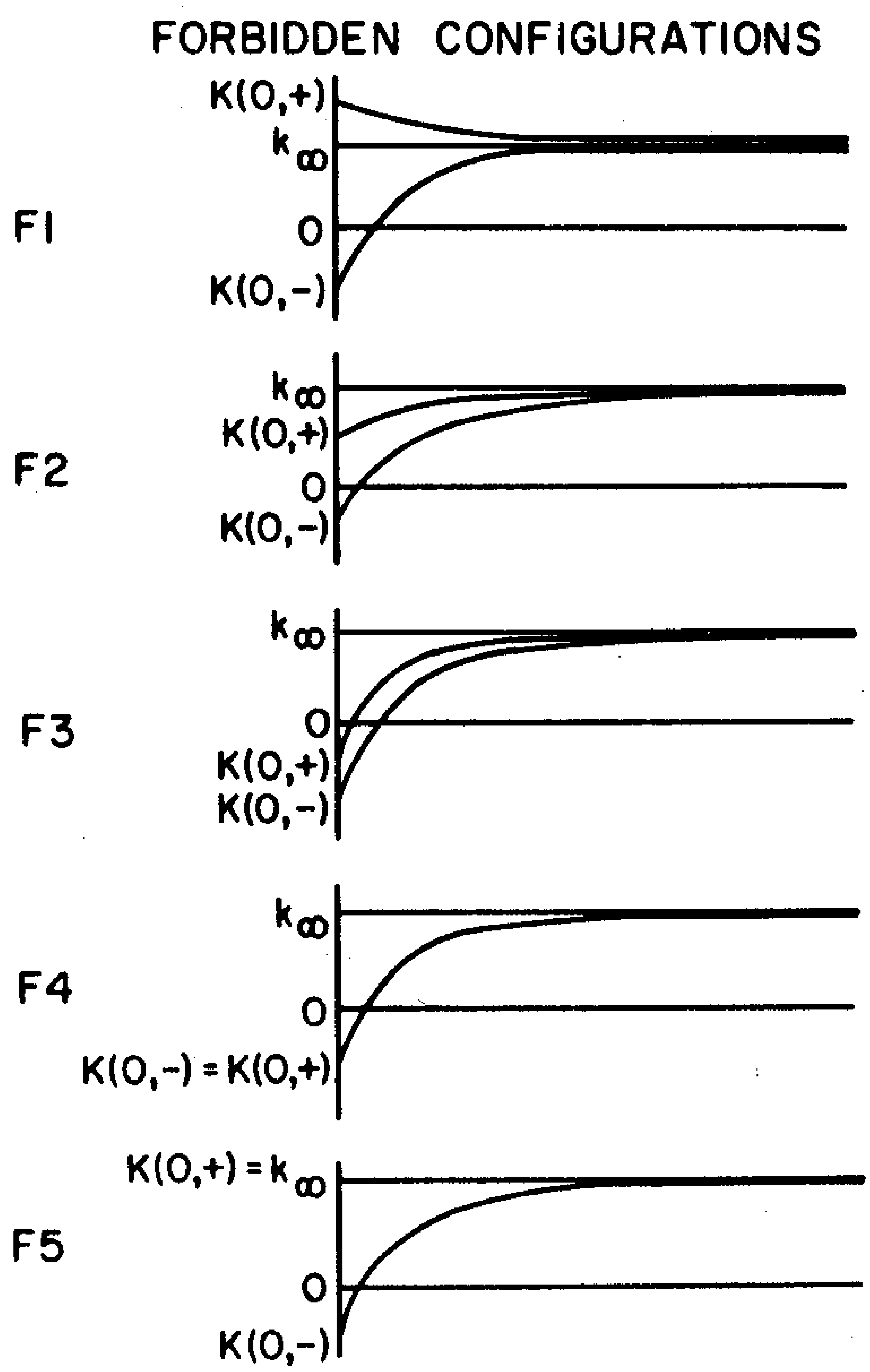

FIG. 10.12

and (2) of Sec. 10.3, it must follow that

$$R_\infty = 1 \quad , \quad k_\infty \geq 0 \quad \text{and} \quad C(\mu_o,-) = C(\mu_o,-) - \mu_o$$

Hence we have the following representations of $H(z,\pm)$:

$$H(z,-) = N^o \left[C(\mu_o,-)e^{-k_\infty z} - C(\mu_o,+)e^{-sz/\mu_o} \right]$$

$$H(z,+) = N^o \left[C(\mu_o,-)e^{-k_\infty z} - C(\mu_o,+)e^{-sz/\mu_o} \right]$$

that is,

$$H(z,+) = H(z,-)$$

for every $z \geq 0$. Moreover, a study of (14)-(16) of Sec. 8.5 shows that if $k_+ = k_-$ (as must be the case in view of $K(z,+) = K(z,-)$ deduced above), then necessarily $k_\infty = 0$. (Here we are identifying $K(z,\pm)$ with $\pm k_\pm$ respectively. The basis for this is (31) of Sec. 9.2.) It follows that, under the present circumstances we must have:

$$H(z,\pm) = N^{\circ} \left[C(\mu_o,-) - C(\mu_o,+) e^{-sz/\mu_o} \right]$$

Now, in forward scattering media, an examination of (3) of Sec. 10.3 shows that it is more likely to have

$$\mu_o \geq C(\mu_o,-)$$

than

$$\mu_o < C(\mu_o,-)$$

Hence if a medium is such that $a = 0$, and that it exhibits backward scattering, then $H(z,\pm)$, by the preceding representations for $H(z,\pm)$, *increase* with increasing z, so that $K(0,+) = K(0,-) < 0$. This unlikely state of affairs is represented by $D_2 3$ in Fig. 10.11; hence the configuration $D_2 3$ is very low on the list of likely configurations encountered in nature, as was to be shown.

Next, if the volume absorption coefficient is very slightly increased from 0 to $a > 0$, then the resulting effect is such that

$$k_\infty > 0$$

$$R_\infty < 1 \quad .$$

Furthermore, by (13) of Sec. 8.5, we would expect that $K(z,-) \neq K(z,+)$. Hence we would expect that

$$k_\infty > 0 > K(0,-) > K(0,+)$$

which is represented by configuration C6 in Fig. 10.8 (see configuration F3 which shows that the reverse inequality between $K(0,+)$ and $K(0,-)$ is impossible). As the value of the volume absorption coefficient a is allowed to increase a bit more, $K(0,-)$ and $K(0,+)$ move upward on the vertical axis, maintaining the above inequality as they approach and assume configuration C5. At this point, as α is further increased, the configuration assumed depends on the external lighting conditions and the volume scattering function σ. If σ is highly anisotropic with high forward scattering values and small backward scattering values, as is the case in most natural hydrosols, then the configurations C4, C3, C2, and C1 are most likely to be realized on the basis of various simple models obtained by assuming the appropriate forms for σ. Thus the phrase "configuration X is more probable than configuration Y," means that the values of the optical parameters associated with configuration X are more likely to be observed in nature than those associated with Y. This ordering of the likelihood of occurrence of optical parameters is suggested by experimental evidence. The configurations therefore are roughly in the order of decreasing probability of occurrence.

The discussion will not go into further detail on the catalog of these configurations. We merely mention that various special models can be obtained by assuming specific, but simple, forms of σ. These easily yield most of the 20 possible configurations. These forms for σ are:

(i) $\sigma(\zeta;\zeta') = \sigma_+\delta(\zeta-\zeta') + \sigma_-\delta(\zeta+\zeta')$, where σ_+, σ_- are fixed constants, and δ is the Dirac delta function (Stick model).

(ii) $\sigma(\zeta;\zeta') = s/4\pi$ where s is the volume total scattering coefficient (Ball model).

Some Special Fine Structure Relations

The models developed in Sec. 10.3 may be used to answer questions of the following kind.

1. What quantitative estimates can be made of the differences:

$$K(0,-) - K(0,+)$$
$$K(0,-) - k_\infty$$
$$K(0,+) - k_\infty \quad ?$$

2. What quantitative estimates can be made of the differences

$$R(0,-) - R_\infty \quad \text{knowing either } R_\infty \text{ or } R(0,-)?$$

3. What can be said about the relative magnitudes of

$$K(z,+),\ K(z,-),\ \text{and } k(z)?$$

4. If $K(0,\pm) < 0$, this implies that in real media $H(z,\pm)$ (respectively) has a maximum at some depth z_{max}. What estimates can be made of z_{max}?

The numerical examples giving the general answers to the first three questions show that the variations of $K(z,\pm)$ and $R(z,-)$ in the hydrosol of Sec. 10.3 are not less than the expected errors of the observed data. Therefore, detailed precise measurements of the light field in natural hydrosols should be generally expected to exhibit the nonlinearity in the H, K, and R plots; the presence of these nonlinearities and their classification (by means of the appropriate catalog of the observed K-configuration) could form part of the detailed description of the hydrosols.

Answer to Question 1. From (8) and (11) of Sec. 10.3, after simplification, we have

$$K(0,-) - K(0,+) = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)\left[A(\mu_o,+) - A(\mu_o,-)\right]}{\left[1 - A(\mu_o,-)\right]\left[1 - A(\mu_o,+)\right]} \tag{3}$$

As an example of the use of this formula we use the values of $A(\mu_o,\pm)$ obtained in the computations for Table 1 in Sec. 10.3 above, whence:

$$K(0,-) - K(0,+) = 0.010/\text{meter}.$$

In addition to (3), we have:

$$K(0,\pm) - k_\infty = - \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right) A(\mu_o,\pm)}{1 - A(\mu_o,\pm)} \qquad (4)$$

For the case of the present medium, we estimate that:

$$K(0,-) - k_\infty = 0.064/\text{meter}$$

$$K(0,+) - k_\infty = 0.054/\text{meter}.$$

Answer to Question 2. From (19) of Sec. 10.3 we have:

$$R(0,-) - R_\infty = \frac{R_\infty \left[A(\mu_o,-) - A(\mu_o,+)\right]}{1 - A(\mu_o,-)}$$

In the present case, we know R_∞ and $A(\mu_o,\pm)$. This leads to the estimate:

$$R(0,-) - R_\infty = - 0.007$$

for the present medium. Therefore the spread $R(0,-) - R_\infty$ of $R(z,-)$ values in the present hydrosol is on the order of 30 percent of $R(0,-)$.

Answer to Question 3. The definition of $k(z)$ is exactly analogous to the definition of $K(z,\pm)$:

$$k(z) = - \frac{1}{h(z)} \frac{dh(z)}{dz} .$$

To obtain an expression involving $K(z,\pm)$ and $k(z)$, we use the notion of the distribution function which links $H(z,\pm)$ and the corresponding components $h(z,\pm)$ of $h(z)$:

$$h(z) = h(z,-) + h(z,+) ,$$

where $h(z,\pm)$ (as in (11) of Sec. 2.7) is the scalar irradiance associated with the downwelling (-) or upwelling (+) stream of radiant energy. From the definitions of $D(z,\pm)$:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} ,$$

and that of $K(z)$, we have:

$$k(z) = -\frac{1}{h(z)}\frac{d}{dz}\left[D(z,-)H(z,-) + D(z,+)H(z,+)\right]$$

$$= \frac{1}{h(z)}\left[D(z,-)H(z,-)K(z,-) + D(z,+)H(z,+)K(z,+)\right]$$

$$- \frac{1}{h(z)}\left[\frac{dD(z,-)}{dz}H(z,-) + \frac{dD(z,+)}{dz}H(z,+)\right] \quad .$$

This is the exact representation of $k(z)$ in terms of the D and K functions.

According to the present model, however,

$$\frac{dD(z,\pm)}{dz} = 0 \quad .$$

This assumption is in good agreement with experimental fact (re: Fig. 10.2). For the present medium,

$$\frac{dD(z,\pm)}{dz} \sim 0.001/\text{meter},$$

as may be verified from Fig. 10.2. The number 0.001 is an upper limit of the derivative values over the indicated depth range. Since $K(z,\pm)$ are usually determined to about 10^{-3} per meter, the contribution to $k(z)$ by the terms containing the derivatives of $D(z,\pm)$ is not significant. Hence we may write:

$$k(z) = \gamma(z)K(z,-) + [1 - \gamma(z)]K(z,+) \quad ,$$

where

$$\gamma(z) = \frac{h(z,-)}{h(z)} < 1 \quad .$$

This representation of $k(z)$ shows that $k(z)$ is expected to be between $K(z,-)$ and $K(z,+)$, regardless of the algebraic signs of $K(z,+)$ and $K(z,-)$. As an example, let $z = 30$ meters. Computations from the data yield the value $k(30) =$ $= 0.187$/meter. Therefore we have, as expected,

$$0.188/\text{meter} - K(30,-) > 0.187 = k(30) > K(30,+) = 0.185 \quad .$$

Answer to Question 4. Configurations C3, C5, C6, $D_1 8$, and $D_2 3$ exhibit all possible ways in which either $K(0,+)$ or $K(0,-)$ may be negative. All except the last configuration exhibits a finite depth z_{max} at which $K(z_{max},+) = 0$ or $K(z_{max},-) = 0$. In these cases, z_{max} is the abscissa of the maximum point on the corresponding H-curve (observe that z_{max} may differ for $K(z,+)$ and $K(z,-)$).

An estimate of z_{max} for the upwelling (+) or downwelling (-) stream can be made directly from (8) and (11) of Sec. 10.3 by setting:

$$K(z_{max},\pm) = 0 \quad ,$$

and solving for z_{max}. Thus, from (8) and (11):

$$0 = K(z_{max},\pm) = \frac{k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}}}{1 - A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}}} ,$$

from which:

$$0 = k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}} .$$

Solving for z_{max}:

$$z_{max}(\pm) = \frac{\ln\left[\frac{\alpha}{k_\infty \mu} A(\mu_o,\pm)\right]}{\frac{\alpha}{\mu_o} - k_\infty} ,$$

where the plus sign refers as usual to the upwelling stream and the minus sign refers to the downwelling stream.

The criterion for the existence of a positive z_{max} value is evidently:

$$\frac{\alpha}{k_\infty \mu_o} A(\mu_o,\pm) > 1 .$$

If, in particular, the argument of the natural logarithm is positive but less than unity then z_{max} has a negative sign, which means that $H(z,+)$ (or $H(z,-)$) has no maximum value. In this case the irradiance simply decreases monotonically for all depths $z > 0$. The conditions under which the preceding inequality holds have yet to be fully explored.

Conclusion

The discussions of the preceding four sections have shown that there exist in natural optical media certain well-ordered, calculable tendencies of growth and decay of the irradiance fields, and their K-functions, with respect to depth. Furthermore, only a few of a large set of theoretical possibilities (configurations C1, C2, C3, and C4) are most likely to be observed in nature, and then principally in shallow depth regions (less than 20 attenuation lengths) of homogeneous hydrosols with calm surfaces.