10.5 A General Proof of the Asymptotic Radiance Hypothesis

In this and the following section we present two proofs of the asymptotic radiance hypothesis, that is, the conjecture concerning the shape of the radiance distribution and its behavior at great depths in natural hydrosols. The two proofs differ in their starting assumptions. The proof given in this section is completely general and uses only the mildest assumptions concerning the functions used to describe natural light fields and their supporting media. The proof is based on the natural mode of solution of the equation of transfer (Chapter 5) and thereby has, despite its analytic complexity and length, the virtue of using only intuitively simple constructs in its development and which, furthermore, begins and ends with directly observable concepts, namely the radiance K-function, the volume attenuation function α, and volume scattering function σ. The proof given in Sec. 10.6 is considerably simpler than that offered in this section because it is assumed at the outset that scalar irradiance h(z) in natural media eventually decreases exponentially with depth, a fact which is *not* assumed and is eventually derived in the present longer proof. This is quite a reasonable assumption, however, being born and sustained on inspection of much experimental evidence, and most students of the subject will thus be content with the shorter arguments of Sec. 10.6. However, those who wish to see the argument developed from first principles, are invited to read on below. The reader wishing only an overview of the arguments of the present proof need only read on through the discussion of (18) below.

At the conclusion of the chapter we will have accumulated four essentially distinct types of proof of the asymtotic radiance hypothesis, namely that given via the characteristic representation of N in Sec. 7.10, those of the present and subsequent section, and finally that in the closing observations of Sec. 10.7. Further, perhaps more elegant proofs should be forthcoming from the functional equations developed throughout Chapter 7. These are left for interested students to pursue.

Introduction

The asymptotic radiance hypothesis was formulated in the field of experimental radiative transfer dealing with the penetration of natural light into the oceans and deep lakes. It may be stated as follows: The form of the radiance distribution about a point in an optical medium approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions at the upper boundary of the medium and which depends only on the inherent optical properties of the medium. Some relatively early references to the hypothesis may be found in the experimental papers of Whitney [315], [316], Poole [209], and Lenoble [154]. Some recent theoretical discussions for particular cases may be found Herman and Lenoble [107], [108]. Subsequently, the mathematical problem underlying the hypothesis took on meaning in a wider set of contexts such as astrophysical optics and neutron transport theory. However, the statement of the hypothesis for these contexts is essentially the same.

In this section a proof of the hypothesis is given for a rather wide class of inhomogeneous spaces known as *eventually separable* spaces, a term which is defined in detail below. The discussion is designed so that the main results are also applicable to the astrophysical and neutron contexts. The approach used is direct in the sense that it is based on a study of the natural mode of solution of the equation of transfer rather than first solving the equation for particular cases and then inspecting the properties of the resulting solutions. Furthermore, the quantities introduced in the study are for the most part directly observable quantities, a feature which reflects the experimental origins of the problem and which keeps sight of possible practical applications of the asymptotic radiance hypothesis. In this way the discussion complements a different approach to the problem, namely the formal-solution approach initiated by Chandrasekhar [43] and extended by Kuscer [147] in the radiative transfer context, and also considered, for example by Davison [62] in the neutron transport context. In particular, the present discussion shows in terms of directly observable quantities that when an asymptotic radiance distribution exists in a medium, it is represented by a formal-solution distribution and is approached in a continuous way by the natural distributions as depth is increased in the medium. Two illustrations of this fact are given. One is based on tables compiled from theoretical calculations made in the neutron transport context, the other is drawn from an experiment which documented the light field in a natural hydrosol. These illustrations will be considered later.

The practical consequences of the asymptotic radiance theorem are many. They take on especial utility in the field of geophysical optics. While an exhaustive discussion of these consequencies is out of place here (see Secs. 10.7 and 10.8), we should observe that the classical two-flow equations of the light field (Chapter 8), are accurate and become exact with increasing depth whenever the hypothesis holds. This results in an enormous simplification of the standard experimental procedures dealing with the determination of the optical properties of natural hydrosols. Finally, the present method allows a means of estimating, with respect to a given preassigned criterion, the optical depth at which the asymptotic distribution has been attained (Sec. 10.7).

Preliminary Definitions

We begin with the general source-free equation of transfer for the radiance function N on a general isotropic* space X as used in geophysical optics ((14) of Sec. 3.15),

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*The arguments developed below go through with minor changes also for nonisotropic media. However, such additional generality does not add materially to the theoretical or practical consequences of the asymptotic radiance hypothesis, and is therefore not postulated at this time.

$$\frac{dN(x,\xi)}{dr} = \xi \cdot \nabla N(x,\xi) = -\alpha(x)N(x,\xi) + N_*(x,\xi) \qquad (1)$$

and recast it into a form which will be most suitable for the present discussion, and which will insure the widest domain of applicability of the present results to related fields such as astrophysical optics and neutron transport theory. Here:

$$N_*(x,\xi) = \int_\Xi N(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$= \int_\Xi \sigma(x;\xi;\xi')N(x,\xi')\,d\Omega(\xi')$$

represents the path function N_* ; and is written twice, as shown, so as to point up the isotropy of X (re Definition 3 of Sec. 7.12). The volume scattering function is σ , the unit sphere of direction vectors ξ is Ξ , and the attenuation function is α .

The present problem is meaningful only in the steady state case, and is most immediately concerned with emission-free arbitrarily stratified plane-parallel media with constant index of refraction. These conditions have been adopted in (1). The introduction of the plane-parallel geometry into (1) results in the usual equation:

$$-\xi \cdot \mathbf{k}\,\frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad , \qquad (2)$$

where $\mathbf{k}$ is the unit outward normal to the plane-parallel medium $X(0,\infty)$. For the present mathematical discussion we observe that the radiance function N is defined on the domain $Z\times\Xi$, where Z is the set of nonnegative real numbers. Furthermore, $\theta = \arccos(\xi \cdot \mathbf{k})$, so that $\xi \in \Xi$ may be represented as usual by a pair of angles $(\theta,\phi) \in \Xi$ (re: Sec. 2.4). Finally, we adopt the parameters μ and $\tau(z)$, where we have written:

$$\text{"}\mu\text{"} \quad \text{for} \quad \xi \cdot \mathbf{k} \quad ,$$

and:

$$\text{"}\tau(z)\text{"} \quad \text{for} \quad \int_0^z \alpha(z')\,dz' \quad ,$$

Recall (Sec. 7.12) that the phase function p , as used in astrophysics, is related to σ by:

$$p = 4\pi\sigma/\alpha \quad .$$

With these notations, (2) takes the form:

$$\boxed{\mu \frac{dN(\tau,\mu,\phi)}{d\tau} = N(\tau,\mu,\phi) - N_q(\tau,\mu,\phi)} \quad , \qquad (3)$$

where we have written:

$$\text{"}N_q(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \frac{1}{4\pi} \int_\Xi p(\tau;\mu,\phi;\mu',\phi')N(\tau,\mu',\phi')d\mu' d\phi'$$

which defines the equilibrium radiance function N_q. Thus (1) reduces, under the above assumptions--which will be considered in force in the sequel--to the standard form for the equation of transfer in plane-parallel media.

The discussion will require consideration of the follo ing scattering-order decomposition of (3):

$$\mu \frac{dN^j(\tau,\mu,\phi)}{d\tau} = N^j(\tau,\mu,\phi) - N_q^j(\tau,\mu,\phi) \quad , \quad j = 1 \,,\ 2 \,,\ \ldots \,, (4)$$

where we have written:

$$\text{"}N_q^j(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \frac{1}{4\pi} \int_\Xi p(\tau;\mu,\phi;\mu',\phi')N^{j-1}(\tau,\mu',\phi')d\mu' d\phi' \ , \qquad (5)$$

and where N^j and N_q^j are positive valued radiance function on $Z \times \Xi$ which refer to radiant flux which has been scattere precisely j times, so that (as in (1) of Sec. 5.4) we have the definitional identity:

$$N(\tau,\mu,\phi) = \sum_{j=0}^{\infty} N^j(\tau,\mu,\phi) \quad . \qquad (6)$$

As outlined in the introduction, the present discussion employs, whenever possible, directly observable quantitites. From the point of view of the experimenter, the depth dependence of the radiance distribution $N(\tau,\cdot,\cdot)$ on Ξ, $\tau \geq 0$, is most conveniently studied by means of the associated functions $K(\tau,\cdot,\cdot)$ on Ξ defined by writing (re: (35) of Sec. 9.2):

$$\text{"}K(\tau,\mu,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N(\tau,\mu,\phi)} \frac{dN(\tau,\mu,\phi)}{d\tau} \quad . \qquad (7)$$

The present discussion will also require consideration of the function $K_q(\tau,\cdot,\cdot)$ on Ξ, $\theta \geq 0$, defined by writing

$$\text{"}K_q(\tau,\mu,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N_q(\tau,\mu,\phi)} \frac{dN_q(\tau,\mu,\phi)}{d\tau} \quad . \qquad (8)$$

Similarly, we write:

$$\text{"}K^j(\tau,\mu,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N^j(\tau,\mu,\phi)}\,\frac{dN^j(\tau,\mu,\phi)}{d\tau}\ , \ j = 0,1,\ldots,\ \tau > 0\ , \tag{9}$$

and:

$$\text{"}K_q^j(\tau,\mu,\phi)\text{"} \ \text{for} \ -\frac{1}{N_q^j(\tau,\mu,\phi)}\,\frac{dN_q^j(\tau,\mu,\phi)}{d\tau}\ , \ j = 1,2,\ldots,\ \tau \geq 0\ . \tag{10}$$

Finally, corresponding to:

$$\text{"}N_q^{(n)}(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \sum_{j=1}^{n} N_q^j(\tau,\mu,\phi) \quad , \tag{11}$$

we write:

$$\text{"}K_q^{(n)}(\tau,\mu,\phi)\text{"} \ \text{for} \ -\frac{1}{N_q^{(n)}(\tau,\mu,\phi)}\,\frac{dN_q^{(n)}(\tau,\mu,\phi)}{d\tau}$$

$$= \left(\frac{\sum_{j=1}^{n} N_q^j(\tau,\mu,\phi)K_q^j(\tau,\mu,\phi)}{\sum_{j=1}^{n} N_q^j(\tau,\mu,\phi)}\right) \tag{12}$$

where, of course:

$$\lim_{n\to\infty} N_q^{(n)}(\tau,\mu,\phi) = N_q(\tau,\mu,\phi) \quad . \tag{13}$$

The equations which govern the behavior of the K-functions play a central role in what follows. The equations have the outward appearance of Riccati differential equations, and it will actually be possible to use to advantage some of the well-known properties of such differential equations.

It is easy to verify the following formulas with the help of (3) and the definitions of K and K_q (see, e.g., (19) of Sec. 10.7 or (7) of Sec. 11.2):

$$\boxed{\frac{dK(\tau,\mu,\phi)}{d\tau} = \Big[K(\tau,\mu,\phi) - K_q(\tau,\mu,\phi)\Big]\Big[K(\tau,\mu,\phi) + \frac{1}{\mu}\Big]} \ . \tag{14}$$

Furthermore, from (4) and the definition of K^j and K_q^j :

$$\boxed{\frac{dK^j(\tau,\mu,\phi)}{d\tau} = \left[K^j(\tau,\mu,\phi) - K_q^j(\tau,\mu,\phi)\right]\left[K^j(\tau,\mu,\phi) + \frac{1}{\mu}\right] \quad j = 1, 2, \ldots, \quad \tau \geq 0} \tag{15}$$

We now can give the motivation for the preceding adoptions of the K - functions in the present approach to the asymptotic radiance problem. Suppose there is some depth in the medium below which the functions $K(\tau,\cdot,\cdot)$, $\tau \geq \tau_0$ are constant functions on Ξ, and whose values are equal to a fixed number k_∞. Then we may write, for all $\tau \geq \tau_0$:

$$N(\tau,\mu,\phi) = N(0,\mu,\phi) \exp\left\{ - \int_0^{\tau_0} K(\tau',\mu,\phi)\,d\tau' - \int_{\tau_0}^{\tau} K(\tau',\mu,\phi)\,d\tau' \right\}$$

$$= N(\tau_0,\mu,\phi) \exp\left\{ - (\tau - \tau_0)k_\infty \right\} \quad .$$

Thus if we write:

$$\text{"}g(\mu,\phi)\text{"} \quad \text{for} \quad N(\tau_0,\mu,\phi)\exp\left\{ \tau_0 k_\infty \right\}$$

we have the following equivalent way of representing the distributions:

$$\boxed{N(\tau,\mu,\phi) = g(\mu,\phi) \exp\left\{ - \tau k_\infty \right\}} \quad . \tag{16}$$

Relation (16) is the starting point of the classical formal procedures referred to above which lead to the determination of a specific radiance distribution g on Ξ. From the point of view of the present approach, however, (16) is an incidental end rather than a means. That is, we will be concerned with the determination of a class of spaces in which the radiance distributions tend continuously to a structure of the kind summarized in (16), and thus as a matter of course, determine a class of spaces in which such a formal procedure for the asymptotic radiance distribution is meaningful.

The preceding heuristic argument leading to (16) supplies the motivation for the following definition of an asymptotic radiance distribution: An *asymptotic radiance distribution* is said to exist if (i) $\lim_{\tau\to\infty} K(\tau,\mu,\phi)$ (henceforth denoted by "$K_\infty(\mu,\phi)$") exists for each $(\mu,\phi) \in \Xi$ and (ii) $K_\infty(\cdot,\cdot)$ is a constant function on Ξ .

It is quite possible for condition (i) of the preceding definition to hold, while condition (ii) does not hold. This state of affairs is encountered, for example, in space in which $s(\tau)/\alpha(\tau) = 0$ for all $\tau \geq 0$, where

$$s(\tau) = \int_{\Xi} \sigma(\tau;\mu',\phi';\mu,\phi)d\mu'd\phi'$$

i.e., in purely absorbing media. However, such spaces are clearly trivial from the present point of view.

Formulation of the Problem

In order to keep the usual operations on the phase function p meaningful, we will assume, as a matter of course, that p is piecewise continuously differentiable with respect to τ, and that p is continuous on $\Xi \times \Xi$ for each $\tau > 0$. Furthermore, we will require that the boundary radiance function $N^{o}(0,\cdot,\cdot)$ on Ξ_- be a nonnegative valued, nontrivial integrable function with respect to the measure Ω. Here, we have

$$\Xi_- = \left\{ \xi : \xi \cdot \mathbf{k} \leq 0 \right\} = \left\{ (\mu,\phi) : -1 \leq \mu \leq 0 \right\} .$$

This Ξ_- differs slightly from that usually used by now including all ξ such that $\xi \cdot \mathbf{k} = 0$. In addition, we define Ξ_+ as the complement of Ξ_- with respect to Ξ. Finally, we observe that subsets of Ξ of solid angle measure zero are of no physical interest, and have no effect on calculations which employ integrations over such sets. For example, the set of all ξ such that $\xi \cdot \mathbf{k} = 0$ is of zero Ω measure. This motivates the following standing assumption to hold throughout this section. Whenever a function f on Ξ is constant on Ξ except for a subset Ξ_o of Ω measure zero, we shall assume that f is replaceable by a constant function $\hat{f}$ on Ξ such that $f(\xi) = \hat{f}(\xi)$ for every ξ in $\Xi - \Xi_o$. We call $\hat{f}$ the (constant) *extension* of f from $\Xi - \Xi_o$ to Ξ.

A *separable medium* is one in which the phase function is independent of position (re: Sec. 7.12). The term "separable" is used to suggest the multiplicative uncoupling of position and directional dependence that σ undergoes in such spaces: $\sigma(x;\xi;\xi') = \alpha(x)p(\xi;\xi')/4\pi$. Separable media form a class of harmlessly inhomogeneous spaces. From the point of view of the equation of transfer (3), such spaces are homogeneous. The present discussion can be carried out in a rather wide class of nonseparable spaces which we will call "eventually separable." We will say that a semi-infinite stratified plane-parallel medium $X(0,\infty)$ is *eventually separable* if, (i) the phase function p on $Z \times \Xi \times \Xi$ has the form $p(\tau;\mu,\phi;\mu',\phi') = p_\infty(\mu,\phi;\mu',\phi') + \phi(\tau;\mu,\phi;\mu',\phi')$ such that p_∞ is independent of τ and not identically zero on $\Xi \times \Xi$, and (ii) $\phi \to 0$ (the zero function) uniformly on $\Xi \times \Xi$, as $\tau \to \infty$.

We can now state the main result: *an asymptotic radiance distribution exists in every plane-parallel medium if and only if the medium is eventually separable.* This statement is understood to hold in media whose equation of transfer is given by (3), and whose boundary conditions are given as above. All of the effort below will be devoted to proving the sufficiency of the eventually separable condition. Simple counter examples show that if a space is not eventually separable, the asymptotic radiance distribution necessarily does not exist. (For example, stack in alternate layers purely absorbing and purely scattering plane parallel media.)

We close this preliminary discussion by making some observations on the K - functions which will be required below. First we observe that from (3), if $\mu = 0$, then $N(\tau,0,\phi) = N_q(\tau,0,\phi)$. Hence, for all $\tau > 0$, $K(\tau,0,\phi) = K_q(\tau,0,\phi)$. Secondly, for each $j,\tau > 0$, $N^j(\tau,\cdot,\cdot)$ is bounded away from zero, is continuous on the compact set Ξ , and hence is uniformly continuous on Ξ and Ξ_- . A similar observation holds for $N_q^j(\tau,\cdot,\cdot)$, $j = 1 , 2 , \ldots$. It follows that $K_q(\tau,\cdot,\cdot)$ and $K_q^j(\tau,\cdot,\cdot)$ are uniformly continuous on Ξ and Ξ_- for all $\tau > 0$ and $j = 1 , 2 , \ldots$. Finally, from (3) and (4) and the definitions of K and K^j ,

$$K(\tau,\mu,\phi) + \frac{1}{\mu} < 0 \quad , \tag{17}$$

$$K^j(\tau,\mu,\phi) + \frac{1}{\mu} < 0 , \; j = 0 , 1 , \ldots , \tag{18}$$

for all $(\mu,\phi) \in \Xi_-$ and all $\tau > 0$. Properties (17) and (18) are particularly useful in conjunction with (14) and (15). For example if $K(\tau,\mu,\phi) \lessgtr K_q(\tau,\mu,\phi)$, then by (14) and (17) it follows that $dK(\tau,\mu,\phi)/d\tau \gtrless 0$, showing that in general K always *tends* toward the equilibrium function K_q . This is a useful fact in practice. Whether or not $K \to K_q$ as $\tau \to \infty$ depends on the relative sizes of $K_q(\mu,\phi) = \lim_{\tau\to\infty} K(\tau,\xi,\phi)$. and $-(1/\mu)$. It follows directly from the properties of the Riccati equation (cf. e.g., [116, p. 312]), that $K \to \min\{K_q(\mu\phi), -(1/\mu)\}$ assuming of course that $K_q(\mu,\phi)$ exists. A similar set of remarks holds for (15) and (18).

The Functions P,Q,R

In order to insure that the main sequence of arguments is uninterrupted by the development of certain required auxiliary relations, these auxiliary relations are gathered here for ready reference.

The first relation needed below gives the connection between the downwelling j - scattered flux at level $\tau > 0$ and the upwelling scattered flux at level τ :

$$N^{j+1}(\tau,\mu,\phi) = \frac{1}{\mu} \int_{\Xi_-} P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi' \tag{19}$$

for all $(\mu,\phi)\in\Xi_+$, and where we have written

$$"P(\tau;\mu,\phi;\mu',\phi')" \quad \text{for} \quad \frac{1}{4\pi}\int_\tau^\infty p(\tau';\mu,\phi;\mu',\phi')\cdot$$

$$\cdot \exp\left\{-(\tau'-\tau)\left[\frac{1}{\mu}-\frac{1}{\mu'}\right]\right\}d\tau' \quad . \tag{20}$$

If the space were separable, i.e., p were independent of τ (or, in the present case, the phase function component $\phi\equiv 0$) then writing "P_∞" for the limit of p as $\tau\to\infty$, we have:

$$P_\infty(\mu,\phi;\mu',\phi') = \frac{1}{4\pi}\,\frac{\mu\mu'}{\mu'-\mu}\,p_\infty(\mu,\phi;\mu',\phi') \quad , \tag{21}$$

where $(\mu,\phi)\in\Xi_+$, $(\mu',\phi')\in\Xi_-$. We observe that for eventually separable spaces,

$$\lim_{\tau\to\infty}\frac{dP(\tau;\mu,\phi;\mu',\phi')}{d\tau} = 0 \tag{22}$$

uniformly on $\Xi_+\times\Xi_-$, and that:

$$\lim_{\tau\to\infty}P(\tau;\mu,\phi;\mu',\phi') = P_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+\times\Xi_-$ and finally, that:

$$\lim_{\tau\to\infty}K^{j+1}(\tau,\mu,\phi) =$$

$$= \lim_{\tau\to\infty}\frac{\displaystyle\int_{\Xi_-}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')K^j(\tau,\mu',\phi')d\mu'd\phi'}{\displaystyle\int_{\Xi_-}P(\tau;\mu'\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi'} \tag{23}$$

for $(\mu,\phi)\in\Xi_+$. A similar expression holds for $K_q^{j+1}(\tau,\mu,\phi)$, which follows from (5) and (10):

$$\lim_{\tau\to\infty}K_q^{j+1}(\tau,\mu,\phi) =$$

$$= \lim_{\tau\to\infty}\frac{\displaystyle\int_{\Xi}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')K^j(\tau,\mu',\phi')d\mu'd\phi'}{\displaystyle\int_{\Xi}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi'} \quad . \tag{24}$$

The next relation required below makes use of the forms of the principles of invariance in generally nonseparable media, in particular, use will be made of:*

$$N(\tau,\mu,\phi) = \frac{1}{\mu}\int_{\Xi_-} R(\tau,\infty;\mu,\phi;\mu',\phi')N(\tau,\mu',\phi')d\mu'd\phi'$$

where $(\mu,\phi) \in \Xi_+$ and $R(\tau,\infty;\cdot;\cdot)$ on $\Xi_+ \times \Xi_-$ is the reflectance function associated with the subset of $Z \times \Xi$ below level $\tau > 0$, i.e., with $X(\tau,\infty)$ (see (31) of Sec. 3.7). If the medium were separable, then for all pairs (τ_1,τ_2) of depths:

$$R(\tau_1,\infty;\mu,\phi;\mu',\phi') = R(\tau_2,\infty;\mu,\phi;\mu',\phi') \quad .$$

In the present case it is possible to verify on the basis of the differential equations for the R and T operators of Sec. 7.1 that:

$$\lim_{\tau\to\infty} \frac{dR(\tau,\infty;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

uniformly on $\Xi_+ \times \Xi_-$, and that:

$$\lim_{\tau\to\infty} R(\tau,\infty;\mu,\phi;\mu',\phi') = R_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+ \times \Xi_-$, where R_∞ on $\Xi_+ \times \Xi_-$ is the reflectance of a homogeneous medium $X(0,\infty)$ with phase function p_∞. (R_∞ may be found using the methods of Sec. 7.6.)

Finally, the integral operator:

$$\int_{\Xi_-} [\quad] Q(\tau,\infty;\mu,\phi;\mu',\phi')d\mu'd\phi' \tag{26}$$

will be used. This operator maps the function $N(\tau,\cdot,\cdot)$ on Ξ_- into the function $N_q(\tau,\cdot,\cdot)$ on Ξ_-. The kernel Q is defined by writing:

"$Q(\tau,\infty;\mu,\phi;\mu',\phi')$" for $p(\tau;\mu,\phi;\mu',\phi')$

$$+ \int_{\Xi_+} p(\tau;\mu,\phi;\mu'',\phi'')R(\tau,\infty;\mu'',\phi'';\mu',\phi')\frac{d\mu''}{\mu''}d\phi''$$

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*This section is adapted from Reference [224] with a minimum of notational change. Hence the reversal of the positions of N and R (and the primed and unprimed arguments) from that throughout the remainder of this work.

The operator (26) is a positive operator.* From the definition of Q, it follows once again from the differential equations for the R operators of Sec. 7.1 that:

$$\lim_{\tau\to\infty} \frac{dQ(\tau,w;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

and that:

$$\lim_{\tau\to\infty} Q(\tau,\infty;\mu,\phi;\mu',\phi') = p_\infty(\mu,\phi;\mu',\phi') + \int_{\Xi_+} P_\infty(\mu,\phi;\mu'',\phi'')R_\infty(\mu'',\phi'';\mu',\phi')\frac{d\mu''}{\mu''}\,d\phi''$$

both uniformly on $\Xi_- \times \Xi_-$.

The Limit of $K_q(\cdot,\mu,\phi)$

We now begin the main steps of the proof. The object of the present discussion is to show that the function $K_q(\cdot,\cdot)$ on Ξ defined by writing:

$$\text{" } K_q(\mu,\phi) \text{ "} \qquad \text{for} \qquad \lim_{\tau\to\infty} K_q(\tau,\mu,\phi)$$

exists and is continuous almost everywhere on Ξ. The discussion begins with some observations on the functions K^j, K_q^1, $K_q^{(n)}$. In particular, we observe that for $(\mu,\phi)\varepsilon\,\Xi$ and every $\tau \geq 0$,

$$K_q^1(\tau,\mu,\phi) = \frac{\left\{\begin{array}{l}\displaystyle\int_{\Xi_-} p(\tau;\mu,\phi;\mu',\phi')N^o(\tau,\mu',\phi')\frac{d\mu'}{\mu'}\,d\phi' \\ \displaystyle + \int_{\Xi_-} p'(\tau;\mu,\phi;\mu',\phi')N^o(\tau,\mu',\phi')\,d\mu' d\phi'\end{array}\right\}}{\displaystyle\int_{\Xi_-} p(\tau;\mu,\phi\ ,\mu',\phi')N^o(\tau,\mu',\phi')\,d\mu' d\phi'}$$

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*For the present discussion, an operation T is said to be *positive* if T maps nonnegative functions into nonnegative functions and $Tf = 0$ (the zero function) implies f is the zero function, where f is a nonnegative valued function on Ξ_-, and the vanishing of f is taken in the sense of Lebesgue (cf., e.g., [111], p. 25).

where p' denotes the derivative of p with respect to τ. The function $N^o(\tau,\cdot,\cdot)$ on Ξ_- is related to the boundary radiance distribution by

$$N^o(\tau,\mu,\phi) = N^o(0,\mu,\phi)e^{\tau/\mu} \quad .$$

Hence each integrand in (27) is integrable on Ξ_-, so that $K_q^1(\tau,\mu,\phi)$ exists and is well defined for every $\tau \geq 0$ and $(\mu,\phi)\varepsilon\Xi$. Furthermore each integrand in (27) satisfies the hypothesis of Lebesgue's bounded convergence theorem, so that by (24):

$$K_q^1(\mu,\phi) = \lim_{\tau\to\infty} K_q^1(\tau,\mu,\phi) \tag{28}$$

exists for every $(\mu,\phi)\varepsilon\Xi$ and in fact $K_q^1(\cdot,\cdot)$ is continuous and therefore bounded on Ξ. The values of $K_q^1(\cdot,\cdot)$ are readily determinable for specific choices of $N^o(0,\cdot,\cdot)$. For example, if we adopt the standard discrete boundary radiance distribution defined by:

$$N^o(0,\mu,\phi) = N^o\delta(\mu-\mu_o)\delta(\phi-\phi_o) \quad , \quad -1 \leq \mu_o < 0 \quad ,$$

then:

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_o} \quad , \quad (\mu,\phi)\ \varepsilon\ \Xi \tag{29}$$

Slightly more generally, if:

$$N^o(0,\mu,\phi) = \sum_{i=o}^{n} N^o(\mu_i)\delta(\mu-\mu_i)\delta(\phi-\phi_i) \quad , \quad -1 \leq \mu_i < 0 \quad ,$$

and if:

$$\mu_o = \min\left\{\mu_i : i = 0, 1, \ldots, n\right\}$$

then:

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_o} \quad , \quad (\mu,\phi)\ \varepsilon\ \Xi \quad . \tag{30}$$

Other simple examples of $N^o(0,\cdot,\cdot)$ may be given, such as step-function representations, various simple continuous functions on Ξ_-, but (29) and (30) will suffice to illustrate the general procedure. In particular they help to illustrate the use of (15) which is required in the next step of the proof and which runs as follows: By means of (15) and (28), we see that for each $(\mu,\phi)\varepsilon\,\Xi_-$ such that

$$K_q^1(\mu,\phi) > -\frac{1}{\mu} \tag{31}$$

we have:

$$\lim_{\tau\to\infty} K^1(\tau,\mu,\phi) = -\frac{1}{\mu}$$

Since $K_q^1(\cdot,\cdot)$ is bounded, the subset of μ's for which (31) holds is a relatively open subset of $[-1,0]$ excluding 0. Finally, from (15) and (28), for each $(\mu,\phi) \varepsilon\, \Xi_-$ such that:

$$K_q^1(\mu,\phi) \leq -\frac{1}{\mu} \quad , \tag{32}$$

we have:

$$\lim_{\tau\to\infty} K^1(\tau,\mu,\phi) = K_q^1(\mu,\phi) \quad .$$

Hence the function $K_\infty^1(\cdot,\cdot)$ on Ξ_- defined by writing:

$$\text{"}K_\infty^1(\mu,\phi)\text{"} \quad \text{for} \quad \lim_{\tau\to\infty} K^1(\tau,\mu,\phi)$$

exists for all $(\mu,\phi) \varepsilon\, \Xi_-$ and is continuous on Ξ_-. A particular illustration of a $K_\infty^1(\cdot,\cdot)$ is given by means of (29).

The main observation to make at this point is the following: In addition to being bounded on Ξ_-, the function $K_\infty^1(\cdot,\cdot)$ has the property that

$$K_\infty^1(\mu,\phi) \leq -\frac{1}{\mu} \tag{33}$$

on Ξ_-. The discussion of $K_\infty^1(\cdot,\cdot)$ is completed by showing that it exists and is continuous on Ξ_+. This is done by applying the preceding arguments to (23). As an example, one may consider (29) once again, which yields $K_\infty^1(\mu,\phi) = -1/\mu_o$ for all $(\mu,\phi)\, \varepsilon\, \Xi_+$.

We now take the general inductive step, that is, we assume that $K_\infty^j(\cdot,\cdot)$, $j \geq 1$ is continuous on Ξ and in particular, $K_\infty^j(\mu,\phi) \leq -1/\mu$ for every $(\mu,\phi)\, \varepsilon\, \Xi_-$. Then by means of (24) and the previously cited convergence arguments, we find that: $K_q^{j+1}(\mu,\phi)$, where we write:

$$\text{"}K_q^{j+1}(\mu,\phi)\text{"} \quad \text{for} \quad \lim_{\tau\to\infty} K_q^{j+1}(\tau,\mu,\phi) \tag{34}$$

exists for every $(\mu,\phi)\, \varepsilon\, \Xi_-$, and $K_q^{j+1}(\cdot,\cdot)$ is continuous on Ξ_-; and in particular,*

$$K_q^{j+1}(\mu,\phi) \leq -\frac{1}{\mu} \; , \; (\mu,\phi)\, \varepsilon\, \Xi_- \quad . \tag{35}$$

---

*To obtain (35), implicit use has been made of the general fact that if $F(x) = [A(z)a(z) + B(x)b(x)]/[A(x) + B(x)]$, and if we have $a(x) \to a(x_o) \leq a_o$ along with $B(x) = o(A(x))$ as $x \to x_o$, then $F(x) \to a(x_o) \leq a_o$, as $x \to x_o$.

Furthermore, by (15):

$$K_\infty^{j+1}(\mu,\phi) = \lim_{\tau\to\infty} K^{j+1}(\tau,\mu,\phi) = K_q^{j+1}(\mu,\phi) \qquad (36)$$

on Ξ_-. Finally, from (23), $K_\infty^{j+1}(\cdot,\cdot)$ exists and is continuous on Ξ_+ and moreover,

$$K_\infty^{j+1}(\mu,\phi) \le -\frac{1}{\mu}, \quad (\mu,\phi)\varepsilon\,\Xi_+ \quad . \qquad (37)$$

Since the induction hypothesis has been demonstrated for the case $j+1$ assuming the case j, and it is true for $j = 1$, the conclusions (34)-(37) then hold for all integers $j = 1, 2, \ldots$. It follows from (12) and the preceding results that:

$$K_q^{(n)}(\mu,\phi) = \lim_{\tau\to\infty} K_q^{(n)}(\tau,\mu,\phi) \le -\frac{1}{\mu}, \qquad (38)$$

exists and is continuous on Ξ, for $n = 2, 3, \ldots$.

Now consider the function $g_q^{(n)}(\cdot,\mu,\phi)$ on Z defined for every $(\mu,\phi)\varepsilon\,\Xi$ by writing:

$$\text{"}g_q^{(n)}(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \frac{N_q^{(n)}(\tau,\mu,\phi)}{N_q(\tau,\mu,\phi)}, \quad n = 1, 2, \ldots. \qquad (39)$$

Clearly $\{g_q^{(n)}(\cdot,\mu,\phi)\}$ is an increasing sequence of functions on Z, such that for every $(\mu,\phi)\varepsilon\,\Xi$,

$$\lim_{n\to\infty} g_q^{(n)}(\cdot,\mu,\phi) = 1,$$

the unit function on Z. From this and the definitions of $K_q^{(n)}$ and K_q, we conclude, first of all, that the sequence $\{K_q^{(n)}(\cdot,\mu,\phi)\}$ of functions converge in the mean to $K_q(\cdot,\mu,\phi)$ on Z. This in turn implies that the convergence to $K_q(\cdot,\mu,\phi)$ is almost uniform on Z for some subsequence $\{K_q^{(n_k)}(\cdot,\mu,\phi)\}$. Hence for every $\varepsilon > 0$, and subset Z_ε of Z,

$$\begin{aligned} \lim_{n_k\to\infty}\lim_{\tau\to\infty} K_q^{(n_k)}(\tau,\mu,\phi) &= \lim_{\tau\to\infty}\lim_{n_k\to\infty} K_q^{(n_k)}(\tau,\mu,\phi) \\ &= \lim_{\tau\to\infty} K(\tau,\mu,\phi) \end{aligned}$$

on $Z' = Z - Z_\varepsilon$, where

$$\int_{Z_\varepsilon} d\tau' < \varepsilon \quad .$$

It follows that the function $K_q(\cdot,\cdot)$ on Ξ has the property:

$$K_q(\mu,\phi) = \lim_{\tau\to\infty} K_q(\tau,\mu,\phi) = \lim_{n_k\to\infty} K_q^{(n_k)}(\mu,\phi) \leq -\frac{1}{\mu} \qquad (40)$$

for every (μ,ϕ) in Ξ . Finally, from (24), (15), (23), and (12) (in that order), we establish the fact that $\{K_q^{(n)}(\cdot,\cdot)\}$ is a sequence of continuous functions whose essential suprema form a nonincreasing sequence of real numbers. It follows that $K_q^{(n)}(\cdot,\cdot)$ converges uniformly a.e., on Ξ to $K_q(\cdot,\cdot)$, and that $K_q(\cdot,\cdot)$ is continuous on $\Xi - \Xi_0$ where Ξ_0 is a subset of Ξ such that $\Omega(\Xi_0) = 0$. Hence $K_q(\cdot,\cdot)$ is continuous almost everywhere on Ξ .

The Limit of $K(\cdot,\mu,\phi)$

The proof is now concluded by showing that $K(\cdot,\mu,\phi)$ satisfies the definition of asymptoticity. By (14) and the result summarized in (40), we have

$$K_\infty(\mu,\phi) = \lim_{\tau\to\infty} K(\tau,\mu,\phi) = K_q(\mu,\phi) \qquad (41)$$

for all $(\mu,\phi) \in \Xi_-$. Hence by our preceding result on $K_q(\cdot,\cdot)$, $K_\infty(\cdot,\cdot)$ is continuous almost everywhere on Ξ_- . It follows [103, p. 242, problem (3)] that there is, for every $\varepsilon > 0$, a *compact* subset $\Xi_-(\varepsilon)$ of $\Xi_- - \Xi_0$ such that $\Omega[(\Xi_- - \Xi_0) - \Xi_-(\varepsilon)] < \varepsilon$ on which $K_\infty(\cdot,\cdot)$ is continuous. We use this fact to establish the existence of a minimal value of $K_\infty(\cdot,\cdot)$ on $\Xi_-(\varepsilon)$. Let (μ_1,ϕ_1) be any direction in $\Xi_-(\varepsilon)$ (there is at least one) defined by the condition:

$$K_\infty(\mu\ ,\phi\ ) = \inf\{K_\infty(\mu,\phi) : (\mu,\phi) \in \Xi_-(\varepsilon)\} \quad ,$$

Errata

and then write:

$$\text{"}g(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \frac{N(\tau,\mu,\phi)}{N(\tau,\mu_1,\phi_1)} \quad ,$$

and observe that g on $\Xi_-(\varepsilon)$ defined by writing

$$\text{"}g(\mu,\phi)\text{"} \qquad \text{for} \quad \lim_{\tau\to\infty} g(\tau,\mu,\phi)$$

is at least bounded and measurable (hence integrable) on $\Xi_-(\varepsilon)$. Then by means of the operator defined in (26), we have:

$$K_q(\mu,\phi) = \frac{\displaystyle\int_{\Xi_-(\varepsilon)} Q_\infty(\mu,\phi;\mu',\phi')g(\mu',\phi')K_\infty(\mu',\phi')d\mu'd\phi'}{\displaystyle\int_{\Xi_-(\varepsilon)} Q_\infty(\mu,\phi;\mu',\phi')g(\mu',\phi')d\mu'd\phi'} \quad , \qquad (42)$$

for all $(\mu,\phi)\varepsilon\,\Xi_-(\varepsilon)$. In particular, (41) holds for $(\mu_1,\phi_1)\varepsilon\,\Xi_-(\varepsilon)$. Using (41), (42) may be rewritten as:

$$\int_{\Xi_-(\varepsilon)} Q_\infty(\mu_1,\phi_1;\mu',\phi')g(\mu',\phi')[K_\infty(\mu',\phi') - K_\infty(\mu_1,\phi_1)]d\mu' d\phi' = 0$$

This operator $Q_\infty g$, as that in (26), being a positive operator, requires that the everywhere nonnegative valued function:

$$K_\infty(\cdot,\cdot) - K_\infty(\mu_1,\phi_1)$$

on $\Xi_-(\varepsilon)$ be the zero function almost everywhere on $\Xi_-(\varepsilon)$. Writing "k_∞" for $K_\infty(\mu_1,\phi_1)$ we have:

$$K_\infty(\mu,\phi) = k_\infty$$

for almost every $(\mu,\phi)\varepsilon\Xi_-(\varepsilon)$. Since ε is arbitrary, this result holds almost everywhere on $\Xi_- - \Xi_0$ and, by extension, everywhere on Ξ_-. An application of (25) to the definitions of $K(\tau,\mu,\phi)$ and $K_\infty(\mu,\phi)$, yields the result that

$$K_\infty(\mu,\phi) = k_\infty$$

for almost every (μ,ϕ) on Ξ_+, so that $K_\infty(\cdot,\cdot)$ is a constant function almost everywhere (and, by our agreed extension, everywhere) on Ξ. *This concludes the proof.*

We observe finally, that, by means of the definition of N_q and (8),

$$K_q(\mu,\phi) = K_\infty(\mu,\phi) = k_\infty$$

everywhere on Ξ.

Notes and Observations

We now make an observation on the physical significance of the number k_∞. We observe that the scalar irradiance function h on Z defined by writing

$$\text{"}h(\tau)\text{"} \quad \text{for} \quad \int_\Xi N(\tau,\mu,\phi)d\mu d\phi$$

has, in analogy to N, a K-function defined by writing:

$$\text{"}k(\tau)\text{"} \quad \text{for} \quad -\frac{1}{h(\tau)}\frac{dh(\tau)}{d\tau} \quad ,$$

which, as we saw in (39) of Sec. 9.2, is represented in terms of $K(\tau,\cdot,\cdot)$ by the formula:

$$k(\tau) = \frac{\int_{\Xi} N(\tau,\mu,\phi)K(\tau,\mu,\phi)\,d\mu d\phi}{\int_{\Xi} N(\tau,\mu,\phi)\,d\mu d\phi}$$

From this and the preceding results, we see that:

$$\lim_{\tau\to\infty} k(\tau) = \frac{\int_{\Xi} g(\mu,\phi)K_\infty(\mu,\phi)\,d\mu d\phi}{\int_{\Xi} g(\mu,\phi)\,d\mu d\phi} = k_\infty \quad .$$

Hence k_∞ is also the limit, as $\tau\to\infty$, of $k(\cdot)$, the K-function for scalar irradiance. The function h is related to the radiant energy density function u by $h = vu$, where v is the speed of light in the medium.

As a second observation, we note how the canonical form of the equation of transfer yields the integral equation for the asymptotic radiance distribution. The equation of transfer (3) may be written in terms of $K(\tau,\cdot,\cdot)$ as follows:

$$N(\tau,\mu,\phi) = \frac{N_q(\tau,\mu,\phi)}{1 + \mu K(\tau,\mu,\phi)}$$

which is the canonical form of the equation of transfer for the slab geometry (Chapter 4). The limit of the canonical form as $\tau\to\infty$ is, by the preceding results (and recall (16)):

$$\boxed{g(\mu,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} p_\infty(\mu,\phi;\mu',\phi')g(\mu',\phi')\,d\mu' d\phi'}{1 + \mu k_\infty}} \quad , \qquad (43)$$

which is the general form used in the formal-solution procedures discussed in the introduction. The real number k_∞ now takes on the additional significance of being an eigenvalue of an eigenvalue problem associated with the above integral equation for g on Ξ. For the kind of boundary conditions adopted in the present section--which as we have noted before, stem from the geophysical origins of the asymptotic radiance problem--the resultant values of k_∞ are nonnegative, and in fact, $0 \le k_\infty < 1$ (cf. (17) and the definition of g).

As a final observation, we relate the present theoretical findings to some independent computations and empirical measurements of asymptotic radiance distributions. Figure 10.13 shows the depth dependence of $K(\cdot,\mu,\phi)$ for several directions $(\mu,\phi)\varepsilon\Xi$. The associated medium is a hypothetical separable half-space, irradiated by normally incident collimated neutron flux, in which scattering is isotropic

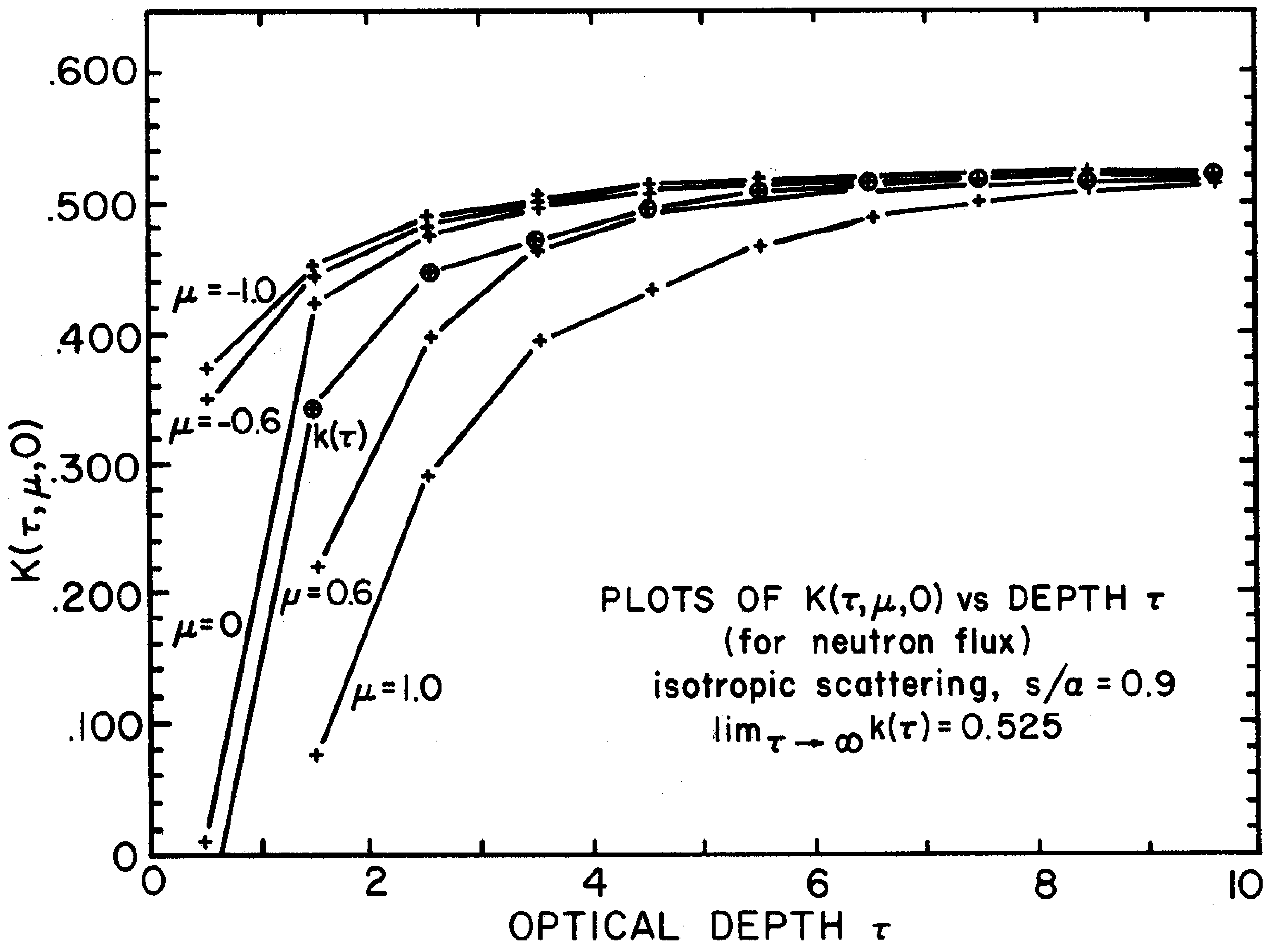

FIG. 10.13 A theoretical example of the asymptotic radiance theorem.

and $s/\alpha = 0.9$. These plots are based on theoretical computations of $N(\tau,\mu,\phi)$ (for neutron flux) compiled in [11]. The plots show clearly that asymptoticity has been essentially attained at $\tau = 10$, for at this depth the function $K(10,\cdot,\cdot)$ is essentially constant on Ξ.

Figure 10.14 shows the depth dependence of $K(\cdot,\mu,\phi)$ for several directions $(\mu,\phi)\varepsilon\Xi$. The associated medium is a natural hydrosol, namely Lake Pend Oreille, Idaho, which at the time of measurement of $N(\tau,\mu,\phi)$, was irradiated by light from a clear sunny sky (angle of sun from zenith was about 40°, hence the associated μ_o was -0.77); scattering was found to be highly anisotropic and s/α approached, with increasing τ, a constant value of about 0.7, indicating the medium was eventually separable. These plots are based on experimental determinations of $N(\tau,\mu,\phi)$ recorded in [298]. All N-measurements were made at about 480 millimicrons. The plots show that asymptoticity is being markedly approached at depth $\tau = 20$ and below. The azimuth angle ϕ has been fixed at 0°, which denotes the vertical plane through the sun. Plots for $\phi \neq 0°$ indicate similar trends to asymptoticity for depths at $\tau = 20$ and below. The vertical K-scale has been exaggerated (relative to that of Fig. 10.13) in order to more clearly show the details of the transition to asymptoticity.

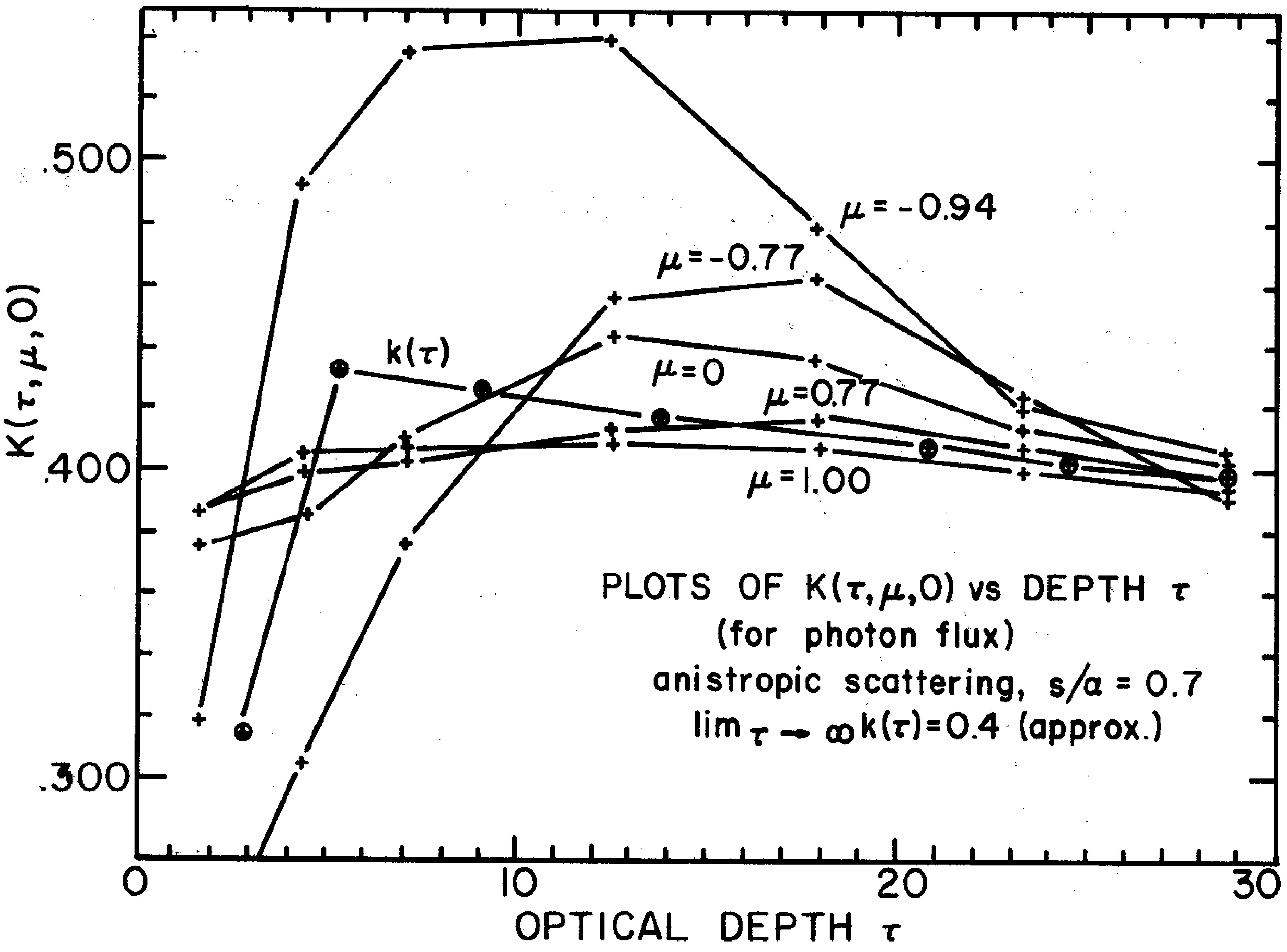

FIG. 10.14 An experimental example of the asymptotic radiance theorem.

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10.6 On the Existence of Characteristic Diffuce Light: A Special Proof of the Asymptotic Radiance Hypothesis

In this section we return to the problem of the asymptotic radiance hypothesis and, as outlined in the introductory remarks to Sec. 10.5, we approach the hypothesis from a basically simpler, more empirical point of view. We shall therefore reintroduce the problem in the following paragraphs from this alternate point of view, and carry out the discussion so that it is virtually independent of that in Sec. 10.5.

Introduction

Recent experimental evidence, recorded in [298], forms the basis for fresh support of the long-standing conjecture that the radiance distribution about a point in an optically deep natural hydrosol approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions and the optical state of the surface of the medium, and which depends only on the inherent optical properties of the medium. This conjecture was apparently given its first definitive formulation by Whitney [315], [316], who referred to the *asymptotic radiance distribution* as *characteristic diffuse light*. (We shall use these two