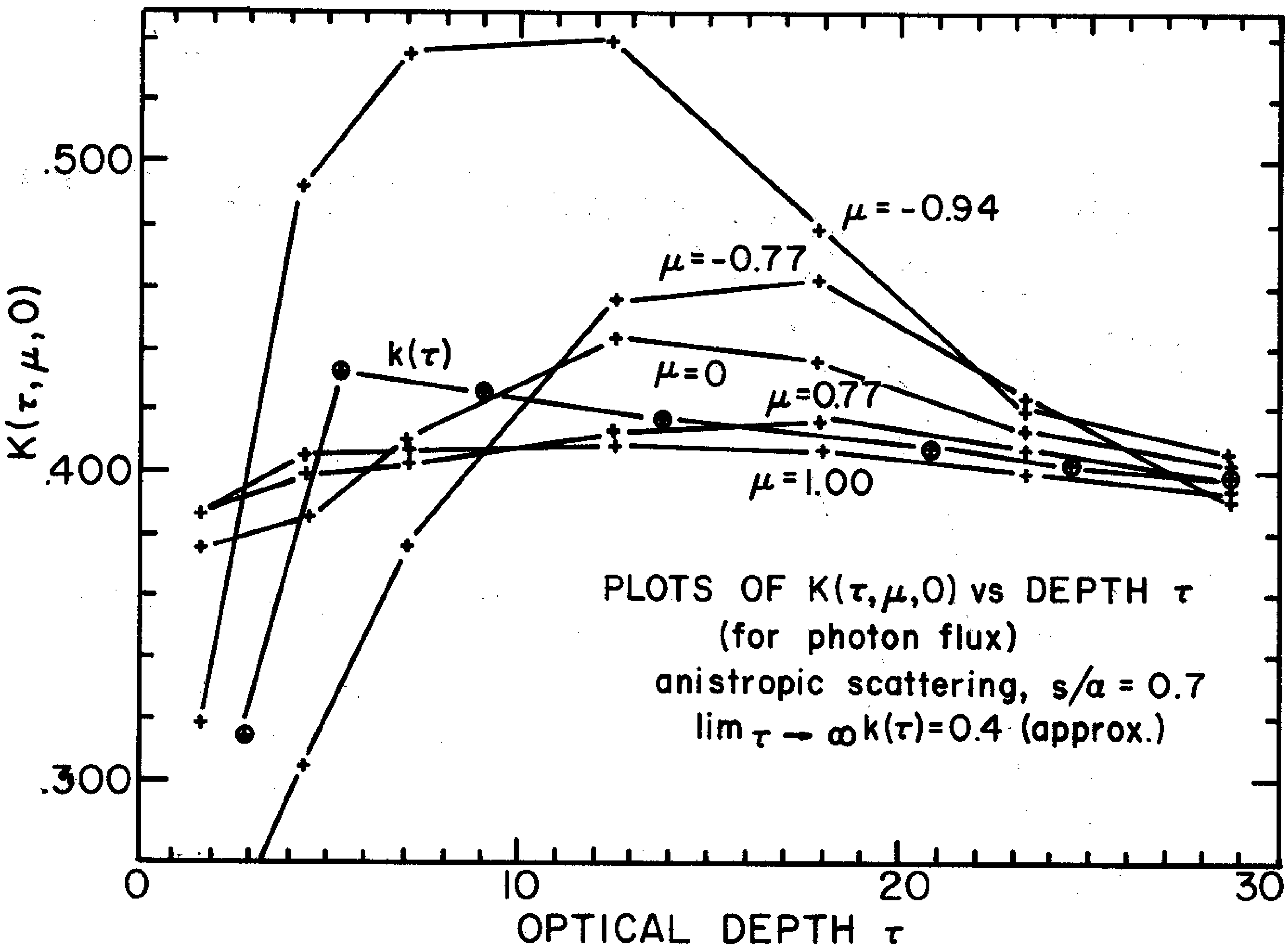

FIG. 10.14 An experimental example of the asymptotic radiance theorem.

10.6 On the Existence of Characteristic Diffuce Light: A Special Proof of the Asymptotic Radiance Hypothesis

In this section we return to the problem of the asymptotic radiance hypothesis and, as outlined in the introductory remarks to Sec. 10.5, we approach the hypothesis from a basically simpler, more empirical point of view. We shall therefore reintroduce the problem in the following paragraphs from this alternate point of view, and carry out the discussion so that it is virtually independent of that in Sec. 10.5.

Introduction

Recent experimental evidence, recorded in [298], forms the basis for fresh support of the long-standing conjecture that the radiance distribution about a point in an optically deep natural hydrosol approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions and the optical state of the surface of the medium, and which depends only on the inherent optical properties of the medium. This conjecture was apparently given its first definitive formulation by Whitney [315], [316], who referred to the *asymptotic radiance distribution* as *characteristic diffuse light*. (We shall use these two names interchangeably in what follows.) In this section we complement the experimental evidence in favor of this conjecture by supplying a simple proof of the existence of characteristic diffuse light in all homogeneous optically deep natural waters. The discussion concludes with a derivation of the integral equation governing the angular structure of the characteristic diffuse light and a brief discussion of an interesting and tractable example for the case of isotropic scattering.

We note in passing that since the time of the formulation of the asymptotic radiance hypothesis by Whitney, its domain of applicability has been widened considerably. The problem of a limiting angular distribution has since been encountered in modern neutron transport theory but basically as an abstract mathematical problem rather than experimental phenomenon. A similar type of problem has long been extant in astrophysical radiative transfer. A general proof of the existence of an asymptotic radiance distribution which covers all these contexts is given in Sec. 10.5.

Despite the widening of the domain of applicability of the hypothesis, it will retain its greatest usefulness in the context of geophysical optics, and in particular in hydrologic optics. For in this field, unlike the others mentioned above, the trend to a characteristic limiting form is a directly observable phenomenon. The existence of such a form is of inestimable importance to all experimental research work dealing with the determination of the optical properties of natural waters. In many important instances, knowledge that an asymptotic radiance distribution exists will obviate the necessity for experimental probings to extremely large depths; for such knowledge will allow, by means of relatively simple formulas, the accurate prediction of the geometrical structure of the light field in the great-depth ranges. Some of these practical consequences of the asymptotic radiance hypothesis are developed in Secs. 10.7 and 10.8.

Physical Background of the Method of Proof

The argument used by Whitney in establishing experimental evidence for the asymptotic radiance hypothesis went basically as follows: He showed that when the experimentally obtained plots of radiance distributions at various large depths were all blown up to the same size (more precisely, the zenith radiances were all normalized to a common value), they formed a set of nearly congruent figures. Now, an interesting feature of such radiance distributions is that they assume the same *shape*, and decrease in size with increasing depth at very nearly the same exponential rate. This fact can be stated precisely as follows:

$$N(z,\theta,\phi) = g(\theta,\phi)e^{-kz} \quad . \tag{1}$$

From this we see that the asymptotic radiance hypothesis is equivalent to the statement that *the directional and depth dependence of radiance distributions multiplicatively uncouple at great depths.* That is, the radiance function N may be represented as the product of two functions: The function g gives the shape or directional structure common to all the distributions, and the exponential function gives the depth dependence of the distributions.

Each factor on the right hand side of (1) has special physical significance. The function g eventually defines the angular form of the characteristic diffuse light. The exponent k of the exponential function has the following interesting interpretation: We define the *scalar irradiance* $h(z)$ at depth z as usual by writing:

$$\text{"}h(z)\text{"} \quad \text{for} \quad \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} N(z,\theta,\phi)\sin\theta \, d\theta \, d\phi \quad . \tag{2}$$

The quantity $h(z)$ is a measure of the volume density of radiant energy at depth z. Measurements of $h(z)$ over the years in many hydrosols have shown that $h(z)$ varies essentially in an exponential manner with depth. That is, semi-log plots of $h(z)$ versus depth show an unmistakable trend toward linearity as depth increases. In any event, $h(z)$ may be accurately represented by a formula of the type:

$$h(z) = h(0)\exp\left\{ - \int_0^z k(z')dz' \right\} \quad , \tag{3}$$

where $k(z)$ is seen to be the logarithmic derivative of $h(z)$. As depth increases, the experimental evidence is that $k(z)$ approaches a constant value. Let us denote this limit value by "k_∞". Now assuming that an asymptotic radiance distribution is approached by the radiance distributions in a particular body of water, we see from (1), (2), and (3), that:

$$h(z) = h(z_o)e^{-k_\infty(z-z_o)} =$$

$$= e^{-kz} \int_{\phi=0}^{2\pi} \int_{\phi=0}^{\pi} g(\theta,\phi)\sin\theta \, d\theta \, d\phi \quad , \tag{4}$$

where z_o is the depth below which we may assume that $k(z) = k_\infty$. From this we conclude that:

$$k = k_\infty \tag{5}$$

Hence under the above assumption (1), we see that at great depths in the water, the size of the radiance distribution plots decrease exponentially with increasing depth, and the rate of this decrease is precisely that of the scalar irradiance (or radiant energy density).

The close connection between the depth dependence of scalar irradiance and that of the radiance distributions, as summarized in (5), suggests the following mode of representation of the radiance distributions for any depth: For each direction (θ,ϕ), we write as in (35) of Sec. 9.2:

$$"K(z,\theta,\phi)" \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} \quad . \tag{6}$$

Then, in analogy to (3), $N(z,\theta,\phi)$ at any depth z may be represented exactly by:

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^z K(z',\theta,\phi)\,dz'\right\} \quad . \tag{7}$$

Now suppose there is some depth z_o below which we have $K(z,\theta,\phi) = k_\infty$ for all directions (θ,ϕ). Then (7) may be written:

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^{z_o} K(z',\theta,\phi)\,dz' - \int_{z_o}^{z} K(z',\theta,\phi)\,dz'\right\}$$

$$= N(z_o,\theta,\phi)\exp\left\{-k_\infty(z - z_o)\right\}$$

If we write:

$$"g(\theta,\phi)" \quad \text{for} \quad N(z_o,\theta,\phi)\exp\left\{k_\infty z_o\right\} \quad ,$$

then we may go on to write:

$$N(z,\theta,\phi) = g(\theta,\phi)^{-k_\infty z} \quad , \qquad \text{Errata} \tag{8}$$

for all depths z below z_o.

The similarity between (1) and (8) is unmistakable. This similarity points out the present method of attack we may follow in order to prove the asymptotic radiance hypothesis: We must show that the quantitites $K(z,\theta,\phi)$ approach a limit as depth is increased, and that this limit is independent of the directions (θ,ϕ). Furthermore, this limit, in accordance with the preceding discussion, should be none other than the limit k_∞ of $k(z)$, as defined in (3). Henceforth, we explicitly assume that k_∞, defined as the limit of $k(z)$ as $z\to\infty$, exists as a nonnegative number.

The Proof

We make use of the steady state source-free equation of transfer for radiance:

$$\frac{dN(z,\theta,\phi)}{dr} = -\alpha N(z,\theta,\phi) + N_*(z,\theta,\phi) \tag{9}$$

where, as usual:

$$N_*(z,\theta,\phi) = \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} N(z,\theta',\phi')\sigma(\theta',\phi';\theta,\phi)\sin\theta' d\theta' d\phi' \quad (10)$$

represents the path function N_* ; σ is the volume scattering function (which governs the law of scattering in the water), and α is the volume attenuation coefficient. The formal solution of (9) is readily obtained and is simply the integral form of the equation of transfer (re: (6) of Sec. 3.13):

$$N(z,\theta,\phi) = N^o(z,\theta,\phi) + \int_0^r N_*(z',\theta,\phi)e^{-\alpha(r-r')}dr' \ . \quad (11)$$

The first term is the residual radiance which represents the component of N consisting of unscattered light. The second term is the path radiance which represents the space light over the path of length r, (Fig. 10.15). This path radiance has been generated by light scattered into the path of sight all along its extent. The formal solution (11) has been written for a general downward direction of flow of light (see Fig. 10.15), so that $N^o(z,\theta,\phi)$ is interpreted as the residual radiance transmitted from the upper boundary of the medium and is of the form:

$$N^o(z,\theta,\phi) = N^o(0,\theta,\phi)e^{-\alpha r}$$

where

$$-r\cos\theta = z \quad .$$

We now turn Equation (11) into a useful inequality by means of the following three steps:

First, since $N(z,\theta,\phi)$ clearly exceeds its path radiance component at all depths, we can write:

$$N(z,\theta,\phi) > \int_0^r N_*(z',\theta,\phi)e^{-\alpha(r-r')}\,dr' \quad .$$

Second, using the definition of N_*, we strengthen the inequality when we write:

$$N(z,\theta,\phi) > \sigma_{min} \int_0^r h(z')e^{-\alpha(r-r')}\,dr'$$

where σ_{min} is the minimum value of the volume scattering function; that is, we have used (10) to deduce that

$$N_*(z,\theta,\phi) > \sigma_{min} \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} N(z,\theta',\phi')\sin\theta' d\theta' d\theta' = \sigma_{min}h(z)$$

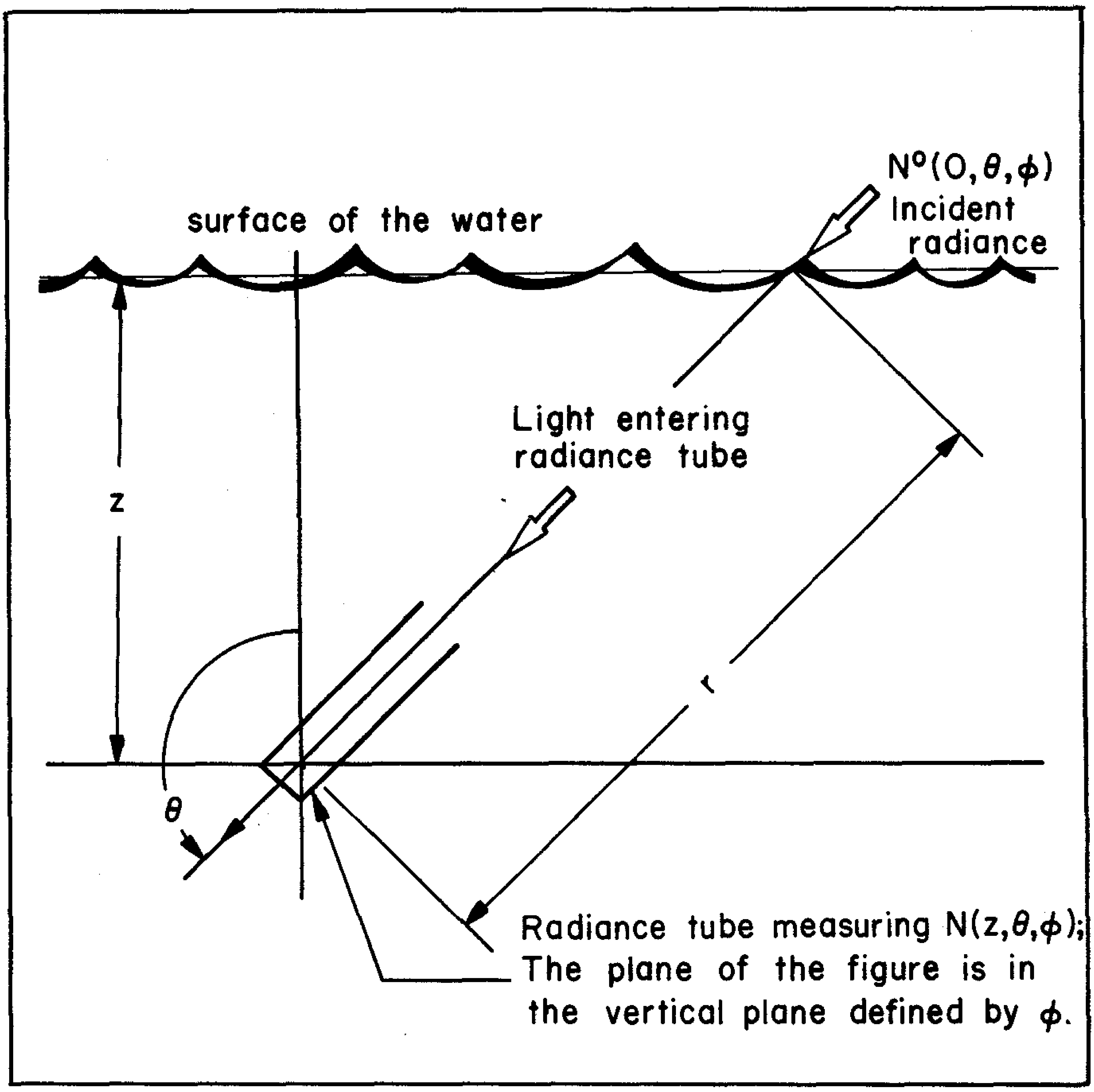

FIG. 10.15 Setting for a simplified proof of the asymptotic radiance theorem.

Finally, since h(z) generally decreases with increasing depth (i.e., k_∞ is positive), we certainly strengthen the inequality by writing:

$$N(z,\theta,\phi) > \sigma_{min} h(z) \int_0^r e^{-\alpha(r-r')} dr' \quad .$$

That is, we have:

$$N(z,\theta,\phi) > \frac{\sigma_{min}}{\alpha} h(z)(1 - e^{-\alpha r}) \qquad (12)$$

for all depths z . From this we see that as depth z increases indefinitely, the exponential rate of decrease

$K(z,\theta,\phi)$ of the radiance cannot eventually exceed, and remain larger by any finite amount, the $k(z)$ of the scalar irradiance. For if it did, the plot of N would eventually fall and remain arbitrarily far below that of h. In other words, the ratio $N(z,\theta,\phi)/h(z)$ would go to zero, with increasing z, contrary to (12). The conclusion of this observation may be stated as follows:

$$\lim_{z\to\infty} K(z,\theta,\phi) \leq \lim_{z\to\infty} k(z) = k_\infty \qquad (13)$$

for all downward directions (θ,ϕ). We now show that strict equality must hold in (13). We achieve this by initially assuming the contrary, that is, we assume that there is a set of directions Ξ_o with positive solid angle measure over which:

$$\lim_{z\to\infty} K(z,\theta,\phi) \leq k_\infty - \varepsilon$$

where ε is an arbitrary small positive number. Then it is clear that the radiances in this set of directions decreases at a definitely smaller rate than the scalar irradiance, so much smaller, in fact that, by our assumption, it is true that for some depth z_1, we must have

$$\int_{\Xi_o} N(z_1,\theta,\phi)\sin\theta\, d\theta\, d\phi > h(z_1) \quad .$$

However, this conclusion clearly contradicts (2) (a part cannot exceed the whole). We have reached a contradiction which leaves only one other possibility, namely that:

$$\lim_{z\to\infty} K(z,\theta,\phi) = k_\infty \qquad (14)$$

for all downward directions (θ,ϕ). In the light of the preceding discussions (cf. (8)) this means that the shapes of the radiance distributions impinging on the upper boundaries of deep layers of water eventually assume a fixed form. But it is known that the shape of the *reflected* radiance distribution at the upper boundary of a scattering layer is determined by the shape of the incident radiance distribution at that boundary (e.g., principle of invariance III of Example 3 of Sec. 3.7, with $N_+(b) = 0$ in the medium $X(0,\infty)$). Hence if the incident radiance distribution approaches a fixed shape, so does that of the reflected distribution. This completes the proof. Errata

We observe that the present proof can also be applied in all natural waters which eventually become homogeneous. That is, the preceding arguments are basically unchanged if the medium is inhomogeneous over any initial finite depth range below the surface. Even more general situations exist which allow asymptotic radiance distributions, namely media in which the ratio σ/α eventually becomes independent of depth (Sec. 10.5).

The Equation for the Characteristic Diffuse Light

Using the equation of transfer, the definition (6), and the relation between z and r, we can write the equation of transfer in the following canonical form (Chapter 4):

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha + K(z,\theta,\phi)\cos\theta} \quad . \tag{15}$$

From (14) and (8) we see that the limiting form of (15) (as depth increases indefinitely) is

$$\boxed{g(\theta,\phi) = \frac{\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi} g(\theta',\phi')\sigma(\theta',\phi';\theta,\phi)\sin\theta'\,d\theta'\,d\phi'}{\alpha + k_\infty\cos\theta}} \tag{16}$$

which is the equation governing the angular form of the characteristic diffuse light (cf. (43) of Sec. 10.5). It is a property of equations of the type shown in (16) that the function g is independent of ϕ for all real physical situations. Thus the characteristic diffuse light is always represented by a surface of revolution whose axis of symmetry is vertical.

The theory of the solution of such equations as (16) is fairly well understood (see e.g., [62]). The present discussions, therefore, will not consider in any detail the general solutions of (16). However, there is one simple special case which is immediately solved and which can shed much light on some of the salient details of the structure of the asymptotic radiance distributions. This is the case of isotropic scattering, where the volume scattering function σ is independent of direction and has the form:

$$\sigma(\theta,\phi;\theta',\phi') = \frac{s}{4\pi} \tag{17}$$

where s is the total scattering coefficient.

To see the resulting structure of the asymptotic radiance distribution, it is convenient in the present case to turn to (15). With the assumption (17) and the definitions (2) and (10), we have

$$N(z,\theta,\phi) = \frac{s}{4\pi}\,\frac{h(z)}{\alpha + K(z,\theta,\phi)\cos\theta} \quad ,$$

which at great depths approaches the form:

$$N(z,\theta,\phi) = \left(\frac{1}{4\pi}\right)\frac{s}{\alpha}\cdot\frac{h(z_o)e^{-k_\infty(z-z_o)}}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta} \quad . \tag{18}$$

Here z_o is the depth below which $h(z)$ is essentially of exponential behavior. Comparing (18) with (8), we see that for the present case,

$$g(\theta,\phi) = \frac{1}{4\pi} \cdot \left(\frac{s}{\alpha}\right) \cdot \frac{h(z_o)e^{k_\infty z_o}}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta} . \qquad (19)$$

We have written (19) in the indicated form to point up the following geometric fact: A polar plot of $g(\theta,\phi)$ is generally a prolate ellipsoid of revolution with vertical axis, and of eccentricity k_∞/α. It is easy to deduce that when there is no absorption in the medium, then $k_\infty = 0$, and the characteristic diffuse light is represented by a sphere. On the other hand, if there is very little scattering as compared to absorption, the figure assumes a very narrow, pencil-like shape. In the limit of no scattering, k_∞ approaches α, and the figure degenerates into a vertical line.

The structure (18) is related to the limiting form for the simple canonical model for the apparent radiance (2) and (6) of Sec. 4.4 and is also related to a formula derived by Poole in Ref. [209]. We conclude with the observation that (19) predicts a different limiting ratio of the horizontal to the upward radiance than that derived by Whitney [316] under the same circumstances (i.e., isotropic scattering). Instead of the ratio 2:1, as suggested by Whitney, the present formula yields:

$$\frac{g(\pi/2,\phi)}{g(0,\theta)} = 1 + \left(\frac{k_\infty}{\alpha}\right) \leq 2 . \qquad (20)$$

In other words, the ratio in (20) is not a fixed magnitude, but depends on the optical properties of the medium in the manner shown.

The distribution (19) can serve as a convenient standard reference distribution against which experimentally determined radiance distribution can be compared. The amount of departure of the experimental distributions from (19) would then serve as a measure of the anisotropy of scattering in the real medium.

<< Table of Contents << Title Page >> Next Page >>

10.7 Some Practical Consequences of the Asymptotic Radiance Hypothesis

We shall now deduce some of the consequences of the asymptotic radiance hypothesis, as stated and proved in Secs. 10.5 and 10.6, for the case of the principal apparent optical properties of natural hydrosols.

It will be recalled that the asymptotic radiance hypothesis asserts that the angular distribution of radiance approaches a fixed form at great depths in eventually homogeneous natural waters. We shall show below that the