Here z is the depth below which $h(z)$ is essentially of exponential behavior. Comparing (18) with (8), we see that for the present case,

$$g(\theta,\phi) = \frac{1}{4\pi} \cdot \left(\frac{s}{\alpha}\right) \cdot \frac{h(z_o)e^{k_\infty z_o}}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta} \quad . \tag{19}$$

We have written (19) in the indicated form to point up the following geometric fact: A polar plot of $g(\theta,\phi)$ is generally a prolate ellipsoid of revolution with vertical axis, and of eccentricity k_∞/α. It is easy to deduce that when there is no absorption in the medium, then $k_\infty = 0$, and the characteristic diffuse light is represented by a sphere. On the other hand, if there is very little scattering as compared to absorption, the figure assumes a very narrow, pencil-like shape. In the limit of no scattering, k_∞ approaches α, and the figure degenerates into a vertical line.

The structure (18) is related to the limiting form for the simple canonical model for the apparent radiance (2) and (6) of Sec. 4.4 and is also related to a formula derived by Poole in Ref. [209]. We conclude with the observation that (19) predicts a different limiting ratio of the horizontal to the upward radiance than that derived by Whitney [316] under the same circumstances (i.e., isotropic scattering). Instead of the ratio 2:1, as suggested by Whitney, the present formula yields:

$$\frac{g(\pi/2,\phi)}{g(0,\theta)} = 1 + \left(\frac{k_\infty}{\alpha}\right) \leq 2 \, . \tag{20}$$

In other words, the ratio in (20) is not a fixed magnitude, but depends on the optical properties of the medium in the manner shown.

The distribution (19) can serve as a convenient standard reference distribution against which experimentally determined radiance distribution can be compared. The amount of departure of the experimental distributions from (19) would then serve as a measure of the anisotropy of scattering in the real medium.

10.7 Some Practical Consequences of the Asymptotic Radiance Hypothesis

We shall now deduce some of the consequences of the asymptotic radiance hypothesis, as stated and proved in Secs. 10.5 and 10.6, for the case of the principal apparent optical properties of natural hydrosols.

It will be recalled that the asymptotic radiance hypothesis asserts that the angular distribution of radiance approaches a fixed form at great depths in eventually homogeneous natural waters. We shall show below that the following consequences can be deduced from the asymptotic radiance hypothesis: The logarithmic derivatives (with respect to depth z) of radiance values $N(z,\theta,\phi)$ approach, with increasing z, a common fixed value k_∞ for all directions (θ,ϕ); further, the logarithmic derivatives of scalar irradiance $h(z)$, its upwelling and downwelling components $h(z,+)$ and $h(z,-)$, along with the derivatives of the upwelling and downwelling irradiances $H(z,+)$ and $H(z,-)$ all approach the common limit k_∞ as depth increases. Further consequences are that the two-D model for the irradiance field in natural waters (Sec. 8.5) becomes exact with increasing depth. These and related results are illustrated by examples drawn from the special case of isotropic scattering. Finally, a formula is developed which allows an estimate of the depth at and below which the actual radiance distributions differ from the asymptotic distribution by no more than a preassigned amount. Thus, the formula may constitute a criterion for asymptoticity in natural hydrosols.

In order to keep the present discussion essentially self contained and useful for references purposes, we shall review the definitions and properties of the various concepts to which the asymptotic radiance hypothesis will be applied.

Basic Formulas: The Irradiance Quartet

As is demonstrated in Chapter 2, the radiance function N is a basic radiometric quantity in terms of which all others can be defined. In particular the downwelling and upwelling irradiances $H(z,-)$ and $H(z,+)$ at depth z in a natural hydrosol are given by:

$$H(z,-) = - \int_{\Xi_-} N(z,\theta,\phi) \cos\theta \, d\Omega \quad , \qquad (1)$$

and

$$H(z,+) = \int_{\Xi_+} N(z,\theta,\phi) \cos\theta \, d\Omega \quad , \qquad (2)$$

where, as usual, we define Ξ_- as the collection of all downward (or inward) directions (θ,ϕ): $\pi/2 < \theta$, $0 \le \phi < 2\pi$, and Ξ_+ is the collection of all upward (or outward) directions (θ,ϕ): $\theta < \pi/2$, $0 \le \phi < 2\pi$, where θ is measured as usual from the outward normal **k** to the medium. We have thus partitioned the set Ξ of all directions into its *upper* (+) *hemisphere* and its *lower* (-) *hemisphere*. For brevity we have written, " $d\Omega$ " for $\sin\theta \, d\theta \, d\phi$, in (1) and (2).

In addition to $H(z,+)$ and $H(z,-)$, underwater optical experiments usually consider the following downwelling and upwelling scalar irradiances:

$$h(z,-) = \int_{\Xi_-} N(z,\theta,\phi)\,d\Omega \quad , \tag{3}$$

and

$$h(z,+) = \int_{\Xi_+} N(z,\theta,\phi)\,d\Omega \quad , \tag{4}$$

and their sum $h(z)$:

$$h(z) = h(z,-) + H(z,+) \quad , \tag{5}$$

which is the scalar irradiance at depth z ($h(z)$ is equal to the product of the speed of light v and radiant density $u(z)$ at depth z). These concepts were introduced and discussed in detail in Chapter 2.

The four quantities $H(z,\pm)$, $h(z,\pm)$, so useful in the formulation of the two-flow theory of Chapter 8, also form the nucleus of a set of modern *experimental quantities* used to document the light field in natural waters. Of course, a complete documentation is obtained only through a systematic determination of the radiance values $N(z,\theta,\phi)$ at all depths z and over all directions (θ,ϕ). Nevertheless, as is seen in the developments of Chapters 8 and 9, this quartet of irradiances contains an extraordinary amount of useful radiometric information compactly packaged.

The D and R Functions

In the absence of detailed knowledge of $N(z,\theta,\phi)$ the basic quartet of irradiance functions defined above can be used to derive most of the information needed for the solution of underwater visibility problems, and image and flux transmission problems in general. In particular, an excellent index of the shape of the radiance distributions at depth z is given by the distribution functions $D(z,\pm)$ represented as:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} \quad . \tag{6}$$

Furthermore, information about the reflectance properties of the water at depth z is furnished by a study of the ratio:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad , \tag{7}$$

which is the experimental counterpart to the classical R_∞ formula as given by classical one-dimensional two-flow analysis of the light field. In fact, the D and R functions defined above, and the functions defined below are all either modern experimental counterparts or logical extensions of the tools provided by the classical two-flow theory of the light field in natural hydrosols. As noted above, the background of these particular radiometric quantities is considered in detail in Chapters 8 and 9 so that the present discussion need not dwell further on their definitions and interrelations. We are concerned here only with the behavior of these quantities at great depths in media satisfying the requirement of the asymptotic radiance hypothesis.

The K Functions

The essentially exponential behavior of the irradiance quantities supplies the motivation for the definitions of the K functions (Chapter 9) assembled here for convenience:

$$K(z,\pm) = -\frac{1}{H(z,\pm)}\frac{dH(z,\pm)}{dz} \quad , \tag{8}$$

$$k(z,\pm) = -\frac{1}{h(z,\pm)}\frac{dh(z,\pm)}{dz} \quad , \tag{9}$$

$$k(z) = -\frac{1}{h(z)}\frac{dh(z)}{dz} \quad . \tag{10}$$

If the various irradiance quantities vary *exactly* in an exponential manner at *all* depths, then the corresponding K functions would be constant functions each assuming a fixed value at all depths. In general, however, (Secs, 10.1-10.4) the depth-dependence of these quantities is nonconstant and it is only after 10-20 attenuation lengths that the exponential features eventually emerge. The preceding representations, however, are designed to characterize the depth-dependence of the irradiances under all conditions.

One of the main consequences derived from the asumptotic radiance hypothesis is that the five K functions represented above all tend to a common limit with increasing depth. We prepare the groundwork leading to this conclusion by reintroducing the K function for the radiance function itself (re (34) of Sec. 9.2): Thus we write:

$$\text{"}K(z,\theta,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} \quad . \tag{11}$$

Just as each of the various irradiance quantitites may be expressed in terms of radiance, so can its corresponding K function be expressed in terms of the K function for radiance:

$$K(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)K(z,\theta,\phi)\cos\theta\, d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi)\cos\theta\, d\Omega} \quad , \tag{12}$$

$$k(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)K(z,\theta,\phi)\,d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi)\,d\Omega} , \qquad (13)$$

$$k(z) = \frac{\int_{\Xi} N(z,\theta,\phi)K(z,\theta,\phi)\,d\Omega}{\int_{\Xi} N(z,\theta,\phi)\,d\Omega} . \qquad (14)$$

The K Characterization of the Hypothesis

The K function $K(z,\theta,\phi)$ for radiance is of fundamental importance in the present discussion of the asymptotic radiance hypothesis. In fact it is the function which gives rise to that form of the hypothesis which is most amenable to exact mathematical analysis. The desired characterization reads as follows ((16) of Sec. 10.5) *for each* $(\theta,\phi)\varepsilon\,\Xi$, *the function* $K(z,\theta,\phi)$ *has a limit, as* $z\to\infty$, *and this limit is independent of* (θ,ϕ). In symbols:

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

exists for every $(\theta,\phi)\varepsilon\,\Xi$, *and is independent of* (θ,ϕ).

The preceding formulation,as explained in Secs. 10.5 and 10.6, is made plausible by the following observations: For every depth z, $N(z,\theta,\phi)$ may be represented exactly by

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^z K(z',\theta,\phi)\,dz'\right\} .$$

Suppose there is some depth z_o below which we have $K(z,\theta,\phi) = k_\infty$, a fixed number for all (θ,ϕ). Then

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^{z_o} K(z',\theta,\phi)\,dz' - \int_{z_o}^{z} K(z',\theta,\phi)\,dz'\right\}$$

$$= N(z_o,\theta,\phi)\exp\left\{-k_\infty(z - z_o)\right\} .$$

Write

$$\text{"}g(z_o,\theta,\phi)\text{"} \quad \text{for} \quad N(z_o,\theta,\phi)\exp\left\{k_\infty z_o\right\} ,$$

then for all $z \geq z_o$,

$$N(z,\theta,\phi) = g(z_o,\theta,\phi)e^{-k_\infty z} \quad . \tag{15}$$

It follows that below z_o , $N(z,\theta,\phi)$ has a fixed angular structure given by $g(z_o,\theta,\phi)$.

The Basic Transfer Equations

One final relation needed below is the reformulation of the equation of transfer in terms of the K function for radiance. This is easily obtained from the standard form of the transfer equation for stratified source-free plane-parallel media:

$$-\cos\theta \frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad ,$$

where:

$$N_*(z,\theta,\phi) = \int_{\Xi} N(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)d\Omega \quad .$$

By means of the definition of $K(z,\theta,\phi)$, the above equation may be rewritten in its canonical form (Chapter 4):

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha(z) + K(z,\theta,\phi)\cos\theta} \quad , \tag{16}$$

The equation of transfer governing $K(z,\theta,\phi)$ is also easily found. From (16), the definition of $K(z,\theta,\phi)$, and the following definition of an analogous K function:

$$K_q(z,\theta,\phi) = -\frac{1}{N_q(z,\theta,\phi)} \frac{dN_q(z,\theta,\phi)}{dz} \quad , \tag{17}$$

where

$$N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z) \quad , \tag{18}$$

we have:

$$\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - K_q(z,\theta,\phi)][K(z,\theta,\phi) + \alpha(z)\sec\theta] \tag{19}$$

and which played a key role in the proof arguments of Sec. 10.5.

This formulation is analogous to the following formulation of the equation of transfer for $N(z,\theta,\phi)$ in which $N_q(z,\theta,\phi)$ is used:

$$\frac{dN(z,\theta,\phi)}{dz} = [N(z,\theta,\phi) - N_q(z,\theta,\phi)][+ \alpha(z)\sec\theta)] \quad .$$

These formulations point up the following physical significance of the equilibrium radiance N_q and its K function K_q: for $\theta > (\pi/2)$ we observe that if $N(z,\theta,\phi) \lesseqgtr N_q(z,\theta,\phi)$ then $dN(z,\theta,\phi)/dz \gtreqless 0$. This follows immediately from the preceding equation. Thus N, $\theta > \pi/2$, always tends toward the equilibrium radiance N_q. Now a similar phenomenon exists between K and K_q. To see this, we observe that the second factor on the right in (19) has the property that:

$$K(z,\theta,\phi) + \alpha(z)\sec\theta < 0 \quad ,$$

for all z and all downward directions (θ,ϕ). Therefore if $K(z,\theta,\phi) \lesseqgtr K_q(z,\theta,\phi)$ then $dK(z,\theta,\phi)/dz \gtreqless 0$, showing that K always tends toward K_q for these directions. This property of the function $K(z,\theta,\phi)$ provided the key to the rigorous proof of the existence of an asymptotic radiance distribution given in Sec. 10.5. A further example of such a use of (19) is given in the final sections below.

Consequences for Directly Observable Quantities: The Equation for the Asymptotic Radiance Distribution

An application of the asymptotic radiance hypothesis to (16) yields the formula for the asymptotic radiance distribution g. In view of the heuristic discussion leading to (15) and the statement of the hypothesis in terms of $K(z,\theta,\phi)$, we see that

$$\lim_{z\to\infty} g(z,\theta,\phi) = \lim_{z\to\infty} N(z,\theta,\phi)\exp\left\{k_\infty z\right\}$$

exists for all (θ,ϕ). We shall denote this limit by "$g(\theta,\phi)$". Hence multiplying each side of (16) by $\exp\{k_\infty z\}$ and passing to the limit as $z\to\infty$, we have:

$$g(\theta,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} g(\theta',\phi')p(\theta',\phi';\theta,\phi)\,d\Omega}{1 + \left(\frac{k_\infty}{\alpha}\right)\cos\theta} \qquad (20)$$

where:

$$p(\theta',\phi';\theta,\phi) = \lim_{z\to\infty} 4\pi\sigma(z;\theta',\phi';\theta,\phi)/\alpha(z)$$

and:

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

and*

$$\alpha = \lim_{z\to\infty}\alpha(z) \quad .$$

The integral equation (20) has the property that the values of its solution g are independent of θ. Thus we may write:

$$"g_o(\theta)" \quad \text{for} \quad 2\pi g(\theta,\phi)$$

and (20) may be simplified to read:

$$\boxed{g_o(\theta) = \frac{\frac{1}{2}\int_{\theta'=0}^{\pi} g_o(\theta')p^{(o)}(\theta';\theta)\sin\theta'\,d\theta'}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta}} \quad , \quad (21)$$

where we have written:

$$"p^{(o)}(\theta';\theta)" \quad \text{for} \quad \frac{1}{2\pi}\int_{\theta'=0}^{2\pi} p(\theta',\phi';\theta,\phi)\,d\phi' \quad .$$

The function g_o describes the essential geometric form of the asymptotic radiance distribution. A graph of g_o is clearly a surface of revolution with vertical axis (in the coordinate system of the plane-parallel medium). Furthermore, the structure of g_o and the value of k_∞ are completely determined by the phase function p (k_∞/α plays the role of an eigenvalue of the equation (21)). Thus g_o is determined completely by the inherent optical properties of the medium by means of equation (21) and therefore is independent of the external lighting conditions. The extension of (21) to the polarized case is immediate, upon starting the preceding derivation with the canonical equation of transfer (8) of Sec. 4.6 for the standard observable radiance vector.

The Limits of the K Functions

From the relations (12)-(14), and the statement of the hypothesis, we conclude that:

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*For most practical situations, the medium is homogeneous or eventually homogeneous, so that this limit exists. Actually, as shown in Sec. 10.5., the asymptotic radiance distribution exists whenever $\lim_{z\to\infty}\sigma/\alpha$ exists, without necessarily requiring that the individual limits $\lim_{z\to\infty}\sigma$ and $\lim_{z\to\infty}\alpha$ exist.

$$\lim_{z\to\infty} K(z,\pm) = k_\infty \quad , \tag{22}$$

$$\lim_{z\to\infty} k(z,\pm) = k_\infty \quad , \tag{23}$$

$$\lim_{z\to\infty} k(z) = k_\infty \quad . \tag{24}$$

The limit (24) is interpreted as follows: The logarithmic derivative (with respect to z) of scalar irradiance $h(z)$ eventually approaches the common limit k of the logarithmic derivatives of radiance distribution $N(z,\theta,\phi)$. The limit (23) shows that the logarithmic derivatives of the upwelling and downwelling irradiances (which as we saw in Sec. 10.2, are measurably distinct at all small depths z) approach a common value, namely k_∞. A similar interpretation holds for the K functions of the upwelling and downwelling scalar irradiances.

The Limits of the D and R Functions

From (6) and the hypothesis, we have immediately:

$$\lim_{z\to\infty} D(z,\pm) = \frac{\int_\pm g_o(\theta)\sin\theta\, d\theta}{\int_\pm g_o(\theta)[\pm\cos\theta]\sin\theta\, d\theta} \quad , \tag{25}$$

which we shall denote by "$D(\pm)$". Furthermore in (25) we have written:

$$\text{"}\int_+\text{" for } \int_{\theta=0}^{\pi/2} \quad , \quad \text{and} \quad \text{"}\int_-\text{" for } \int_{\theta=\pi/2}^{\pi} \quad .$$

In (25) of Sec. 9.2 it was shown that $R(z,-)$ can be represented quite generally in terms of the K functions and the distribution functions as follows:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)} \quad ,$$

where:

$$a(z,\pm) = D(z,\pm)a(z) \quad ,$$

and where $a(z)$ is the value of the volume absorption function of the medium at depth z. Let us write:

$$\text{"}R_\infty\text{" for } \lim_{z\to\infty} R(z,-)$$

It follows from the preceding representation for $R(z,-)$ that R_∞ exists and is given by:

$$R_\infty = \frac{k_\infty - a(-)}{k_\infty + a(+)} \quad , \tag{26}$$

where:

$$\lim_{z\to\infty} a(z,\pm) = \lim_{z\to\infty} D(z,\pm)a(z) = D(\pm)a \quad .$$

and where we have written:

$$"a(\pm)" \quad \text{for} \quad D(\pm)a \quad .$$

Further limit relations may be determined by systematically going through the set of directly observable quantities discussed in Chapter 9. The preceding results will serve to illustrate the general procedure of obtaining the desired limit expressions.

We observe that (26) is similar to the classical expression for R_∞ as given by the two-D model for the irradiance field ((102) of Sec. 8.7). This similarity is not coincidental; it is, rather a consequence of the fact that under the asymptotic radiance hypothesis, the general two-D model becomes exact with increasing depth. We now consider this fact in more detail.

Consequences for Some Simple Theoretical Models: The Two-D Model for Irradiance Fields

In Chapter 8 a study of the classical two-flow equations for $H(z,+)$ and $H(z.-)$ showed that these equations were exact if and only if the distribution functions $D(z,+)$ and $D(z,-)$ were independent of depth (Sec. 8.5). Under the asymptotic radiance hypothesis it was seen in (25) that the distribution functions become independent of depth at great depths. It follows that the two-D equations for undecomposed irradiance $H(z,+)$ and $H(z,-)$ become exact at great depths whenever the hypothesis holds.

In Chapter 8 a formulation of the equations for $H^*(z,+)$ and $H^*(z,-)$ (the decomposed irradiances associated with diffuse light) was made in which each stream of flux was assigned a fixed distribution factor $D^*(+)$, $D^*(-)$ (the two-D theory for decomposed irradiance). This formulation was justified on the basis of experimental evidence which showed that $D(z,+)$ and $D(z,-)$ were essentially fixed (generally distinct) constants. In the light of the present analysis, the use of the two-D theory for decomposed irradiance is also given further justification on theoretical grounds whenever the asymptotic radiance hypothesis holds.

The two-D model gives explicit formulas for $H^*(z,+)$ and $H^*(z,-)$. In view of the preceding observations, these expressions become exact with increasing depth z. Using the equations developed in Sec. 8.6 it may be shown that for every depth z in an infinitely deep medium $X(0,\infty)$:

$$H^*(z,-) = N^\circ C(\mu_o,-)\left[e^{-k_\infty z} - e^{-\alpha z/\mu_o}\right] \quad , \tag{27}$$

$$H^*(z,+) = N^\circ \left[C(\mu_o,-)\,\frac{g_-(+)}{g_+(-)}\,e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}\right] \quad . \tag{28}$$

Observe that we have set $k_\infty = -k_-$, where k_- is given in (12) of Sec. 8.5. The physical setting associated with (27) and (28) is an infinitely deep plane-parallel slab irradiated by collimated flux incident at the upper boundary at an angle $\theta_o = \arccos \mu_o$ from the outward unit normal . The response to an arbitrary incident distribution is obtained by integrating (27) and (28) over Ξ_- (see Sec. 10.3). $C(\mu_o,\pm)$ are constants for each μ_o, determined by the optical parameters and boundary conditions; and:

$$g_-(\pm) = 1 \mp \frac{a(\mp)}{k_\infty} \quad ,$$

where $a(\pm)$ are as defined in (26). The observable irradiances $H(z,\pm)$ are, by definition,

$$H(z,\pm) = H^\circ(z,\pm) + H^*(z,\pm) \quad ,$$

where:

$$H^\circ(z,+) = 0 \quad ,$$

$$H^\circ(z,-) = N^\circ \mu_o e^{-\alpha z/\mu_o} \quad .$$

The expressions for the observable irradiances are given in (1) and (2) of Sec. 10.3. The preceding model yields the following prediction of the limit of $R(z,-)$:

$$R_\infty = \lim_{z\to\infty} R(z,-) = \lim_{z\to\infty} \frac{H(z,+)}{H(z,-)} = \frac{g_-(+)}{g_-(-)} = \frac{k_\infty - a(-)}{k_\infty + a(+)} \quad ,$$

which agrees with (26), the exact limit given by general radiative transfer theory. These observations show that in any medium in which the asymptotic radiance hypothesis holds, if discussions are restricted to the class of all possible *two-flow* models of the light field, the model which attains maximal accuracy is that given by the two-D theory.

Critique of Whitney's "General Law"

After conjecturing that the radiance distributions assume a fixed shape at great depths, L. V. Whitney made use of the conjecture to deduce a so-called "general law of the diminution of light intensity in natural waters" (cf. ref. [316]). An examination of the differential equations formulating this law reveals that they are incomplete: They fail to account for the contribution to the downwelling irradiance by the backscattered fraction of the upwelling irradiance. As a result, the solutions of the differential equations are generally inadequate to cope with the contribution from one half of the light field, namely the component associated with the upwelling flux. Furthermore, some (convenient, but incorrect) assumptions were made about the depth rate of change of the mean free path for unscattered light at various depths. On this basis the equations were integrated, holding fixed the mean free path for directly transmitted light. Both of these inadequacies of an otherwise satisfactory theory have been remedied in the two-D theory of the light field. The equations (27) and (28) (or their observable counterparts (1) and (2) of Sec. 10.3) represent the concomitant effects of both upwelling and downwelling streams. Finally, the awkwardness stemming from the change with depth of the mean free path of directly transmitted light has been avoided by considering only collimated incident flux of radiance N° at the upper boundary.

The Simple Model for Radiance Distributions

In (2) of Sec. 4.4 a simple model for radiance distributions was derived in the form of the canonical representation of the apparent radiance. One key assumption in the establishment of the model was the depth-independence of the K function for radiance. In view of the preceding observations, it is concluded that the simple canonical model becomes exact with increasing depth in all media in which the asymptotic radiance hypothesis holds.

Further Consequences of Asymptoticity

We conclude with some examples drawn from the case of a plane-parallel medium which exhibits isotropic scattering and in which the asymptotic radiance hypothesis holds. In this way we obtain some general ideas about the shape of g_0, as governed by (21), and the order of magnitudes of the quantities $D(\pm)$, R_∞, and k_∞ one may expect in *real* media. Finally, it is possible to give, in the present context, a simple heuristic proof of the hypothesis, and at the same time derive a formula which will provide a means of determining the depth in a medium below which asymptoticity has essentially been attained. We shall now consider these matters in turn.

The Standard Ellipsoid

When scattering is isotropic, the phase function takes on the form:

$$p(\theta',\phi';\theta,\phi) = \rho = s/\alpha \quad ,$$

where s is the volume total scattering coefficient and "ρ" is an abbreviation of "s/α", and both denote the scattering-attenuation ratio. Using this phase function in (21) we see that $g_o(\theta)$ takes on a particularly simple form,

$$g_o(\theta) = \frac{\rho}{2} \cdot \frac{g_o}{1 + \varepsilon \cos\theta} \tag{29}$$

where we have written

$$\text{"}\varepsilon\text{"} \quad \text{for} \quad k_\infty/\alpha \quad ,$$

and

$$\text{"}g_o\text{"} \quad \text{for} \quad \int_{\theta=0}^{\pi} g_o(\theta)\sin\theta\, d\theta \quad .$$

Physical significance can be attached to g_o by returning to the definition of $g(\theta,\phi)$ and integrating over Ξ (see (15) and (20)). The result is:

$$g_o = \lim_{z\to\infty} h(z)e^{k_\infty z} \quad .$$

Hence if there is some depth z_o below which one may consider that for practical purposes asymptoticity has been attained, then the preceding relation can be written:

$$g_o = h(z_o)e^{k_\infty z_o} \quad .$$

Expression (29) represents a prolate spheroid of revolution whose axis of symmetry is vertical. The eccentricity of the ellipsoid is $\varepsilon = k_\infty/\alpha$. This ellipsoid may serve as a convenient reference against which distributions from real media may be compared. To effect a comparison one must know the ρ and ε of the medium. Since ε and ρ are generally related, it suffices in principle to know only ρ and the phase function. This is illustrated below after a necessary preliminary discussion of $D(\pm)$ and R_∞.

Expressions for $D(\pm)$ and R_∞

By means of (25) and (29) we find that (see also (57) and (58) of Sec. 2.11):

$$D(\pm) = \frac{\varepsilon \ln(1 \pm \varepsilon)}{\varepsilon \mp \ln(1 \pm \varepsilon)} \quad . \tag{30}$$

Furthermore, from (7) and (29)(i.e., (29) replaces $N(z,\theta,\phi)$ in (7)), we have:

$$R_\infty = \frac{\ln(1 + \varepsilon) - \varepsilon}{\ln(1 - \varepsilon) + \varepsilon} \quad . \tag{31}$$

The same result could be obtained by using (26) and the preceding form for $D(\pm)$.

Values of $D(\pm)$ and R_∞ as functions of ε, $0 < \varepsilon < 1$ are given in Table 1. It is easy to verify that for the extreme values 0 and 1 of ε the corresponding values of $D(\pm)$ and R_∞ are:

$$\lim_{\varepsilon \to 0} D(\pm) = 2 \quad ;$$

$$\lim_{\varepsilon \to 1} D(+) = \frac{\ln 2}{1 - \ln 2} = 2.259 \quad ;$$

$$\lim_{\varepsilon \to 1} D(-) = 1 \quad .$$

$$\lim_{\varepsilon \to 0} R_\infty = 1 \quad ,$$

$$\lim_{\varepsilon \to 1} R_\infty = 0 \quad .$$

Table 1 of Sec. 8.5 gives values of $D(z,\pm)$ for a real medium under varying external conditions. A comparison of these real values with those summarized in Table 1 below reveals the following information: The $D(z,-)$ values are significantly less than the standard $D(-)$ values; the $D(z,+)$ values are significantly greater than the standard $D(+)$ values. Since all natural waters exhibit anisotropic scattering we can infer the following features of the structure of asymptotic radiance distributions in all natural waters: When compared with the standard ellipsoid, the plots of $g_0(\theta)$ for real media must necessarily be narrower in the angular range $\theta > \pi/2$ (upwelling light).

The amount of departure of the $g_0(\theta)$ for a real medium from the standard ellipsoid may be taken as a measure of the anisotropy of scattering in the medium.

TABLE 1

Distribution and reflectance factors for standard ellipsoid

ε	D(-)	D(+)	R_∞
0.100	1.9664	2.0319	0.8750
0.200	1.9286	2.0622	0.7640
0.300	1.8881	2.0911	0.6642
0.400	1.8438	2.1185	0.5733
0.500	1.7943	2.1143	0.4895
0.600	1.7381	2.1692	0.4110
0.700	1.6722	2.1928	0.3361
0.800	1.5906	2.2157	0.2622
0.900	1.4775	2.2377	0.1841
0.950	1.3911	2.2483	0.1379

The Determination of ε

The quantity $\varepsilon (= k_\infty/\alpha)$ is functionally related to ρ. In the case of isotropic scattering the relation is well known and of a particularly simple structure (cf. Ref. [43]). In general, ε is determined by viewing it as an eigenvalue of the integral equation (20). There is an alternate way, however, to characterize ε which, while not the most analytically direct way, is perhaps of greatest value in generating an insight into the physical signficance of ε and also of supplying a link between ε and the directly observable quantities of the light in real media. This alternate characterization of ε stems from the following functional relation which holds between $K(z,\pm)$ and the various scattering and absorption function of an arbitrary medium ((31) of Sec. 9.2):

$$1 = \frac{b(z,-)}{K(z,-) - a(z,-)} - \frac{b(z,+)}{K(z,+) + a(z,+)} .$$

As depth is increased each term, as a result of the asymptotic property of the light field, tends toward a well-defined limit, so that as $z \to \infty$, the above relation tends to:

$$1 = \frac{b(-)}{k_\infty - D(-)a} - \frac{b(+)}{k_\infty + D(+)a} .$$

This may be rewritten as:

$$1 = \frac{\beta(-)}{\varepsilon - (1 - \rho)D(-)} - \frac{\beta(+)}{\varepsilon + (1 - \rho)D(+)} , \qquad (32)$$

which is the *general characteristic equation for* ε. Here we have written:

$$"\beta(\pm)" \quad \text{for} \quad \frac{\frac{1}{4\pi}\int_{\Xi_{\mp}}\int_{\Xi_{\pm}} g_o(\theta')p(\theta',\phi';\theta,\phi)d\Omega' \; d\Omega}{\pm\int_{\Xi_{\pm}} g_o(\theta')\cos\theta' \, d\Omega} \quad .$$

In the case of isotropic scattering:

$$\beta(\pm) = \frac{\rho}{2} D(\pm) \quad ,$$

and (32) reduces to the following simple form after the explicit expressions for $D(\pm)$, as given by (30), are substituted in it:

$$\rho = \frac{2\varepsilon}{\ln\left[\frac{1+\varepsilon}{1-\varepsilon}\right]} \quad . \tag{33}$$

This is the well-known characteristic equation for ε in the isotropic case. As ρ varies from 0 to 1, ε varies from 1 to 0. Hence, for all ρ, $0 \le \rho \le 1$; $0 \le \varepsilon \le 1$. Whenever scattering is present, i.e., whenever $\rho^- > 0$, then the useful inequality $k_\infty < \alpha$ holds. Actually, the inequalities $0 \le k_\infty/\alpha \le 1$ hold in general (Sec. 10.6). This fact is made plausible by an inspection of (21) keeping in mind that the function g_o is bounded in all physically meaningful situations, so that the denominator cannot vanish.

An Heurisitc Proof of the Hypothesis Errata

We now present a brief argument which makes plausible the assertion of the hypothesis, namely that $K(z,\theta,\phi) \to k_\infty$ for all (θ,ϕ). For simplicity we will assume that the space is homogeneous and that scattering is isotropic. The resulting line of argument, while restricted to this special setting, can be made completely rigorous. The setting is that depicted in Fig. 10.15.

Under the present assumptions, we see that (18) may be written

$$N_q(z,\theta,\phi) = \frac{1}{4\pi}\rho h(z) \quad ,$$

so that:

$$K_q(z,\theta,\phi) = k(z) \quad .$$

Thus (19) reduces to

$$\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - k(z)][K(z,\theta,\phi) + \alpha \sec \theta] \quad .$$

The preceding discussion of this equation showed that $K(z,\theta,\phi)$ always tends toward $k(z)$ for downward directions. Hence if $k(z)$ approaches a limit, $K(z,\theta,\phi)$ also tends toward this limit. More explicitly, suppose there is some depth z_o below which $k(z)$ is essentially constant and equal to k_∞. Then the preceding equation is a simple Riccati equation for $K(z,\theta,\phi)$ whose general solution is:

$$K(z,\theta,\phi) = \frac{k_\infty + \alpha \sec \theta \; C \exp\{(k_\infty + \alpha \sec \theta) z\}}{1 - C \exp\{(k_\infty + \alpha \sec \theta) z\}} \qquad (34)$$

where:

$$C = \frac{K(0,\theta,\phi) - k_\infty}{K(0,\theta,\phi) + \alpha \sec \theta} \quad .$$

Since:

$$k_\infty + \alpha \sec \theta < 0$$

for all $\theta > \pi/2$, it follows immediately from (34) that

$$\lim_{z\to\infty} K(z,\theta,\phi) = k_\infty$$

for all $\theta > \pi/2$. This means that the shape of the downwelling radiance distribution becomes fixed at great depths. It follows from the principles of invariance that the reflected upwelling radiance distribution also becomes fixed, so that the shape of the entire radiance distribution becomes fixed at great depths.

A Criterion for Asymptoticity

According to (34), $K(z,\theta,\phi)$ approaches k_∞ with *least* speed when $\theta = \pi$ (i.e., for the directly downward direction, as in Fig. 10.15). Hence when $K(z,\pi,\phi)$ has come within a given distance of k_∞, we can conclude that the other values $K(z,\theta,\phi)$, $\pi/2 \le \theta < \pi$ are within the same neighborhood of k_∞. From (34) it follows that

$$K(z,\pi,\phi) - k_\infty = \frac{(k_\infty - \alpha) C \exp (k_\infty - \alpha) z}{1 - C \exp (k_\infty - \alpha) z} \quad . \qquad (35)$$

Thus a preassigned value of the difference on the left side determines an associated value of z. Although (35) is exact only at great depths, and applies only in the present (isotropic) context, it nevertheless supplies a useful approximate method for estimating the depths at which $K(z,\pi,\phi) - k_\infty$ has attained a given small value.

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