10.8 Simple Formulas for the Volume Absorption Coefficient in Asymptotic Light Fields

Introduction

As a result of the work of the preceding sections, it is now a well-established fact that the properties of the light field in optically deep homogeneous stratified media (such as deep cloud layers, oceans, lakes, etc.) become extremely regular and predictable at great depths. The conceptual and practical consequences of this fact were begun to be explored in Sec. 10.7; and the discussion there entered only in the first stages of exploration. In this section we derive some further consequences from this regularity property of deep light fields in the context of natural hydrosols. In particular, we use the regularity of the deep light field to derive several simple, exact, formulas relating the volume absorption coefficient to the common lighting value k_∞ of the K-functions for irradiance. In this way we supplement the exact formula

$$a(z) = \frac{1}{h(z)} \frac{d\overline{H}(z,+)}{dz} \tag{1}$$

for the volume absorption function α (as given in (18) of Sec. 8.8) with several alternate formulas which are especially suitable for engineering calculations and as handy rules of thumb relating a and k_∞. These formulas are as follows:

I. $$\boxed{a = k_\infty \frac{\overline{H}(z,-)}{h(z)} \quad , \; z \geq z_o} \quad ,$$

II. $$\boxed{a = k_\infty \frac{(1 - R_\infty)}{D(-) + R_\infty D(+)}} \quad ,$$

III. $$\boxed{a = \frac{3}{4} k_\infty} \quad ,$$

where (as defined earlier in Sec. 10.7):

k_∞ is the common limit, as $z \to \infty$, of the K-functions $K(z,\pm)$, $k(z)$ for irradiance and scalar irradiance, respectively.

$D(\pm)$ are the limits, as $z \to \infty$, of the distribution functions $D(z,\pm)$.

R_∞ is the limit, as $z \to \infty$, of the reflectance function $R(z,-) = H(z,+)/H(z,-)$.

$\overline{H}(z,-)$ $= H(z,-) - H(z,+)$, the net downward irradiance at any depth $z > z_0$, where z_0 is the depth below which the light field has essentially attained its asymptotic structure.

The details of the derivation of I, II, and II, will now be given.

Short Derivation of I

The short derivation of I starts with (1), and the fact that there exists a depth z_0 below which the logarithmic derivatives of $H(z,-)$, $H(z,+)$, and $h(z)$ are constant and equal to a common value k (see (22), (24), and (35) of Sec. 10.7). Therefore:

$$\frac{d\overline{H}(z,+)}{dz} = \frac{dH(z,+)}{dz} - \frac{dH(z,-)}{dz}$$

$$= - k_\infty H(z,+) + k_\infty H(z,-) = k_\infty \overline{H}(z,-) \quad , \quad (2)$$

for all $z \geq z_0$. Hence:

$$a = k_\infty \frac{\overline{H}(z,-)}{h(z)} \quad .$$

Long Derivation of I

The long derivation of I is essentially an exercise in the use of the integrated form of the divergence relation for the light field vector ((33) of Sec. 8.8)

$$\overline{P}(s,-) = a\, v\, U(M) \quad , \quad (3)$$

where M is *any* regularly or irregularly shaped region of the optical medium, S is its boundary, and $\overline{P}(S,-)$ is the net inward flux across S into M. $U(M)$ is the radiant energy content of M, v is the speed of light in M, and a is the required value of the volume absorption coefficient.

It is interesting to observe that (3) yields a value of a in *any* homogeneous medium, regardless of the structure of the light field:

$$\boxed{a = \frac{\overline{P}(S,-)}{v\, U(M)}} \quad (4)$$

The numerator of (4) can be obtained by traversing the boundary of M with flat plate collectors or other flux-measuring devices. The denominator is obtained by probing the interior of M with a spherical collector (to find h(p) at each point p) and integrating the values over M.

In the present case, the extreme regularity of the asymptotic light field allows one to estimate U(M) *knowing only one value* of the scalar irradiance at a boundary point of M. This fact holds also for $\overline{P}(S,-)$. Specifically, consider a region M in the form of a vertical column of unit cross section, and bounded by two parallel planes at depth z_1, and z_2, such that $z_0 \le z_1 \le z_2$. The medium is homogeneous and stratified; hence:

$$\overline{P}(s,-) = \overline{H}(z_1,-) + \overline{H}(z_2,+) \quad . \tag{5}$$

The net fluxes over the vertical sides of the column cancel by virtue of the stratified light field. By hypothesis, we have:

$$H(z_2,\pm) = H(z_1,\pm)e^{-k_\infty(z_2 - z_1)} \quad , \tag{6}$$

so that:

$$\overline{P}(s,-) = \overline{H}(z_1,-)\left[1 - e^{-k_\infty(z_2 - z_1)}\right] \quad . \tag{7}$$

Furthermore:

$$\begin{aligned} vU(M) &= \int_{z_1}^{z_2} h(z)dz \\ &= h(z_1)\int_{z_1}^{z_2} e^{-k_\infty(z - z_1)}\,dz \\ &= \frac{h(z_1)}{k_\infty}\left[1 - e^{-k_\infty(z_2 - z_1)}\right] \quad . \end{aligned} \tag{8}$$

Inserting (7) and (8) into the general formula (4), we have the desired result

$$a = k_\infty \frac{\overline{H}(z_1,-)}{h(z_1)} \quad , \quad z_1 \ge z_0 \quad .$$

Derivation of II

The formula II can be obtained directly from I by recalling that:

$$\overline{H}(z,-) = H(z,-) - H(z,+) \quad ,$$

$$h(z) = h(z,-) + h(z,+) \quad ,$$

and invoking the definitions of $R(z,-)$, and $D(z,-)$. That is, in general:

$$\overline{H}(z,-) = H(z,-) - R(z,-)H(z,-) = H(z,-)[1 - R(z,-)]$$

and

$$h(z) = D(z,-)H(z,-) + D(z,+)H(z,+) \quad ;$$

so that when $z \geq z_o$, we have:

$$a = k_\infty \frac{(1 - R_\infty)}{D(-) + R_\infty D(+)} \quad ,$$

which is the desired alternate formula. We observe in passing that II is a limiting form of the exact formula:

$$a(z) = \frac{K(z,-) - R(z,-)K(z,+)}{D(z,-) + R(z,-)D(z,+)} \quad , \tag{9}$$

the basis for which is (25) of 9.2. Clearly, as $z \to \infty$, equation (9) takes the limiting form II. Furthermore if we assume $D(\pm) = 2$, as is done in the classical one-D two-flow theory of the light field, then II reduces to the relation:

$$a = \frac{k_\infty}{2} \cdot \frac{1 - R_\infty}{1 + R_\infty} \quad . \tag{10}$$

Applied Numerology: A Rule of Thumb

Formula III is to be taken as a convenient rule of thumb, and as such, is subject to possible revision whenever specific optical media are under study. Yet for many purposes it is quite adequate, a fact which is based on the following observed regularities in the values of R_∞ and $D(\pm)$ in natural waters: R_∞ is usually found to be in the neighborhood of 0.02, give or take 0.01 for wavelengths near 500 μm . Furthermore for the same wavelength vicinity, $D(\pm)$ appears to be such that the sum $D(+) + D(-)$ is usually very nearly equal to 4; and the ratio $D(+)/D(-)$ is usually very nearly equal to 2, over great ranges of depths and in many media. Solving these two simultaneous equations yields, to two significant figures:

$$D(-) = 4/3$$
$$D(+) = 8/3 \tag{11}$$

which agrees very well with experimental results (cf., e.g., Table 1 of Sec. 8.5). It follows that, to the nearest rational number with small integers for numerator and denominator, we have from II:

$$\boxed{a = \frac{3}{4} k_\infty} \tag{12}$$

or:

$$k_\infty = \frac{4}{3} a \quad . \tag{13}$$

Any similarity between the appearance of the fraction 4/3 in (13) and the index of refraction of water must be viewed as an amusing coincidence. Equation (13), incidentally, points up once again the kinship of k_∞ with the absorption mechanisms in optical media (see the discussion of (5) of Sec. 9.2 and (29) of Sec. 9.3).

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10.9 Bibliographic Notes for Chapter 10

The developments of Secs. 10.1 to 10.4 are based on the work of [245].

The problem of the asymptotic light field in natural hydrosols was first clearly recognized by Whitney (re: [315] and [316]). The mathematical formulations and solutions of the problem as in Secs. 10.5, 10.6, and 10.7 are based on the researches in [224], [225], [244], and [226], respectively. Important references to the asymptotic radiance hypothesis in the hydrologic optics context may be found in [107], [108], and [209]. References to the asymptotic radiance hypothesis in the astrophysical context may be found in [43] and [147]; references to the neutron diffusion setting are made in [62]. Section 10.8 is based in the main on [230].

Experimental data in [298] exhibit clearly the asymptotic property of radiance fields in a real optical medium and were instrumental in the empirical establishment of the hypothesis.