11.1 Transport Equations for Radiometric Concepts

In this section we will present the transport equations for the following six radiometric quantities used in the study of plane-parallel media: radiance function $N(z,\theta,\phi)$, upwelling and downwelling irradiance functions $H(z,\pm)$, upwelling and downwelling scalar irradiance functions $h(z,\pm)$, and the scalar irradiance function $h(z)$.

Each of these transport equations is cast into a form which explicitly exhibits a certain attenuation function and equilibrium function associated with the radiometric concept it governs. It is the isolation and emphasis of these two concepts which is the earmark of the universal radiative transport equation. Thus, for example, the customary form of the equation of transfer for radiance is recast so that it explicitly exhibits the special attenuation function $-\alpha(z)/\cos\theta$ and the equilibrium function $N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z)$. Similarly, the unified irradiance equations governing $H(z,\pm)$ are recast into forms which explicitly exhibit the corresponding attentuation functions $\mp [a(z,\pm) + b(z,\pm)]$ and equilibrium functions $H_q(z,\pm)$. These two reformulations for the transport equations of $N(z,\theta,\phi)$ and $H(z,\pm)$ are already known (see Sec. 10.7 for the case of N, and Sec. 8.3 for the case of H); however, the reformulations are now viewed with the purpose of seeing what mathematical and physical characteristics are held in common by these transport equations. It turns out that the common characteristics are the *attenuation* and *equilibrium* functions associated with each of the radiometric concepts governed by these equations and that each of these transport equations is but a special case of a more general equation, to be determined.

The discussion of the present section continues with the derivation of the exact transport equations for $h(z,\pm)$ and $h(z)$. It is shown that each of these functions also may have associated with it an *attenuation* function and an *equilibrium* function. In this way we show that the six radiometric quantities used in the study of plane-parallel media have an important set of properties common to all: The notion of an associated attenuation function and an associated equilibrium function, and finally that the transport equation for each of these six radiometric concepts, is subsumed under one general equation.

We now proceed to substantiate the preceding assertions by considering in turn each of the six radiometric concepts and its associated transport equation.

Equation of Transfer for Radiance

The equation of transfer for radiance ((3) of Sec. 3.15) in source-free stratified plane-parallel media is of the form:

$$- \cos\theta \frac{dN(z,\theta,\phi)}{dz} = - \alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad , \quad (1)$$

where:

$$N_*(z,\theta,\phi) = \int_{\Xi} N(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)\, d\Omega \quad .$$

Equation (1) is the most basic of all transport equations and, as we have seen repeatedly in the preceding chapters, can often be used in its full generality in the several different branches of applied radiative transfer theory such as astrophysical optics, and in the two subdisciplines of geophysical optics: hydrologic optics and meteorologic optics.

The reformulation of (1) which is of immediate interest is obtained by using the notion of equilibrium radiance:

$$N_q(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha(z)} \quad , \tag{2}$$

for by means of this function, (1) may be written:

$$\boxed{\frac{dN(z,\theta,\phi)}{dz} = \frac{\alpha(z)}{\cos\theta}\,[N(z,\theta,\phi) - N_q(z,\theta,\phi)]} \tag{3}$$

Equation (3) is the desired reformulation of (1).* For our present purposes we draw special attention to the two functions:

$$\text{(i)} \quad -\frac{\alpha(z)}{\cos\theta}$$

$$\text{(ii)} \quad N_q(z,\theta,\phi) \tag{4}$$

Function (i) is the *attenuation function* for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ). Function (ii) is the *equilibrium function* for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ).

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*An alternate formulation of (3) is possible by adopting the optical depth parameter $\tau(= \int_0^z \alpha(z')dz')$. Such a formulation using τ has been found of especial use, e.g., in Chapter 10. However, for our present purposes, Eq. (3) is more appropriate.

Transport Equations for $H(z,\pm)$

The transport equations for $H(z,\pm)$ (or more accurately the *two-flow equations* for the irradiance field) are of the form (Chapter 8):

Errata

$$\mp \frac{dH(z,\pm)}{dz} = - [\alpha(z,\pm) + b(z,\pm)]H(z,\pm) + b(z,\mp)H(z,\mp) \qquad (5)$$

Associated with $H(z,-)$ and $H(z,+)$ are the equilibrium functions $H_q(z,-)$ and $H_q(z,+)$, respectively. These equilibrium functions are defined by writing:

$$\text{"}H_q(z,\pm)\text{"} \quad \text{for} \quad \frac{b(z,\mp)H(z,\mp)}{a(z,\pm) + b(z,\pm)} \quad . \qquad (6)$$

By means of these functions the equations in (5) may be written:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [a(z,\pm) + b(z,\pm)][H(z,\pm) - H_q(z,\pm)]} \quad . \qquad (7)$$

The equations in (7) are the desired reformulations of (5). For our present purposes we draw special attention to the two sets of functions:

$$\text{(i)} \quad \mp [a(z,\pm) + b(z,\pm)]$$

$$\text{(ii)} \quad H_q(z,\pm) \qquad (8)$$

Set (i) gives the *attenuation function* for the upwelling (+) and downwelling (-) irradiances $H(z,\pm)$. Observe that, by (11) of Sec. 8.3, the terms in (i) can be represented by $\pm\tau(z,\pm)$. Set (ii) gives the *equilibrium function* for the upwelling (+) and downwelling (-) irradiances $H(z,\pm)$.

Transport Equations for $h(z,\pm)$

The exact transport equations for $h(z,+)$ and $H(z,-)$ apparently have never been even remotely discussed in the literature. The reason for this gap in the family transport equations for the common radiometric concepts is two-fold. First, and perhaps most important, in the classical one-D theory, there has never been an explicit need for the transport equations for $h(z,\pm)$; the ordinary irradiances $H(z,\pm)$ were considered adequate in the early studies of the light field in stratified media. However, with the advent of more precise and detailed studies of the irradiance field (Chapters 8, 9, 10), the functions $h(z,\pm)$ have finally assumed a legitimate and useful role in modern radiative transfer theory. Second, there is no simple or intuitively obvious way of obtaining the *exact* transport equations for $h(z,\pm)$ from first principles (that is, obtaining *de novo* derivations starting only with the definition of $h(z,\pm)$ and the basic volume absorption and volume scattering functions) as is the case for the irradiances $H(z,\pm)$. Neither is there any simple way of obtaining the requisite transport equations directly from the equation of transfer for radiance (again in contradistinction to case for $H(z,\pm)$). In the present paragraph we derive the exact transport equations for $h(z,\pm)$ by a simultaneous use of (a): The connections between these functions and $H(z,\pm)$, provided by the distribution functions $D(z,\pm)$; and (b): the exact transport equations for $H(z,\pm)$.

We begin with the derivation of the transport equation for $h(z,-)$. By definition of $D(z,-)$,

$$h(z,-) = D(z,-)H(z,-) \quad . \tag{9}$$

Taking the derivative of each side with respect to z:

$$\frac{dh(z,-)}{dz} = D(z,-)\,\frac{dH(z,-)}{dz} + H(z,-)\,\frac{dD(z,-)}{dz} \quad .$$

By means of (5), this may be written:

$$\frac{dh(z,-)}{dz} = D(z,-)\left\{[a(z,-) + b(z,-)]H(z,-) + b(z,+)H(z,+)\right\} + H(z,-)\,\frac{dD(z,-)}{dz} \quad .$$

Using the definitions of $D(z,-)$ and $D(z,+)$ $(= h(z,+)/H(z,+))$ and denoting the derivatives with respect to z by a prime (which will be used interchangeably with d/dz in all that follows), the preceding equation may be written:

$$h'(z,-) = \left\{-\,[a(z,-) + b(z,-)] + \frac{D'(z,-)}{D(z,-)}\right\} h(z,-) + \frac{D(z,-)}{D(z,+)}\, b(z,+)h(z,+) \quad , \tag{10}$$

which is the general transport equation for $h(z,-)$.

Now, as in the case of $N(z,\theta,\phi)$ and $H(z,\pm)$, we may associate with $h(z,-)$ an equilibrium function $h_q(z,-)$ where we write:

$$\text{"}h_q(z,-)\text{"} \quad \text{for} \quad \frac{\dfrac{D(z,-)}{D(z,+)}\, b(z,+)h(z,+)}{[a(z,-) + b(z,-)] - \dfrac{D'(z,-)}{D(z,-)}} \quad . \tag{11}$$

An alternate representation of $h_q(z,-)$ is:

$$h_q(z,-) = \frac{D^2(z,-)b(z,+)h(z,+)}{D(z,+)D(z,-)[a(z,-) + b(z,-)] - D'(z,-)D(z,+)} \quad .$$

With this definition of $h_q(z,-)$, the transport equation (10) may be written:

$$\boxed{\frac{dh(z,-)}{dz} = \left[- (a(z,-) + b(z,-)) + \frac{D'(z,-)}{D(z,-)}\right] [h(z,-) - h_q(z,-)]} \tag{12}$$

Equation (12) is the reformulation of (11) which is of central interest in the present study, and as before we call special attention to the two functions:

$$\begin{aligned} &\text{(i)} \;\; + \left[a(z,-) + b(z,-)\right] - \frac{D'(z,-)}{D(z,-)} \\ &\text{(ii)} \;\; h_q(z,-) \end{aligned} \tag{13}$$

The function (i) is the *attenuation function* for $h(z,-)$. The function (ii) is the *equilibrium function* for $h(z,-)$.

The derivation of the transport equation for $h(z,+)$ proceeds in a similar manner to that leading to (12) and (13) in the case of $h(z,-)$. Therefore, the reader may easily verify first of all that:

$$\begin{aligned} -\frac{dh(z,+)}{dz} = &\left\{[a(z,+) + b(z,+)] + \frac{D'(z,+)}{D(z,+)}\right\} h(z,+) \\ &+ \frac{D(z,+)}{D(z,-)} b(z,-)h(z,-) \quad . \end{aligned} \tag{14}$$

Next, if we write:

$$\text{"}h_q(z,+)\text{"} \quad \text{for} \quad \frac{\frac{D(z,+)}{D(z,-)} b(z,-)h(z,-)}{\left[a(z,+) + b(z,+)\right] + \frac{D'(z,+)}{D(z,+)}} \tag{15}$$

then we have also:

$$h_q(z,+) = \frac{D^2(z,+)b(z,-)h(z,-)}{D(z,+)D(z,-)[a(z,+) + b(z,+)] + D'(z,+)D(z,-)}$$

so that (14) may be written:

$$-\frac{dh(z,+)}{dz} = -\left[(a(z,+)+b(z,+)) + \frac{D'(z,+)}{D(z,+)}\right]\left[h(z,+) - h_q(z,+)\right] , \tag{16}$$

which is the desired reformulation of (14). We draw special attention to the functions:

$$\text{(i)}\ -[a(z,+)+b(z,+)] - \frac{D'(z,+)}{D(z,+)}$$

$$\text{(ii)}\ h_q(z,+) \tag{17}$$

The function (i) is the *attenuation function* for $h(z,+)$. The function (ii) is the *equilibrium function* for $h(z,+)$.

We pause to observe the similarity of the functions in (8) (the set for $H(z,\pm)$) and with those in (13) and (17) (the set for $h(z,\pm)$). These sets coincide when $D'(z,\pm) = 0$, i.e., when $H(z,\pm)$ and $h(z,\pm)$ differ multiplicatively by a constant factor. That is, under this condition, (i) of (8) reduces to (i) of (13) and (17), and

$$\frac{H_q(z,\pm)}{h_q(z,\pm)} = D(z,\pm) = D(\pm), \quad \text{for all } z .$$

The physical significance of the condition $D'(z,\pm) = 0$ is now clear from the study of the two-D model for irradiance fields in Chapter 8, in particular from the introductory discussions of Sec. 8.5.

Transport Equation for Scalar Irradiance

To obtain the transport equation for the scalar irradiance function $h(z)$, we begin by decomposing $h(z)$ into its upwelling and downwelling components:

$$h(z) = h(z,+) + h(z,-) .$$

Then by using the definitions of the distribution functions:

$$D(z) = \frac{h(z,\pm)}{H(z,\pm)} \quad , \text{ Errata } .$$

$h(z)$ may be represented in terms of $D(z,\pm)$ and $H(z,\pm)$:

$$h(z) = D(z,-)H(z,-) + D(z,+)H(z,+) .$$

Taking the derivative of $h(z)$, we have

$$\frac{dh(z)}{dz} = D(z,-)\frac{dH(z,-)}{dz} + H(z,-)\frac{dD(z,-)}{dz}$$

$$+ D(z,+)\frac{dH(z,+)}{dz} + H(z,+)\frac{dD(z,+)}{dz} \quad .$$

We now make use of the exact transport equations for $H(z,\pm)$:

$$\frac{dh(z)}{dz} = D(z,-)\left\{-[a(z,-)+b(z,-)]H(z,-) + b(z,+)H(z,+)\right\}$$

$$+ H(z,-)D'(z,-) + H(z,+)D'(z,+)$$

$$+ D(z,+)\left\{[a(z,+)+b(z,+)]H(z,+) - b(z,-)H(z,-)\right\} \quad .$$

The next step is to convert the products $D(z,\pm)H(z,\pm)$ into the equivalent functions $h(z,\pm)$ and write $h'(z)$ as a linear combination of $h(z,+)$, $h(z,-)$:

$$\frac{dh(z)}{dz} = -[a(z,-)+b(z,-)]h(z,-) + \frac{D(z,-)}{D(z,+)}b(z,+)h(z,+)$$

$$+ \frac{D'(z,-)}{D(z,-)}h(z,-) + \frac{D'(z,+)}{D(z,+)}$$

$$+ [a(z,+)+b(z,+)]h(z,+) - \frac{D(z,+)}{D(z,-)}b(z,-)h(z,-) \quad .$$

Collecting coefficients of $h(z,\pm)$:

$$\frac{dh(z)}{dz} = A_-(z)h(z,-) + A_+(z)h(z,+) \quad , \tag{18}$$

where we have written:

$$\text{"}A_-(z)\text{"} \quad \text{for} \quad -[a(z,-)+b(z,-)] + \frac{D'(z,-) - D(z,+)b(z,-)}{D(z,-)}$$

and:

$$\text{"}A_+(z)\text{"} \quad \text{for} \quad [a(z,+)+b(z,+)] + \frac{D'(z,+) + D(z,-)b(z,+)}{D(z,+)}$$

Evidently (18) is unchanged if we write:

$$\frac{dh(z)}{dz} = A_-(z)h(z,-) + A_-(z)h(z,+)$$

$$+ A_+(z)h(z,+) + A_+(z)h(z,-)$$

$$- [A_-(z)h(z,+) + A_+(z)h(z,-)] \quad .$$

But then this equation may be reduced to:

$$\frac{dh(z)}{dz} = [A_-(z) + A_+(z)]h(z) - [A_-(z)h(z,+) + A_+(z)h(z,-)] \tag{19}$$

which is the transport equation for h(z).

By writing:

$$\text{"}h_q(z)\text{"} \quad \text{for} \quad \frac{A_-(z)h(z,+) + A_+(z)h(z,-)}{A_-(z) + A_+(z)} ,$$

Equation (19) is expressible as:

$$\boxed{\frac{dh(z)}{dz} = [A_-(z) + A_+(z)]\ [h(z) - h_q(z)]} \quad . \tag{20}$$

For our present purposes, Equation (20) is of central interest, and we mark for future reference:

$$\begin{array}{l} \text{(i)}\ -\ [A_-(z) + A_+(z)] \\ \text{(ii)}\ h_q(z) \end{array} \tag{21}$$

Expression (i) is the *attenuation function* for h(z). Expression (ii) is the *equilibrium function* for h(z).

Preliminary Unification and Preliminary Statement of the Equilibrium Principle

We have now reached a point in our discussion where we may consolidate the results obtained so far. The consolidation will serve two purposes: It will yield a preliminary view of the structure of the universal transport equation, and secondly, it will prepare the way for a discussion of the transport equations for the apparent optical properties to be taken up in the next section.

We turn now to the transport equations discussed so far, in particular the equations (3), (7), (12), (16), and (20). These six equations have a common mathematical structure, and the various components of the structure are associated with physical concepts common to the respective radiometric concepts. Specifically, let the general symbol " $\mathcal{P}(z)$" denote any one of the following six radiometric concepts:

$$\mathcal{P}(z): \quad \begin{cases} N(z,\theta,\phi) \\ H(z,\pm) \\ h(z,\pm) \\ h(z) \end{cases}$$

Furthermore, let "$\mathcal{P}_\alpha(z)$" denote the associated attenuation function for $\mathcal{P}(z)$. Finally, let "$\mathcal{P}_q(z)$" denote the associated equilibrium function for $\mathcal{P}(z)$. Then each of the six transport equations developed above is precisely of the form:

$$\boxed{\frac{d\mathcal{P}(z)}{dz} = -\mathcal{P}_\alpha(z)\,[\mathcal{P}(z) - \mathcal{P}_q(z)]} \quad . \tag{22}$$

We now may make a key observation on the dynamic behavior of the five radiometric concepts which are associated with a general direction of flow (h(z) is the only one of the preceding concepts which, by definition, is not associated with any particular directed pencil of radiation or general hemispherical flow). If "$\mathcal{P}(z)$" stands for any one of these five concepts: $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, then it is easy to verify that on the basis of (22):

$$\text{If } \mathcal{P}(z) > \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d|z|} < 0 \quad ,$$

and: (23)

$$\text{if } \mathcal{P}(z) < \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d|z|} > 0 \quad ,$$

where the symbol '$d\mathcal{P}(z)/d|z|$' is defined as follows, we write:

$$\text{"}\frac{d\mathcal{P}(z)}{d|z|}\text{"} \quad \text{for} \quad \frac{d\mathcal{P}(z)}{dz}$$

if $\mathcal{P}(z)$ is associated with the direction of increasing z (downwelling direction)

and:

$$\text{"}\frac{d\mathcal{P}(z)}{d|z|}\text{"} \quad \text{for} \quad \frac{d\mathcal{P}(z)}{d(-z)}$$

if $\mathcal{P}(z)$ is associated with the direction of decreasing z (upwelling direction).

In other words, the equations (23) simply state that as the geometric form of the radiation represented by $\mathcal{P}(z)$ travels *in its assigned direction*, the magnitude of $\mathcal{P}(z)$ always changes in such a way that it tends to approach the magnitude of its equilibrium function $\mathcal{P}_q(z)$. This observation forms the core of the general equilibrium principle formulated below.

<< Table of Contents << Title Page >> Next Page >>

11.2 Transport Equations for Apparent Optical Properties

The notion of "apparent optical property" is discussed in detail in Chapter 9. The following list consists of the ten more important apparent optical properties associated with plane-parallel media, as developed in Chapter 9: