Furthermore, let "$\mathcal{P}_\alpha(z)$" denote the associated attenuation function for $\mathcal{P}(z)$. Finally, let "$\mathcal{P}_q(z)$" denote the associated equilibrium function for $\mathcal{P}(z)$. Then each of the six transport equations developed above is precisely of the form:

$$\boxed{\frac{d\mathcal{P}(z)}{dz} = -\mathcal{P}_\alpha(z)\ [\mathcal{P}(z) - \mathcal{P}_q(z)]} \quad . \tag{22}$$

We now may make a key observation on the dynamic behavior of the five radiometric concepts which are associated with a general direction of flow (h(z) is the only one of the preceding concepts which, by definition, is not associated with any particular directed pencil of radiation or general hemispherical flow). If "$\mathcal{P}(z)$" stands for any one of these five concepts: $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, then it is easy to to verify that on the basis of (22):

$$\text{If } \mathcal{P}(z) > \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d\,|z|} < 0 \quad ,$$

and: (23)

$$\text{if } \mathcal{P}(z) < \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d\,|z|} > 0 \quad ,$$

where the symbol '$d\mathcal{P}(z)/d\,|z|$' is defined as follows, we write:

$$\text{"}\frac{d\mathcal{P}(z)}{d\,|z|}\text{"} \quad \text{for} \quad \frac{d\mathcal{P}(z)}{dz}$$

if $\mathcal{P}(z)$ is associated with the direction of increasing z (downwelling direction)

and:

$$\text{"}\frac{d\mathcal{P}(z)}{d\,|z|}\text{"} \quad \text{for} \quad \frac{d\mathcal{P}(z)}{d(-z)}$$

if $\mathcal{P}(z)$ is associated with the direction of decreasing z (upwelling direction).

In other words, the equations (23) simply state that as the geometric form of the radiation represented by $\mathcal{P}(z)$ travels *in its assigned direction*, the magnitude of $\mathcal{P}(z)$ always changes in such a way that it tends to approach the magnitude of its equilibrium function $\mathcal{P}_q(z)$. This observation forms the core of the general equilibrium principle formulated below.

11.2 Transport Equations for Apparent Optical Properties

The notion of "apparent optical property" is discussed in detail in Chapter 9. The following list consists of the ten more important apparent optical properties associated with plane-parallel media, as developed in Chapter 9:

$$\begin{cases} K(z,\xi) \\ K(z,\pm) \\ R(z,\pm) \\ D(z,\pm) \\ k(z,\pm) \\ k(z) \end{cases}$$

We shall show in this section that a transport equation may be assigned to each of the above K - functions. As noted, we can assign a transport equation to either of the reflectance functions $R(z,\pm)$ and to the distribution functions $D(z,\pm)$, and in fact we will exhibit the transport equation for $R(z,-)$ and go on to deduce, by means of this equation, an interesting property about the depth behavior of $R(z,-)$. We will not, however, exhibit the transport equation for $R(z,+)$ and $D(z,\pm)$ for the following reasons: By definition, $R(z,+) = 1/R(z,-)$, so that once a transport equation is obtained for $R(z,-)$, one for $R(z,+)$ would be superfluous. The reason for not obtaining transport equations for the optical properties $D(z,\pm)$ is more subtle and may be inferred from the preceding formulations by recalling that the transport equations for $H(z,\pm)$, $h(z,\pm)$ make implicit or explicit use of the distribution functions. If we were to deduce the transport equations for $D(z,\pm)$ we would see that the quantities $H(z,\pm)$ or $h(z,\pm)$ would be explicitly involved in them. Therefore, a logical circularity would creep into the final set of transport equations if we insisted on obtaining transport equations for $D(z,\pm)$ in addition to those of $H(z,\pm)$ and $h(z,\pm)$. In order to avoid such a circularity we must decide on the elimination of one of the three sets of quantities: $H(z,\pm)$, $h(z,\pm)$, $D(z,\pm)$. Such a decision is easy to reach after we note that $H(z,\pm)$ and $H(z,\pm)$ are the fundamental observables in natural light fields, and that the $D(z,\pm)$ simply act as analytical liaisons between these quantities. Therefore, we will agree that $D(z,\pm)$ are to continue to act as the connecting links between the irradiance and scalar irradiance concepts, and that they are to enter into the calculations solely in the capacity of dimensionless mathematical parameters. Their usual physical interpretation will, of course, be retained, namely that they are measures of the directional variation of the radiance distribution at a general depth z. (In this connection, see Sec. 8.5.)

Canonical Forms of Transport Equations for K Functions

The procedure for obtaining the the transport equation for the six K-functions is facilitated by the preceding results, in particular by means of the six transport equations for $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, and $h(z)$. If "$\mathcal{P}(z)$" denotes for any of these six functions, then the corresponding K - function $K(\mathcal{P})$ is defined by writing:

$$\text{"}K(\mathcal{P})\text{"} \quad \text{for} \quad -\frac{1}{\mathcal{P}(z)}\frac{d\mathcal{P}(z)}{dz} \quad . \tag{1}$$

Using the generic equation (22) of Sec. 11.1 and the definition (1), we have:

$$-\mathcal{P}(z)K(\mathcal{P}) = -\mathcal{P}_\alpha[\mathcal{P}(z) - \mathcal{P}_q(z)] \quad .$$

Solving this for $\mathcal{P}(z)$, we obtain the *canonical form* of the transport equation for $\mathcal{P}(z)$:

$$\boxed{\mathcal{P}(z) = \frac{\mathcal{P}_q(z)}{1 - \left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right]}} \quad , \tag{2}$$

The canonical form for the radiance function of Chapter 4 is thus extended to wider contexts (see, e.g., (5) of Sec. 4.7).

This canonical form of the transport equation serves as the common starting point for the derivation of the equations governing individual K - functions. Thus, by taking the formal logarithmic derivative of each side of (2):

$$\frac{d\ln\mathcal{P}(z)}{dz} = \frac{d\ln\mathcal{P}_q(z)}{dz} - \frac{d}{dz}\ln\left[1 - \frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right] \quad ,$$

and writing, in analogy to (1):

$$\text{"}K_q(\mathcal{P})\text{"} \quad \text{for} \quad -\frac{1}{\mathcal{P}_q(z)}\frac{d\mathcal{P}_q(z)}{dz} \quad , \tag{3}$$

we have:

$$-K(\mathcal{P}) = -K_q(\mathcal{P}) + \frac{\frac{d}{dz}\left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right]}{1 - \frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}} \quad ,$$

whence:

$$\frac{d}{dz}\left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right] = \left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)} - 1\right]\left[K(\mathcal{P}) - K_q(\mathcal{P})\right] \quad . \tag{4}$$

As it stands, (4) may be taken as the transport equation for $K(\mathcal{P})$. However, by suitable transformations of variables, we can reduce (4) to the general form of the universal transport equation. We next consider such transformations.

Dimensionless Transport Equation for $K(\mathscr{P})$

At the present point of the discussions (namely (4)), we have two alternative routes open to a universal transport equation: One route starts with the adoption of a generalized notion of *optical depth* defined by writing:

$$"\tau(z)" \quad \text{or} \quad "\tau" \quad \text{for} \quad \int_0^z \mathscr{P}_\alpha(z')dz' \quad ,$$

along with a *relativization* of $K(\mathscr{P})$ and $K_q(\mathscr{P})$ with respect to $\mathscr{P}_\alpha(z)$; thus we write:

$$"\tilde{K}(\mathscr{P})" \quad \text{for} \quad \frac{K(\mathscr{P})}{\mathscr{P}_\alpha} \quad ,$$

and:

$$"\tilde{K}_q(\mathscr{P})" \quad \text{for} \quad \frac{K_q(\mathscr{P})}{\mathscr{P}_\alpha} \quad .$$

Then (4) may be written in the *dimensionless form:*

$$\boxed{\frac{d\tilde{K}(\mathscr{P})}{d\tau} = [\tilde{K}(\mathscr{P}) - 1][\tilde{K}(\mathscr{P}) - \tilde{K}_q(\mathscr{P})]} \tag{5}$$

Equation (5) has the advantage of simplicity of structure and is therefore ideal for formal work. For example, the dimensionless form of (5) for the case $\mathscr{P}(z) = N(z,\theta,\phi)$ was used in the proof of the asymptotic radiance hypothesis in Sec. 10.5. However, (5) has the disadvantage of not showing the explicit effects on the associated K - functions produced by inhomogeneities of the medium nor of the way in which the K - functions vary with *geometrical depth*, the natural measure of depth used in experimental work. Therefore, we will actually take the second route which consists in adopting *geometrical depth* and which uses *un*relativized K - functions. This results in a mathematically more cumbersome transport equation but is actually of greater use in practical applications. By adopting the alternative route, we are now obliged to consider each of the K - functions in turn. The common starting point is (2) in which the explicit forms of $\mathscr{P}_\alpha(z)$ and $\mathscr{P}_q(z)$ for the various concepts have been substituted.

Transport Equation for $K(z,\theta,\phi)$

From (2) we have

$$N(z,\theta,\phi) = \frac{N_q(z,\theta,\phi)}{1 + \cos\theta \, \dfrac{K(z,\theta,\phi)}{\alpha(z)}} \quad , \tag{6}$$

in which we have set $\mathcal{P}_\alpha(z) = -\alpha(z)/\cos\theta$, $\mathcal{P}_q(z) = N_q(z,\theta,\phi)$, so that $K(\mathcal{P}) = K(z,\theta,\phi)$ and $K_q(\mathcal{P}) = K_q(z,\theta,\phi)$, using the definitions in (4) of Sec. 11.1. Taking the logarithmic derivative of each side of (6) and solving for $dK(z,\theta,\phi)/dz$:

$$\frac{dK(z,\theta,\phi)}{dz} = K^2(z,\theta,\phi) + \left[\frac{\alpha(z)}{\cos\theta} - K_q(z,\theta,\phi) + \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz}\right] K(z,\theta,\phi) - K_q(z,\theta,\phi)\,\frac{\alpha(z)}{\cos\theta} \quad .$$

The right-hand side of this equation may be factored into the product of two functions yielding the desired form of the transport equation for $K(z,\theta,\phi)$:

$$\boxed{\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - \mathcal{K}_\alpha(z,\theta,\phi)][K(z,\theta,\phi) - \mathcal{K}_q(z,\theta,\phi)]} \quad , \tag{7}$$

where $\mathcal{K}_\alpha$ and $\mathcal{K}_q$ are defined in context by the following two equations:

$$\begin{aligned} \mathcal{K}_\alpha(z,\theta,\phi) + \mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta} + K_q(z,\theta,\phi) - \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz} \\ \mathcal{K}_\alpha(z,\theta,\phi)\,\mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta}\,K_q(z,\theta,\phi) \end{aligned} \tag{8}$$

The functions $\mathcal{K}_\alpha(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$ appearing in (7) are, respectively the attenuation and equilibrium functions for $K(z,\theta,\phi)$. They are defined as shown by the pair of simultaneous equations in (8), whose solutions are:

$$\left.\begin{matrix} 2\mathcal{K}_q \\ \\ 2\mathcal{K}_\alpha \end{matrix}\right\} = -\left[\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right] \pm \left[\left(\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right)^2 + \frac{4K_q\alpha}{\cos\theta}\right]^{1/2}$$

The quantities K_q and $\mathcal{K}_q$ should not be confused with each other. K_q is the logarithmic derivative of N_q (see definition (3)) whereas $\mathcal{K}_q$ is the sought-for equilibrium function for $K(z,\theta,\phi)$ in the general context. Observe, however, that if the medium were homogeneous, then:

$$\mathcal{K}_\alpha(z,\theta,\phi) = -\frac{\alpha}{\cos\theta} \quad ,$$

$$\mathcal{K}_q(z,\theta,\phi) = K_q(z,\theta,\phi) \quad .$$

More generally, in eventually homogeneous media (i.e., media in which $\alpha'(z) \to 0$ as $z \to \infty$)

$$\mathcal{K}_\alpha(z,\theta,\phi) \to -\frac{\alpha}{\cos\theta} \quad ,$$

$$\mathcal{K}_q(z,\theta,\phi) \to K_q(z,\theta,\phi) \to k_\infty \quad .$$

This follows from the asymptotic radiance theorem and its various consequences discussed in Chapter 10.

Transport Equations for $K(z,\pm)$

The appropriate form of (2) in the case of the K-functions $K(z,\pm)$ is obtained by substituting the attenuation and equilibrium functions for $H(z,\pm)$ in (2):

$$H(z,\pm) = \frac{H_q(z,\pm)}{1 \pm \dfrac{K(z,\pm)}{[a(z,\pm) + b(z,\pm)]}} \quad .$$

Taking the logarithmic derivatives of each side, solving for $K'(z,\pm)$, and factoring the quadratic in $K(z,\pm)$, we have

$$\frac{dK(z,-)}{dz} = [K(z,-) - \mathcal{K}_\alpha(z,-)][K(z,-) - \mathcal{K}_q(z,-)] \quad , \qquad (9)$$

$$\frac{dK(z,+)}{dz} = [K(z,+) - \mathcal{K}_\alpha(z,+)][K(z,+) - \mathcal{K}_q(z,+)] \quad . \qquad (10)$$

For $K(z,+)$ the functions $\mathcal{K}_\alpha(z,+)$, $\mathcal{K}_q(z,+)$ are defined in context by the equations:

$$\mathcal{K}_\alpha(z,+) + \mathcal{K}_q(z,+) = [a(z,-) + b(z,-)] + K_q(z,-) - (\ln[a(z,-) + b(z,-)])'$$

$$\mathcal{K}_\alpha(z,+)\mathcal{K}_q(z,+) = [a(z,-) + b(z,-)]K_q(z,-)$$

Similarly, for $K(z,-)$:

$$\mathcal{K}_\alpha(z,-) + \mathcal{K}_q(z,-) = -[a(z,+) + b(z,+)] + K_q(z,+) - (\ln[a(z,+) + b(z,+)])'$$

$$\mathcal{K}_\alpha(z,-)\mathcal{K}_q(z,-) = -[a(z,+) + b(z,+)]K_q(z,+) \quad .$$

These simultaneous equations may be solved to obtain explicit expressions for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_q$'s. We will not do this here, but simply point out that, in all eventually homogeneous media, as $z \to \infty$,

$$\mathcal{K}_\alpha(z,\pm) \to \mp[a(z,\pm) + b(z,\pm)] \quad ,$$

$$\mathcal{K}_q(z,\pm) \to K_q(z,\pm) \to k_\infty \quad .$$

This follows from the asymptotic radiance theorem and its various consequences studied in Chapter 10. As in the case of $K_q(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$, care should be taken so as not to confuse $K_q(z,\pm)$ with $\mathcal{K}_q(z,\pm)$. The former is defined in (3), the latter by the preceding simultaneous equations.

Transport Equations for k(z,±) and k(z)

Starting with the general canonical equation (2), we have for $h(z)$:

$$h(z) = \frac{h_q(z)}{1 + \dfrac{k(z)}{[A_+(z) + A_-(z)]}} \quad .$$

Similarly, for $h(z,\pm)$:

$$h(z,\pm) = \frac{h_q(z,\pm)}{1 \pm \dfrac{k(z,\pm)}{[a(z,\pm) + b(z,\pm)] - \dfrac{D'(z,\pm)}{D(z,\pm)}}} \quad .$$

The existence of these canonical equations for $h(z,\pm)$ and $h(z)$ is sufficient to prove the existence of the appropriate transport equations for $k(z,\pm)$ and $k(z)$ by following the procedure illustrated in the preceding two paragraphs. The results are

$$\frac{dk(z,\pm)}{dz} = [k(z,\pm) - k_\alpha(z,\pm)][k(z,\pm) - k_q(z,\pm)] \quad , \quad (11)$$

$$\frac{dk(z)}{dz} = [k(z) - k_\alpha(z)][k(z) - k_q(z)] \quad . \quad (12)$$

The exact forms for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_q$'s will not be worked out; this may be left as an exercise for the interested reader. The important point to observe is that we have now proved that for all six K-functions, the generic transport equation is:

$$\frac{dK(\mathcal{P})}{dz} = [K(\mathcal{P}) - \mathcal{K}_\alpha(\mathcal{P})][K(\mathcal{P}) - \mathcal{K}_q(\mathcal{P})] \quad (13)$$

Equations (22) of Sec. 11.1 and (13) form the two major sets of transport equations considered in this chapter. These two equations cover all twelve transport equations for $\mathcal{P}$ and $K(\mathcal{P})$ considered so far.

As in the case of (22) of Sec. 11.1, it is easy to verify on the basis of (13) that:

$$\text{If} \quad K(\mathcal{P}) > \mathcal{K}_q(\mathcal{P}), \quad \text{then} \quad \frac{dK(\mathcal{P})}{d|z|} < 0 \quad ,$$

and (14)

$$\text{if} \quad K(\mathcal{P}) < \mathcal{K}_q(\mathcal{P}), \quad \text{then} \quad \frac{dK(\mathcal{P})}{d|z|} > 0 \quad ,$$

which show that $K(\mathcal{P})$ always tends toward* its equilibrium function $\mathcal{K}_q(\mathcal{P})$.

We now turn to consider the last of the standard transport equations, namely that for $R(z,-)$.

Transport Equation for R(z,-)

By definition of $R(z,-)$:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad .$$

Taking the logarithmic derivative of each side, and applying the definitions of $K(z,+)$ and $K(z,-)$, we have:

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad .$$

Using the following representations (18) and (19) of Sec. 9.2 of $K(z,\pm)$:

$$K(z,\pm) = \mp [a(z,\pm) + b(z,\pm)] \pm b(z,\mp)R(z,\pm) \quad ,$$

the derivative of $R(z,-)$ may be cast into the form:

$$\frac{dR(z,-)}{dz} = - b(z,+)R^2(z,-) + [a(z,-) + a(z,+) + b(z,-) + b(z,+)] \cdot$$

$$\cdot R(z,-) - b(z,-) \quad .$$

The right-hand side, which is a quadratic in $R(z,-)$, may be factored:

$$\frac{dR(z,-)}{dz} = - b(z,+)[R(z,-) - R_\alpha(z,-)][R(z,-) - R_q(z,-)] . \quad (15)$$

---

*The term "tends toward" has a precise meaning here: If f_1 and f_2 are two real-valued functions defined on some common domain $\mathcal{D}$ of the reals then f_1 *tends toward* f_2 at $x \in \mathcal{D}$ if sign $[f_2(x) - f_1(x)]$ = sign $f_1'(x)$ where "sign" means the same as "sign of." As an earlier example of this, see (4) of Sec. 9.4.

Equation (15) is the required transport equation for $R(z,-)$, in which $R_\alpha(z,-)$ is the *attenuation function* for $R(z,-)$ and $R_q(z,-)$ is the *equilibrium function* for $R(z,-)$ (compare with (2) of Sec. 9.4). These functions are defined in context by the following system of simultaneous equations:

$$R_\alpha(z,-) + R_q(z,-) = \frac{a(z,-) + a(z,+) + b(z,-) + b(z,+)}{b(z,+)}$$

$$R_\alpha(z,-)R_q(z,-) = \frac{b(z,-)}{b(z,+)} \quad . \tag{16}$$

As in the case of the K - functions, these may be solved for $R_\alpha(z,-)$ and $R_q(z,-)$:

$$\left.\begin{matrix} 2R_\alpha(z,-) \\ \\ 2R_q(z,-) \end{matrix}\right\} = \left\{R(z,-) + \frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} - \frac{1}{b(z,+)}\,[K(z,-) - K(z,+)]\right\}$$

$$\pm \left[\{\ \}^2 - 4\,\frac{b(z,-)}{b(z,+)}\right]^{1/2} \quad . \tag{17}$$

R_α goes with the plus sign, R_q with the minus sign.

We observe that, in eventually homogeneous media, as $z \to \infty$:

$$R_\alpha(z,-) \to \frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} \to \frac{1}{R_\infty}\,\frac{b(-)}{b(+)}$$

$$R_q(z,-) \to R(z,-) \to R_\infty \quad . \tag{18}$$

These facts follow from (17) and the asymptotic radiance theorem of Sec. 10.7.

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11.3 Universal Radiative Transport Equation and the Equilibrium Principle

For the purposes of this section, let us refer to the thirteen quantities studied so far as the *standard concepts* (namely $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, $h(z)$, $K(z,\theta,\phi)$, $K(z,\pm)$, $k(z,\pm)$, $k(z)$, and $R(z,-)$). A *directed standard concept* is any of the preceding standard concepts except $h(z)$ and $K(z)$.

The evidence gathered in the preceding discussions may now be assembled in the form of: