Equation (15) is the required transport equation for $R(z,-)$, in which $R_\alpha(z,-)$ is the *attenuation function* for $R(z,-)$ and $R_q(z,-)$ is the *equilibrium function* for $R(z,-)$ (compare with (2) of Sec. 9.4). These functions are defined in context by the following system of simultaneous equations:

$$R_\alpha(z,-) + R_q(z,-) = \frac{a(z,-) + a(z,+) + b(z,-) + b(z,+)}{b(z,+)}$$

$$R_\alpha(z,-)R_q(z,-) = \frac{b(z,-)}{b(z,+)} \quad . \tag{16}$$

As in the case of the K - functions, these may be solved for $R_\alpha(z,-)$ and $R_q(z,-)$:

$$\left.\begin{matrix} 2R_\alpha(z,-) \\ \\ 2R_q(z,-) \end{matrix}\right\} = \left\{R(z,-) + \frac{1}{R(z,-)}\frac{b(z,-)}{b(z,+)} - \frac{1}{b(z,+)}[K(z,-) - K(z,+)]\right\}$$

$$\pm \left[\{\ \}^2 - 4\frac{b(z,-)}{b(z,+)}\right]^{1/2} \quad . \tag{17}$$

R_α goes with the plus sign, R_q with the minus sign.

We observe that, in eventually homogeneous media, as $z \to \infty$:

$$R_\alpha(z,-) \to \frac{1}{R(z,-)}\frac{b(z,-)}{b(z,+)} \to \frac{1}{R_\infty}\frac{b(-)}{b(+)}$$

$$R_q(z,-) \to R(z,-) \to R_\infty \quad . \tag{18}$$

These facts follow from (17) and the asymptotic radiance theorem of Sec. 10.7.

11.3 Universal Radiative Transport Equation and the Equilibrium Principle

For the purposes of this section, let us refer to the thirteen quantities studied so far as the *standard concepts* (namely $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, $h(z)$, $K(z,\theta,\phi)$, $K(z,\pm)$, $k(z,\pm)$, $k(z)$, and $R(z,-)$). A *directed standard concept* is any of the preceding standard concepts except $h(z)$ and $K(z)$.

The evidence gathered in the preceding discussions may now be assembled in the form of:

THE UNIVERSAL RADIATIVE TRANSPORT EQUATION AND THE EQUILIBRIUM PRINCIPLE. *Let X be an arbitrarily stratified source-free plane-parallel medium with arbitrary incident lighting conditions. Let "$\mathcal{C}(z)$" denote any one of the standard concepts. Then associated with $\mathcal{C}(z)$ are two functions $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_q(z)$, the attenuation and equilibrium functions for $\mathcal{C}(z)$, respectively. The standard concept $\mathcal{C}(z)$ together with $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_q(z)$ satisfy the functional relation:*

$$\frac{d\mathcal{C}(z)}{dz} = \mu(z)\,[\delta\mathcal{C}(z) - \mathcal{C}_\alpha(z)]\,[\mathcal{C}(z) - \mathcal{C}_q(z)] \quad , \tag{1}$$

where $\mu(z)$ and δ are known parameters depending on $\mathcal{C}(z)$. The relation (1) is the ***universal radiative transport equation.*** (It is *degenerate* if $\delta = 0$; and *normalized* if $\mu = 1$, $\delta = 1$.)

If $\mathcal{C}(z)$ is a directed standard concept and $\mu(z) > 0$, *then:*

$$\frac{d\mathcal{C}(z)}{d|z|} \gtrless 0 \quad \text{whenever} \quad \mathcal{C}(z) \lesseqgtr \mathcal{C}_q(z) \quad ; \tag{2}$$

and if $\mathcal{C}(z)$ is any standard concept, and X is eventually homogeneous, then:

$$\begin{aligned} \mathcal{C}_\alpha(\infty) &= \lim_{z\to\infty}\mathcal{C}_\alpha(z) \qquad \textit{exists}, \\ \mathcal{C}_q(\infty) &= \lim_{z\to\infty}\mathcal{C}_q(z) \qquad \textit{exists}, \end{aligned} \tag{3}$$

and:

$$\lim_{z\to\infty}\mathcal{C}(z) = \mathcal{C}_q(\infty) \quad . \tag{4}$$

The proof of the statements (1), (2), (3), and (4) have essentially been covered in the preceding discussions either directly (as in the case of (1)), or indirectly by references to the appropriate portions of the present work (as in the case of (2)-(4)). Table 1 below gives the explicit forms of $\mu(z)$ and δ for the thirteen standard concepts: An examination of Table 1 shows that if R(z,-) is removed from the list of standard concepts, a considerable simplification is effected in the form of (1). However, in the interests of completeness we have included R(z,-) within the purview of (1), and we note that by a change of z-scale, the equation is normalizable.

TABLE 1

Standard Cases of the Universal Radiative Transport Equation

Standard Concept	Values of μ,δ
$N(z,\theta,\phi)$ $H(z,\pm)$ $h(z,\pm)$ $h(z)$	$\mu(z) = 1$ $\delta = 0$ (degenerate)
$K(z,\theta,\phi)$ $K(z,\pm)$ $k(z,\pm)$ $k(z)$	$\mu(z) = 1$ $\delta = 1$ (normalized)
$R(z,-)$	$\mu(z) = -b(z,+)$ $\delta = 1$ (normalizable)

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11.4 Some Additional Transport Equations Subsumed by the Universal Transport Equation

The standard transport equations enumerated in Table 1 of Sec. 11.3 constitute the most frequently used equations in general radiative transfer theory. This list, however, by no means exhausts the various ramifications of the universal transport equation as given by (1) of Sec. 11.3. An additional set of transport equations which fall under the domain of the degenerate universal transport equation will now be mentioned. This set is associated with less frequently used--but no less important--radiometric concepts than those of the standard type. We will consider in particular the following radiometric quantities:

(i) n-ary radiance N^n

(ii) n-ary radiant energy U^n

(iii) path function N_*

(iv) vector irradiance $\mathbf{H}$

(i) *The transport equation governing* N^n in plane-parallel media is ((1) of Sec. 5.2):

$$- \cos\theta \frac{dN^n(z,\theta,\phi)}{dz} = -\alpha(z)N^n(z,\theta,\phi) + N_*^n(z,\theta,\phi) \qquad (1)$$