TABLE 1

Standard Cases of the Universal Radiative Transport Equation

Standard Concept	Values of μ,δ
$N(z,\theta,\phi)$ $H(z,\pm)$ $h(z,\pm)$ $h(z)$	$\mu(z) = 1$ $\delta = 0$ (degenerate)
$K(z,\theta,\phi)$ $K(z,\pm)$ $k(z,\pm)$ $k(z)$	$\mu(z) = 1$ $\delta = 1$ (normalized)
$R(z,-)$	$\mu(z) = -b(z,+)$ $\delta = 1$ (normalizable)

11.4 Some Additional Transport Equations Subsumed by the Universal Transport Equation

The standard transport equations enumerated in Table 1 of Sec. 11.3 constitute the most frequently used equations in general radiative transfer theory. This list, however, by no means exhausts the various ramifications of the universal transport equation as given by (1) of Sec. 11.3. An additional set of transport equations which fall under the domain of the degenerate universal transport equation will now be mentioned. This set is associated with less frequently used--but no less important--radiometric concepts than those of the standard type. We will consider in particular the following radiometric quantities:

(i) n-ary radiance N^n

(ii) n-ary radiant energy U^n

(iii) path function N_*

(iv) vector irradiance $\mathbf{H}$

(i) *The transport equation governing* N^n in plane-parallel media is ((1) of Sec. 5.2):

$$-\cos\theta \frac{dN^n(z,\theta,\phi)}{dz} = -\alpha(z)N^n(z,\theta,\phi) + N_*^n(z,\theta,\phi) \quad (1)$$

where (re: (6) of Sec. 5.1):

$$N_*^n(z,\theta,\phi) = \int_{\Xi} N^{n-1}(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)d\Omega \quad . \qquad (2)$$

Here N^n, $n = 1, 2, \ldots,$ is the n-ary scattered radiance (re: (11) of Sec. 5.1), i.e., radiance consisting of photons having been scattered precisely n-times with respect to those comprising the initial radiance N^o entering the medium. In any particular problem, it is assumed that $N^o(z,\theta,\phi)$ is given. From this, $N_*^1(z,\theta,\phi)$ is obtainable by means of (2). Then $N_*^1(z,\theta,\phi)$ is known, and (51) becomes a differential equation in $N^1(z,\theta,\phi)$, which is easily solved in principle. This solution, we recall, is the basis of the definition (4) of Sec. 5.1. Numerical solutions of $N^1(z,\theta,\phi)$ may be readily obtained by means of a computer programmed for (1) (cf. concluding remarks in Sec. 5.6). Once $N^1(z,\theta,\phi)$ is known for all z and (θ,ϕ), (2) yields $N_*^2(z,\theta,\phi)$ and (1) may be solved for $N^2(z,\theta,\phi)$. By repeating this process, we are led to obtain $N^n(z,\theta,\phi)$ knowing $N^{n-1}(z,\theta,\phi)$. The total (observable) radiance $N(z,\theta,\phi)$ is defined by writing ((3) of Sec. 5.2):

$$\text{"}N(z,\theta,\phi)\text{"} \quad \text{for} \quad \sum_{n=o}^{\infty} N^n(z,\theta,\phi) \quad .$$

For our present purposes we write:

$$\text{"}N_q^n(z,\theta,\phi)\text{"} \quad \text{for} \quad \frac{N_*^n(z,\theta,\phi)}{\alpha(z)} \quad ,$$

so that (1) may be written:

$$\boxed{\frac{dN^n(z,\theta,\phi)}{dz} = \frac{\alpha(z)}{\cos\theta}\,[N^n(z,\theta,\phi) - N_q^n(z,\theta,\phi)]} \quad . \qquad (3)$$

When written in this form, (3) closely parallels the form of (3) of Sec. 11.1, so that we conclude, as in (4) of Sec. 11.1:

(a) $-\frac{\alpha(z)}{\cos\theta}$ is the attenuation function for $N^n(z,\theta,\phi)$

(b) $N_q^n(z,\theta,\phi)$ is the equilibrium function for $N^n(z,\theta,\phi)$.

In this way the transport equation for $N^n(z,\theta,\phi)$ is subsumed by (1) of Sec. 11.3, in which $\mu(z) = 1$, $\delta = 0$.

(ii) *The time-dependent transport equation governing* U^n in a medium with no net flux across its boundary is usually written in terms of a time parameter t instead of a space parameter z ((24) of Sec. 5.8):

$$\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s} \tag{4}$$

where $T_\alpha = 1/v\alpha$, $T_s = 1/vs$. However, if "v" denotes the speed of light in X, then we may introduce a new variable r by writing

"r" for vt ,

so that (4) becomes:

$$\frac{dU^n(r)}{dr} = -\alpha U^n(r) + sU^{n-1}(r) \quad . \tag{5}$$

The symbol "$U^n(r)$" denotes the n-ary radiant energy content of a sphere of radius r about a point source (in a space X) which emits radiant flux in some prescribed manner starting from time t = 0 . The space is assumed homogeneous so that $\alpha(z) = \alpha$ for every z in the space. By writing:

"$U_*^n(r)$" for $sU^{n-1}(r)$

and:

"$U_q^n(r)$" for $\frac{U_*^n(r)}{\alpha}\left(= \rho U^{n-1}\right)$

where "ρ" denotes s/α, (5) may be then written:

$$\boxed{\frac{dU^n(r)}{dr} = -\alpha\left[U^n(r) - U_q^n(r)\right]} \tag{6}$$

Hence:

(a) α is the *attenuation function* for U^n

(b) U_q^n is the *equilibrium function* for U^n,

and (6) is subsumed by (1) of Sec. 11.3.

(iii) *The transport equation governing* N_* has the form (re: (9) of Sec. 5.2):

$$-\cos\theta \frac{dN_*(z,\theta,\phi)}{dz} = -\alpha N_*(z,\theta,\phi) + N_{**}(z,\theta,\phi) \tag{7}$$

where

$$N_{**}(z,\theta,\phi) = \int_\Xi N_*(z,\theta',\phi')\sigma(\theta',\phi';\theta,\phi)d\Omega \quad . \tag{8}$$

Equation (7) holds in all homogeneous isotropic media (thus the reason for explicity dropping z - notation in α and σ; on the other hand, σ may be arbitrary). If we write:

$$"N_{*q}(z,\theta,\phi)" \quad \text{for} \quad \frac{N_{**}(z,\theta,\phi)}{\alpha}$$

then:

$$\frac{dN_*(z,\theta,\phi)}{dz} = \frac{\alpha}{\cos\theta}[N_*(z,\theta,\phi) - N_{*q}(z,\theta,\phi)] \quad ; \tag{9}$$

therefore

(a) $-\frac{\alpha}{\cos\theta}$ is the *attenuation function* for N_* ,

(b) N_{*q} is the *equilibrium function* for N_* .

(iv) *The steady-state, source-free transport equation for vector irradiance* **H** has the form (the case n = 2 in (55) of Sec. 8.6):

$$\frac{d}{dz}\bar{H}(z,\mathbf{n},\Xi_o) = -[a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)]\bar{H}(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi'_o)\bar{H}(z,\mathbf{n},\Xi'_o) \tag{10}$$

Here we write:

$$"\bar{H}(z,\mathbf{n},\Xi_o)" \quad \text{for} \quad \mathbf{n}\cdot\mathbf{H}(z,\Xi_o) \quad ,$$

which is the component of $\mathbf{H}(z,\Xi_o)$ along the direction of the unit inward normal **n** to a unit area at depth z . $\mathbf{H}(z,\Xi_o)$ is the vector irradiance generated by radiant flux at z arriving from the general subregion Ξ_o of the unit sphere Ξ. If $\Xi_o = \Xi$, then $\mathbf{H}(z,\Xi_o) = \mathbf{H}(z)$ the usual vector irradiance at z. The quantity $\bar{H}(z,\mathbf{n},\Xi'_o)$ is the associated (net) irradiance on the unit area contributed by the complement Ξ'_o of Ξ_o with respect to Ξ . Because of the assumed stratification, $\mathbf{H}(z,\Xi_o)$ (and hence all its components) depends only on z . By writing:

$$"\bar{H}_q(z,n,\Xi_o)" \quad \text{for} \quad \frac{b(z,\mathbf{n},\Xi'_o)\,\bar{H}(z,\mathbf{n},\Xi'_o)}{a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)} \quad , \tag{11}$$

we may write (10) as:

$$\frac{d\bar{H}(z,\mathbf{n},\Xi_o)}{dz} = -[a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)][\bar{H}(z,\mathbf{n},\Xi_o) - \bar{H}_q(z,\mathbf{n},\Xi_o)] \tag{12}$$

so that:

(a) $a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)$ is the *attenuation function* for $\bar{H}(z,\mathbf{n},\Xi_o)$,

(b) $\bar{H}_q(z,\mathbf{n},\Xi_o)$ is the *equilibrium function* for $\bar{H}(z,\mathbf{n},\Xi_o)$.

The transport equation (10) is a generalization of the standard two-flow equations for $H(z,+)$ and $H(z,-)$ (Chapter 8). (In the latter case, for example, $\Xi_o = \Xi_-$, the downward hemisphere, and $\mathbf{n} = -\mathbf{k}$, where $\mathbf{k}$ is the unit outward normal to the plane-parallel medium.)

Each of the four preceding transport equations may be cast into a *canonical form* (see (2) of Sec. 11.2) by introducing the appropriate K-function for the associated radiometric quantity (see general definition (1) of Sec. 11.4). Therefore, a transport equation for each of these K-function exists, and is of the form (1) of Sec. 11.3. The equilibrium principle holds for N^n, N_*, $\mathbf{H}$, and U^n.

Summary and Conclusion

To summarize, the domain of applicability of the universal transport equation (1) of Sec. 11.3 is quite wide. In fact its domain covers the totality of radiometric functions used and known to date in radiative transfer theory (the 17 distinct types of radiometric concepts and their corresponding K-functions discussed above--34 concepts in all). By means of it, the general mathematical structure of the light field in plane-parallel media can be contained in a single unifying framework, and the necessity of invoking individual discussions and principles for each of the many radiometric quantities, at least on a logical level, is now obviated. All this from the interaction principle. Hence:

*Frustra fit per plura quod potest fieri per pauciora.**

William of Ockham (ca. 1300-1347)

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11.5 Bibliographic Notes for Chapter 11

The concept of a universal radiative transport equation was introduced in [240]. The mathematical vehicle of the universal radiative transport equation is that of a Riccati differential equation in factored form:

$$\frac{df(x)}{dx} = \mu(x)[\delta f(x) - f_\alpha(x)][f(x) - f_q(x)]$$

*It will be futile to employ more [principles] when it is possible to employ fewer.