to 12.14 a statistical theory of radiative transfer across a dynamic air-water surface is developed and applied to some illustrative examples. The chapter concludes with a brief study of possible devices which may be used in the laboratory to simulate the optical properties of randomly moving surfaces of natural hydrosols.

12.1 Reflectance and Transmittance Properties of the Static Surface

We begin the discussions of the reflectance and transmittance properties of the static surface with the simplest and most useful of the laws of geometric optics for our present purposes, namely:

The Geometric Law of Reflection

Figure 12.1(a) depicts a portion Y of an optical medium near a plane boundary interface S which separates Y into parts X' and X in which the indices of refraction are respectively n' and n. The surface S is a *mathematical* surface, i.e., one which has no thickness and serves merely to separate X' from X. A narrow beam of radiant flux in X' is incident along a direction ξ' at point x on the interface S. If $\mathbf{n}$ $(= \mathbf{k})$ is the unit normal to the surface S and is directed, as shown in (a) of Fig. 12.1, from X to X', then the part of the incident beam that is reflected at x back into X' is directed along ξ where ξ', ξ and $\mathbf{n}$ are related by the *law of reflection*:

$$\boxed{\frac{\xi - \xi'}{|\xi - \xi'|} = \mathbf{n}} \tag{1}$$

where "$|\xi - \xi'|$" denotes the magnitude of the difference $\xi - \xi'$ of the two unit vectors ξ' and ξ. From (1) we find (since $\xi + \xi'$ is perpendicular to $\xi - \xi'$) on dotting $(\xi + \xi')$ into each side:

$$\xi \cdot \mathbf{n} = - \xi' \cdot \mathbf{n} \quad , \tag{2}$$

which shows that the angles between the reflection direction ξ and $\mathbf{n}$, and between the incident direction ξ' and $\mathbf{n}$ are equal. Suppose we write:

$$\text{"}\theta'\text{"} \quad \text{for} \quad \arccos(-\xi' \cdot \mathbf{n})$$

and

$$\text{"}\theta\text{"} \quad \text{for} \quad \arccos(\xi \cdot \mathbf{n})$$

then (2) implies that:

$$\theta = \theta' \quad . \tag{3}$$

In addition, (1) summarizes the fact that ξ', ξ and $\mathbf{n}$ lie in a common plane, the *plane of incidence*. The significance

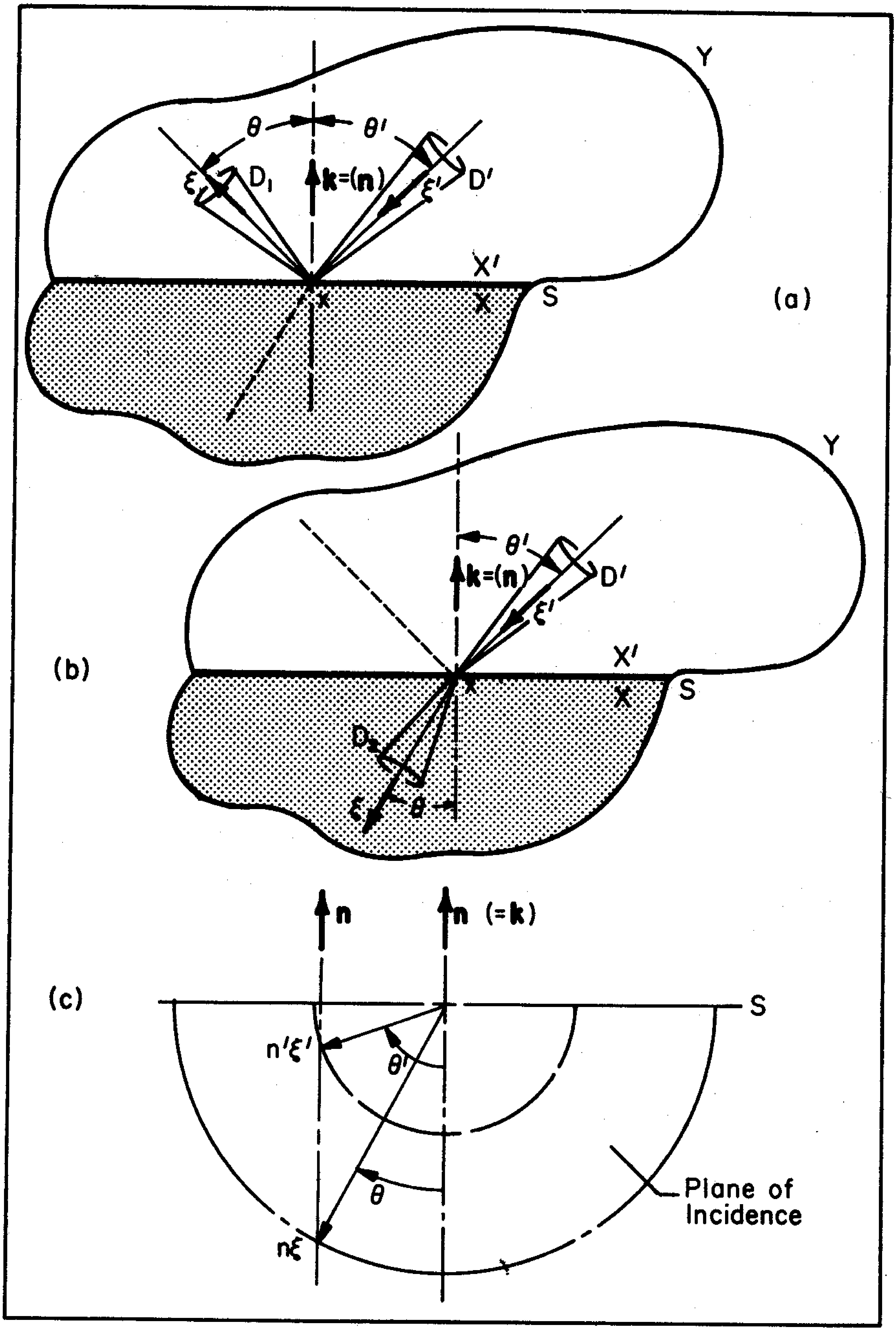

FIG. 12.1 Geometry for reflection and refraction laws for surface S .

of the law of reflection for our present purposes is that knowledge of any two of the three unit vectors ξ', ξ, and $\mathbf{n}$ is sufficient to determine the third. Thus e.g., knowing ξ' and ξ, we can find $\mathbf{n}$. An important application of this will occur later in Sec. 12.10 when we are devising ways of inferring the instantaneous orientation $\mathbf{n}$ of a wave facet's normal on the dynamic surface of a natural hydrosol, having measured ξ' and ξ.

The Geometric Law of Refraction

Fig. 12.1(b) depicts the refraction of a ray incident along ξ' at x on the interface S between X' and X, and refracted along ξ. The directions ξ, ξ', $\mathbf{n}$ are related by the *law of refraction:*

$$\boxed{(n\xi - n'\xi') \times \mathbf{n} = 0} \qquad (4)$$

where n' and n are again the indices of refraction of X' and X, respectively. This law summarizes two important facts: first, the refraction direction ξ along with ξ' and $\mathbf{n}$ lie in a common plane, which is the *plane of incidence* (cf. Fig. 12.2(a)). Second, since:

$$\xi' \times \mathbf{n} = \sin\theta'$$

$$\xi \times \mathbf{n} = \sin\theta \quad ,$$

where θ' and θ are defined above, (4) implies:

$$n' \sin\theta' = n \sin\theta \qquad (5)$$

which is the scalar version of (4) (and the most common representation of the law of refraction) known as *Snell's Law.* A simple graphical interpretation of (5) is shown in (c) of Fig. 12.1. The salient geometric fact to observe in connection with (5) is that if $n \geq n'$, then $\theta \leq \theta'$. Table 1 is a tabulation of angle pairs θ', θ related by (5), corresponding to the *relative index* of refraction $m = 4/3$ where we have written:

$$\text{"m" for } n/n' \qquad (6)$$

and where n' is the index of refraction of the incident medium and n that of the refracting medium. The most common pair of media to which Table 1 is applied is the air-water pair, for which $n' = 1$, $n = 4/3$.

TABLE 1

Snell's Law (m = 4/3)

θ'	θ	θ'	θ	θ'	θ
0°00'	0°00'	7°00'	5°15'	14°00'	10°27'
10'	08'	10'	22'	10'	35'
20'	15'	20'	30'	20'	42'
30'	23'	30'	37'	30'	49'
40'	30'	40'	45'	40'	57'
50'	38'	50'	52'	50'	11°04'
1°00'	45'	8°00'	6°00'	15°00'	12'
10'	53'	10'	07'	10'	19'
20'	1°00'	20'	15'	20'	26'
30'	08'	30'	22'	30'	34'
40'	15'	40'	29'	40'	41'
50'	23'	50'	37'	50'	48'
2°00'	30'	9°00'	44'	16°00'	56'
10'	38'	10'	52'	10'	12°03'
20'	45'	20'	59'	20'	11'
30'	53'	30'	7°07'	30'	18'
40'	2°00'	40'	14'	40'	25'
50'	08'	50'	22'	50'	33'
3°00'	15'	10°00'	29'	17°00'	40'
10'	23'	10'	37'	10'	47'
20'	30'	20'	44'	20'	55'
30'	38'	30'	51'	30'	13°02'
40'	45'	40'	59'	40'	09'
50'	52'	50'	8°06'	50'	17'
4°00'	3°00'	11°00'	14'	18°00'	13°24'
10'	07'	10'	21'	10'	31'
20'	15'	20'	29'	20'	39'
30'	22'	30'	36'	30'	46'
40'	30'	40'	44'	40'	53'
50'	37'	50'	51'	50'	14°01'
4°00'	45'	12°00'	8°58'	19°00'	08'
10'	52'	10'	9°06'	10'	15'
20'	4°00'	20'	13'	20'	23'
30'	07'	30'	21'	30'	30'
40'	15'	40'	28'	40'	37'
50'	22'	50'	36'	50'	45'
6°00'	4°30'	13°00'	43'	20°00'	52'
10'	37'	10'	50'	10'	59'
20'	45'	20'	58'	20'	15°06'
30'	52'	30'	10°05'	30'	14'
40'	5°00'	40'	13'	40'	21'
50'	07'	50'	20'	50'	28'

TABLE 1

Snell's Law (m = 4/3)--Continued.

θ'	θ	θ'	θ	θ'	θ
21°00'	15°36'	28°00'	20°37'	35°00'	25°29'
10'	43'	10'	44'	10'	36'
20'	50'	20'	51'	20'	42'
30'	57'	30'	58'	30'	49'
40'	16°05'	40'	21°05'	40'	56'
50'	12'	50'	12'	50'	26°03'
22°00'	19'	29°00'	19'	36°00'	09'
10'	26'	10'	26'	10'	16'
20'	34'	20'	33'	20'	23'
30'	41'	30'	40'	30'	30'
40'	48'	40'	47'	40'	36'
50'	55'	50'	54'	50'	43'
23°00'	17°02'	30°00'	22°01'	37°00'	50'
10'	10'	10'	08'	10'	57'
20'	17'	20'	15'	20'	27°03'
30'	24'	30'	22'	30'	10'
40'	31'	40'	29'	40'	17'
50'	38'	50'	36'	50'	23'
24°00'	46'	31°00'	43'	38°00'	30'
10'	53'	10'	50'	10'	37'
20'	18°00'	20'	57'	20'	43'
30'	07'	30'	23°04'	30'	50'
40'	14'	40'	11'	40'	57'
50'	22	50'	18'	50'	28°03'
25°00'	29'	32°00'	25'	39°00'	10'
10'	36'	10'	32'	10'	16'
20'	43'	20'	39'	20'	23'
30'	50'	30'	46'	30'	30'
40'	57'	40'	53'	40'	36'
50'	19°05'	50'	24°00'	50'	43'
26°00'	12'	33°00'	07'	40°00'	49'
10'	19'	10'	13'	10'	56'
20'	26'	20'	20'	20'	29°02'
30'	33'	30'	27'	30'	09'
40'	40'	40'	34'	40'	16'
50'	47'	50'	41'	50'	22'
27°00'	54'	34°00'	48'	41°00'	29'
10'	20°02'	10'	55'	10'	35'
20'	09'	20'	25°01'	20'	41'
30'	16'	30'	08'	30'	48'
40'	23'	40'	15'	40'	54'
50'	30'	50'	22'	50'	30°01'

TABLE 1

Snell's Law (m = 4/3)--Continued.

θ'	θ	θ'	θ	θ'	θ
42°00'	30°07'	49°00'	34°28'	56°00'	38°27'
10'	14'	10'	34'	10'	32'
20'	20'	20'	40'	20'	37'
30'	27'	30'	46'	30'	43'
40'	33'	40'	52'	40'	48'
50'	39'	50'	58'	50'	53'
43°00'	46'	50°00'	35°04'	57°00'	59'
10'	52'	10'	10'	10'	39°04'
20'	59'	20'	16'	20'	09'
30'	31°05'	30'	22'	30'	14'
40'	11'	40'	27'	40'	19'
50'	18'	50'	33'	50'	25'
44°00'	24'	51°00'	39'	58°00'	30'
10'	30'	10'	45'	10'	35'
20'	37'	20'	51'	20'	40'
30'	43'	30'	57'	30'	45'
40'	49'	40'	36°02'	40'	50'
50'	55'	50'	08'	50'	55'
45°00'	32°02'	52°00'	14'	59°00'	40°00'
10'	08'	10'	19'	10'	05'
20'	14'	20'	25'	20'	10'
30'	20'	30'	31'	30'	15'
40'	27'	40'	37'	40'	20'
50'	33'	50'	42'	50'	25'
46°00'	39'	53°00'	48'	60°00'	30'
10'	45'	10'	53'	10'	35'
20'	51'	20'	59'	20'	40'
30'	58'	30'	37°05'	30'	45'
40'	33°04'	40'	10'	40'	50'
50'	10'	50'	16'	50'	55'
47°00'	16'	54°00'	21'	61°00'	41°00'
10'	22'	10'	27'	10'	04'
20'	28'	20'	32'	20'	09'
30'	34'	30'	38'	30'	14'
40'	40'	40'	43'	40'	19'
50'	46'	50'	49'	50'	23'
48°00'	33°52'	55°00'	54'	62°00'	28'
10'	58'	10'	38°00'	10'	07'
20'	34°04'	20'	05'	20'	11'
30'	34°04'	30'	11'	30'	15'
40'	16'	40'	16'	40'	19'
50'	22'	50'	21'	50'	23'

TABLE 1

Snell's Law (m = 4/3)--Continued.

θ'	θ	θ'	θ	θ'	θ
63°00'	41°56'	70°00'	44°49'	77°00'	46°57'
10'	42°01'	10'	52'	10'	47°00'
20'	05'	20'	56'	20'	02'
30'	10'	30'	59'	30'	04'
40'	14'	40'	45°03'	40'	07'
50'	19'	50'	06'	50'	09'
64°00'	23'	71°00'	10'	78°00'	11'
10'	27'	10'	13'	10'	14'
20'	32'	20'	17'	20'	16'
30'	36'	30'	20'	30'	18'
40'	41'	40'	24'	40'	20'
50'	45'	50'	27'	50'	23'
65°00'	49'	72°00'	30'	79°00'	25'
10'	54'	10'	33'	10'	27'
20'	58'	20'	37'	20'	29'
30'	43°02'	30'	40'	30'	31'
40'	06'	40'	43'	40'	33'
50'	11'	50'	46'	50'	35'
66°00'	15'	73°00'	50'	80°00'	37'
10'	19'	10'	53'	10'	39'
20'	23'	20'	56'	20'	41'
30'	27'	30'	59'	30'	42'
40'	31'	40'	46°02'	40'	44'
50'	35'	50'	05'	50'	46'
67°00'	40'	74°00'	08'	81°00'	48'
10'	44'	10'	11'	10'	50'
20'	48'	20'	14'	20'	51'
30'	52'	30'	17'	30'	53'
40'	56'	40'	20'	40'	55'
50'	44°00'	50'	23'	50'	56'
68°00'	04'	75°00'	25'	82°00'	58'
10'	07'	10'	28'	10'	59'
20'	11'	20'	31'	20'	48°01'
30'	15'	30'	34'	30'	02'
40'	19'	40'	36'	40'	04'
50'	23'	50'	39'	50'	05'
69°00'	27'	76°00'	42'	83°00'	07'
10'	30'	10'	44'	10'	08'
20'	34'	20'	47'	20'	09'
30'	38'	30'	50'	30'	10'
40'	41'	40'	52'	40'	12'
50'	45'	50'	55'	50'	13'

TABLE 1

Snell's Law (m = 4/3)--Continued.

θ'	θ	θ'	θ	θ'	θ
84°00'	48°14'	86°00'	48°26'	88°00'	48°33'
10'	15'	10'	27'	10'	33'
20'	16'	20'	27'	20'	34'
30'	18'	30'	28'	30'	34'
40'	19'	40'	29'	40'	34'
50'	20'	50'	29'	50'	35'
85°00'	21'	00'	30'	89°00'	35'
10'	22'	10'	31'	10'	35'
20'	23'	20'	31'	20'	35'
30'	23'	30'	32'	30'	35'
40'	24'	40'	32'	40'	35'
50'	25'	50'	33'	50'	35'
				90°00'	35'

The Fresnel Laws for Reflectance

The laws of reflection (1) and refraction (4) may be derived from Maxwell's equations for electromagnetic waves in dielectric media in a very simple manner (see, e.g., [292]). The derivations automatically yield not only (1) and (4) but the amount of radiant flux reflected back into X' (as in Fig. 12.1(a)) and refracted into X (as in Fig. 12.1(b)). We consider now the laws, originally derived by Fresnel from Maxwell's equations, which govern the amount of *reflected* radiant flux.

In explaining the basis of Fresnel laws, it is necessary to revert momentarily from the radiometric picture of light to the electromagnetic picture of light (re Sec. 2.2). Figure 12.2 gives a perspective view of the situation in Fig. 12.1. Recall that the three vectors ξ', ξ, $\mathbf{n}$ (= $\mathbf{k}$) lie in a common plane, the *plane of incidence*. Along the incident ray direction ξ' moves an electric vector $\mathbf{E}'$ which, by the transverse nature of electromagnetic waves, oscillates in a plane normal to ξ'. A small circular patch of this plane, for three orientations, is depicted in Fig. 12.2. Now, in analyzing what happens to $\mathbf{E}'$ as it strikes S at x, it is found convenient and possible (because of the linearity of the Maxwellian theory) to resolve $\mathbf{E}'$ into the equivalent sum of two components whose magnitudes are $E'_\perp$ and $E'_{||}$, and which are, respectively, perpendicular and parallel to the plane of incidence, and which still lie in the plane of $\mathbf{E}'$. If it is known how $\mathbf{E}'_\perp$ is reflected and refracted at x, and similarly for $\mathbf{E}'_{||}$, then the behavior of $\mathbf{E}'$ at x is completely determined. It can be shown (see, e.g., [292]) that the magnitude of the reflected perpendicular component $E_\perp$ is related to the magnitude $E'_\perp$ at

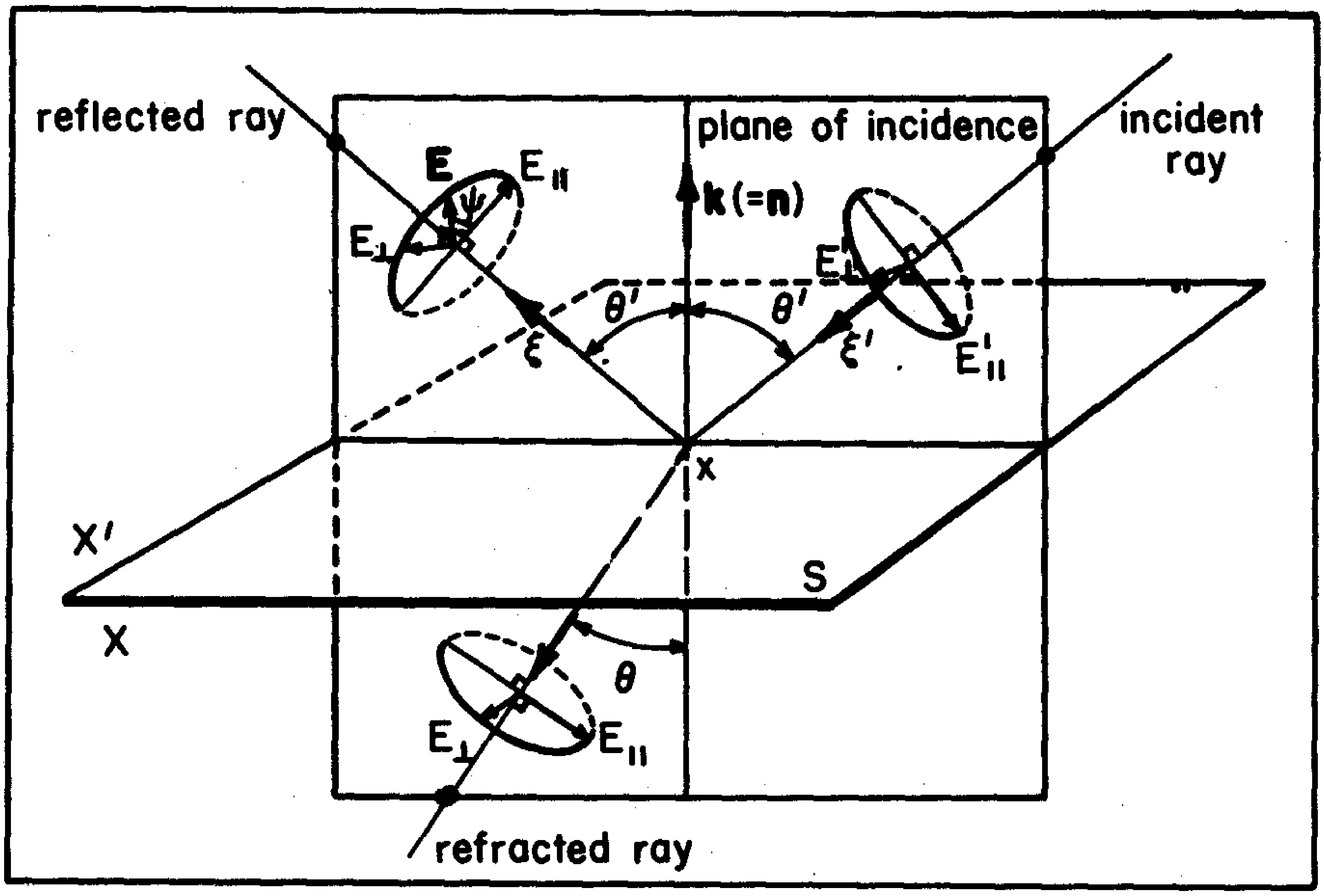

FIG. 12.2 Direction space conventions for general reflectance calculations.

x by:

$$E_\perp = -\frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)} E'_\perp \quad .$$

Further, the reflected parallel component $E_\parallel$ is related to $E'_\parallel$ at x by:

$$E_\parallel = \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)} E'_\parallel$$

where θ is the angle of the refracted ray and where θ and θ' are related by Snell's law (5).

We can use the preceding relations along with the classical results (4) and (5) to predict the connection between incident and reflected radiance as would be observed using radiance meters at x. Thus if $N(x,\xi')$ is the incident radiance at x and $N(x,\xi)$ the reflected radiance, we first note that*

---

*To within a fixed factor, involving the dielectric constant of X for a given wavelength. As long as one works in an arbitrary, but fixed homogeneous medium, the connections (7) are adequate, since instruments can be appropriately calibrated.

$$\left.\begin{aligned} N(x,\xi') &= |\bar{E}'(x)|^2 \,/\, 4\pi \\ \text{and} & \\ N(x,\xi) &= |\bar{E}(x)|^2 \,/\, 4\pi \quad . \end{aligned}\right\} \qquad (7)$$

here $|\bar{E}'(x)|^2$ is the mean square amplitude of the incident electric vector $\mathbf{E}'$, where the average is taken over some suitable interval T of time (say on the order of a hundredth of a second). Similar definitions hold for $N(x,\xi)$. For a derivation of the general relations in (7), along with a discussion of the conditions under which they generally can be used, the reader is referred to Sec. 124 of Ref. [251]. It suffices to observe here that the main conditions of validity of (7) hold in virtually all natural radiometric environments lighted by the sun or by most man-made artificial sources in either the atmosphere or the hydrosphere. These main conditions are made explicit below.

Next we observe that for the case of steady unpolarized light, $\mathbf{E}(x,t)$ arrives at x at time t with steady sinusoidal frequency, but with random orientation, over the interval T ($\cong 10^{-2}$ sec.). Let $E'(x,t)$ be the magnitude of $\mathbf{E}(x,t)$. If $E'(x)$ is the maximum value attained by $E'(x,t)$, then $E'(x,t) = E'(x) \cos \omega t$, and:

$$E'_{\perp}(x,t) = E'(x,t) \sin \psi(t) \qquad (8)$$

$$E'_{\parallel}(x,t) = E'(x,t) \cos \psi(t) \qquad (9)$$

where $\psi(t)$ is the angle $\mathbf{E}(x,t)$ makes with the plane of incidence at time t (Fig. 12.2). The reflected vector $\mathbf{E}(x,t)$ is then given at each instant by:

$$\mathbf{E}(x,t) = E_{\parallel}(x,t)\mathbf{e}_{\parallel} + E_{\perp}(x,t)\mathbf{e}_{\perp}$$

Errata

where $\mathbf{e}_{\parallel}$ and $\mathbf{e}_{\perp}$ are unit vectors perpendicular and parallel to the plane of incidence and such that $\mathbf{e}_{\parallel} \times \mathbf{e}_{\perp} = \xi$. In view of (8),(9) we have:

$$\begin{aligned} \mathbf{E}(x,t) &= + \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)} E'_{\parallel}(x,t)\mathbf{e}_{\parallel} - \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)} E'_{\perp}(x,t)\mathbf{e}_{\perp} \\ &= \left[- \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)} \cos \psi(t)\mathbf{e}_{\parallel} + \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)} \sin \psi(t)\mathbf{e}_{\perp} \right] E'(x,t) \end{aligned} \qquad (10)$$

It follows that, since $\psi(t)$ oscillates randomly over all values in the interval $0 \leq \psi(t) \leq 2\pi$, and with great frequency during T (about 6×10^{14} cycles per second) the mean square value $|\bar{E}(x)|^2$ of $\mathbf{E}(x,t)$ over T is:

$$|\bar{E}(x)|^2 = \frac{1}{2} \left[\frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)} + \frac{\tan^2(\theta' - \theta)}{\tan^2(\theta' + \theta)} \right] |\bar{E}'(x)|^2 \quad . \qquad (11)$$

The conditions on the randomness of $\psi(t)$ and on the great frequency ω, which lead to (11), are also those that enter into the derivation of (7), so that (7) and (11) combine to yield:

$$\boxed{N(x,\xi) = N(x,\xi')r(\xi',\xi)} \qquad (12)$$

where we write:

"$r(\mathbf{n},\xi',\xi)$" or "$r(\xi',\xi)$" or "$r(\theta')$" for

$$\frac{1}{2}\left\{\frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)} + \frac{\tan^2(\theta' - \theta)}{\tan^2(\theta' + \theta)}\right\} \quad . \qquad (13a)$$

The number $r(\mathbf{n},\xi',\xi)$, (or $r(\xi',\xi)$, if the unit outward normal $\mathbf{n}$ to the surface is understood) is the *Fresnel reflectance* of the surface S for unpolarized electromagnetic fields. Equation (12) is the most important and frequently used form of the radiance reflectance law. Tabulations of $r(\theta)$ are given in Table 2, and are adapted from [183]. The relative index of refraction m is not explicitly shown in the notation. If it is needed explicitly, we could write "$r(m,\xi',\xi)$" for $r(\xi',\xi)$, or "$r(m,\theta')$" for $r(\theta')$, as convenience indicates. To convert from radiance to degrees, use the relation,1 radian = 57.296 degrees ≅ 57.30 degrees. (Reference [183] also tabulates reflectances for linearly polarized light so that, together with (14) below, reflectances for arbitrary incident orientations of the E-vector are determinable.)

Errata

The radiance reflectance law (12) can be supplemented by the law of reflection for linearly polarized radiance, with fixed orientation of the $\mathbf{E}$ vector at angle ψ, as in Fig. 12.2. The result is:

$$\boxed{N(x,\xi) = N(x,\xi')r(\xi',\xi;\psi)} \qquad (13)$$

where we write:

"$r(\xi',\xi;\psi)$" or "$r(\theta';\psi)$" for

$$\left[\frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)}\cos^2\psi + \frac{\tan^2(\theta' - \theta)}{\tan^2(\theta' + \theta)}\sin^2\psi\right] \quad . \qquad (14)$$

In general, both the incident radiant flux and the reflected radiant flux at an interface S between two media X' and X of different indices of refraction will be partially polarized, so that (11) and (13) are ideal special cases. By using the operational definitions of polarized radiance given in Sec. 2.10, the detailed empirical study of reflected, refracted and scattered polarized radiance fields is possible in natural optical media. However, for many practical purposes formulas (11) and (13) serve adequately (separately or jointly) to give quantitative estimates of the reflected radiance at interfaces S. Observe that $r(\xi',\xi;\psi)$ in (14) may

TABLE 2

Fresnel Reflection m = 4/3

(Superscripts refer to the number of decimal zeros before the tabulated entry. Thus "^{1}20408 163" stands for 0.02408 163. This holds for all entires down the table until the next superscript at: ^{0}10834 505.)

θ (radians)	r	θ (radians)	r
0.00	^{1}20408 163	0.80	28708 037
0.02	20408 165	0.82	29875 821
0.04	20408 191	0.84	31192 993
0.06	20408 306	0.86	32677 961
0.08	20408 616	0.88	34351 455
0.10	20409 273	0.90	36236 826
0.12	20410 474	0.92	38360 401
0.14	20412 465	0.94	40751 873
0.16	20415 541	0.96	43444 754
0.18	20420 053	0.98	46476 892
0.20	20426 410	1.00	49891 050
0.22	20435 082	1.02	53735 576
0.24	20446 607	1.04	58065 163
0.26	20461 597	1.06	62941 719
0.28	20480 744	1.08	68435 355
0.30	20504 828	1.10	74625 517
0.32	20534 725	1.12	81602 284
0.34	20571 419	1.14	89467 841
0.36	20616 011	1.16	98338 183
0.38	20669 735	1.18	^{0}10834 505
0.40	20733 967	1.20	11963 817
0.42	20810 245	1.22	13238 780
0.44	20900 287	1.24	14678 768
0.46	21006 009	1.26	16305 844
0.48	21129 552	1.28	18145 151
0.50	21273 303	1.30	20225 368
0.52	21439 928	1.32	22579 233
0.54	21632 403	1.34	25244 160
0.56	21854 051	1.36	28262 953
0.58	22108 585	1.38	31684 645
0.60	22400 153	1.40	35565 483
0.62	22733 394	1.42	39970 095
0.63	23113 495	1.44	44972 871
0.66	23546 261	1.46	50659 613
0.68	24038 191	1.48	57129 513
0.70	24596 565	1.500	64497 533
0.72	25229 540	1.505	66495 071
0.74	25946 265	1.510	68559 275
0.76	26757 005	1.515	70692 527
0.78	27673 281	1.520	72897 299

TABLE 2

Fresnel Reflection m = 4/3--Continued.

θ (radians)	r		θ (radians)	r	
1.525	75176	165	1.550	87779	435
1.530	77531	800	1.555	90563	010
1.535	79966	989	1.560	93441	860
1.540	82484	628	1.565	96419	529
1.545	85087	730	1.570	99499	715
			π/2	1.00000	000

also be written as:

$$r(\xi',\xi;\psi) = r(\xi',\xi;0)\cos^2\psi + r(\xi',\xi;\pi/2)\sin^2\psi \qquad (15)$$

For brevity we usually write "$r_\perp(\theta)$" for $r(\xi',\xi;\pi/2)$ and "$r_\parallel(\theta)$" for $r(\xi',\xi;0)$. These values are tabulated in [183].

The Fresnel Laws for Transmittance

Having found the quantitative law for reflection of radiance ((12) or (13)) we can deduce with relative ease the associated law for the transmission of radiance across the interface S (Fig. 12.1(b)). Suppose the radiant flux content of a beam incident at x via a solid angle D' is P(S',D') where S' is a small plane surface normal to ξ' at x . In terms of radiance this is:

$$P(S',D') = N(S',D')A(S')\Omega(D')$$

Now the flux comprising the reflected radiance leaves S at x in a set of directions D_1 such that:

$$\Omega(D') = \Omega(D_1) \quad ,$$

which is a simple consequence of (1). Furthermore, the projection of S' along ξ' down onto S defines on S a patch of surface S" which, when subsequently projected on a plane perpendicular to ξ clearly defines another patch of surface S_1, such that $A(S_1) = A(S)$. It then follows from (12) that the connection between the incident and reflected radiant flux $P(S_1,D_1)$ at x is:

$$\boxed{P(S_1,D_1) = P(S',D')r(\xi',\xi)} \qquad (16)$$

Since no absorption of radiant flux takes place at x, it is now clear that an amount $P(S_2,D_2)$, where

$$\boxed{P(S_2,D_2) = (1 - r(\xi',\xi))P(S',D')} \quad , \qquad (17)$$

is transmitted through the interface S along the various refraction directions within the refracted direction set D_2. Here S_2 is the projection of S' on a plane normal to ξ, in the manner that S_1 was defined. It is this amount of flux that now goes on to comprise the transmitted radiance in X. It follows that the n^2-law for radiance ((14) of Sec. 2.6) Errata now takes the form:

$$\boxed{\frac{N(x,\xi)}{n_2^2} = \frac{N(x,\xi')}{n_1^2}\, t(\xi',\xi)} \qquad (18)$$

where we have written:

$$\text{"}t(\xi',\xi)\text{"} \quad \text{for} \quad 1 - r(\xi',\xi) \quad . \qquad (19)$$

A complete derivation of (18) can be based on the discussion following (4) of Sec. 2.6. Equation (18) also holds for the polarized case, in which case we would use $r(\xi',\xi;\psi)$ in (17).

Example 1: Reflectance Under Uniform Radiance Distributions

As an illustration of the use of the Fresnel reflectance law (12), we shall develop an exact formula for the reflectance of an interface between two media of relative index of refraction $m \geq 1$, as irradiated by unpolarized radiant flux from the side of index of refraction 1. To point up the fact that irradiation is incident in this direction we call the associated reflectance the *external reflectance*. If the flux was incident from the side with index m, then the reflectance would be *internal reflectance*. Figure 12.3 (a) depicts the point x on a surface S irradiated by radiant flux streaming onto x over the hemisphere $\Xi_-(\mathbf{x}) = \Xi(\mathbf{k}')$, where $\mathbf{k}'$ is the unit inward normal to S at x. Then by (8) of Sec. 2.5, the irradiance on S at x is

$$H(x,\mathbf{k}') = \int_{\Xi_-(x)} N(x,\xi')\ \xi' \cdot \mathbf{k}' d\Omega(\xi') \qquad (20)$$

where $\Xi(\xi)$ in (8) of Sec. 2.5 is now specifically of the form $\Xi_-(x)$, as introduced in Sec. 3.3, which is customarily used for work with actual surfaces. Now according to (2), the radiance along direction ξ' is reflected along direction ξ, where:

$$\xi' \cdot \mathbf{k}' = \xi \cdot \mathbf{k} \qquad (21)$$

and where $\mathbf{k}$ is the unit outward normal to S at x. Using this and (12), we have:

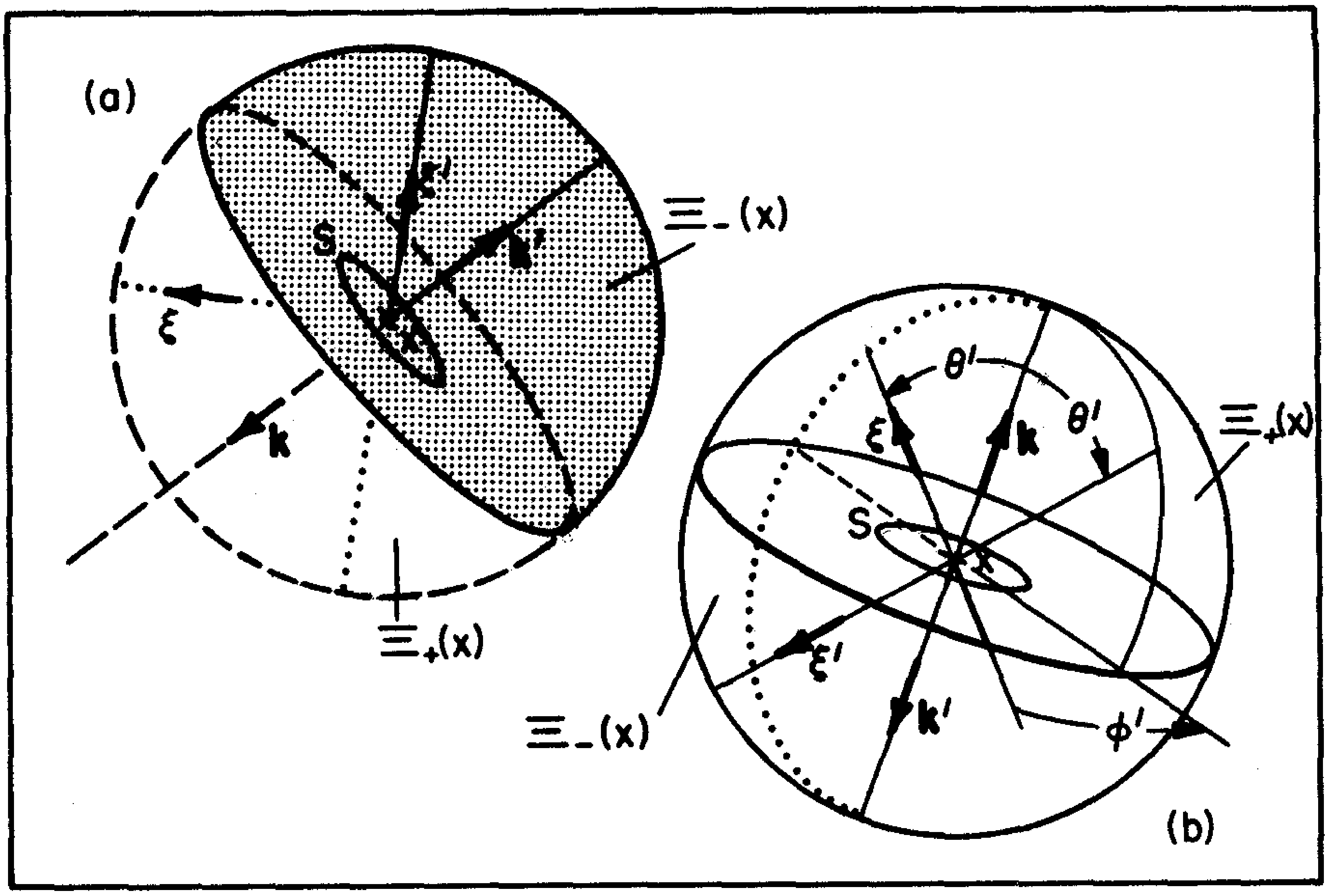

FIG. 12.3 Direction-space conventions for general reflectance calculations.

$$N(x,\xi)\;\xi\cdot\mathbf{k} = N(x,\xi')r(\xi',\xi)\xi'\cdot\mathbf{k}' \tag{22}$$

This equation relates the *irradiance* on S induced at x by N(x,ξ'), to the resultant *radiant emittance* of S at x associated with the reflected flux. Hence the total radiant emittance associated with $H(x,\mathbf{k}')$ is:

$$W(x,\mathbf{k}) = \int_{\Xi_-(x)} N(x,\xi')r(\xi',\xi)\xi'\cdot\mathbf{k}'\; d\Omega(\xi') \tag{23}$$

where $W(x,\mathbf{k})$ is defined in (22) of Sec. 2.4. Then, by (19) of Sec. 3.3, we have as the external reflectance $r_-(x)$ of S at x for irradiance:

$$r_-(x) = W(x,\mathbf{k})/H(x,\mathbf{k}') \tag{24}$$

Combining this with (20) and (23),

$$\boxed{r_-(x) = \frac{\displaystyle\int_{\Xi_-(x)} N(x,\xi')r(\xi',\xi)\xi'\cdot\mathbf{k}'d\Omega(\xi')}{\displaystyle\int_{\Xi_-(x)} N(x,\xi')\xi'\cdot\mathbf{k}'\; d\Omega(\xi')}}\;. \tag{25}$$

In the present example, we require N(x,ξ') to be independent of ξ'; so that (25) reduces to:

$$r_-(x) = \frac{\displaystyle\int_{\Xi_-(x)} r(\xi',\xi)\xi'\cdot\mathbf{k}'d\Omega(\xi')}{\displaystyle\int_{\Xi_-(x)} \xi'\cdot\mathbf{k}'d\Omega(\xi')} \quad . \qquad (26)$$

Equation (26) can be readied for evaluation by introducing an appropriate coordinate system. Such a coordinate system is depicted in (b) of Fig. 12.3, which serves as a transition diagram between the standard orientations of Figs. 12.1, 12.2, and the general situation depicted in Fig. 12.3 (a). Thus, in the framework of Fig. 12.3(b), (26) becomes:

$$r_-(x) = \frac{\displaystyle\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi/2} r(\theta')\cos\theta'\sin\theta'\,d\theta'\,d\phi'}{\displaystyle\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi/2} \cos\theta'\sin\theta'\,d\theta'\,d\phi'} \;.(27)$$

The transition from the solid angle measure Ω to the θ',ϕ' representation of $d\Omega(\xi')$ is given in (9) of Sec. 2.5, and $r(\theta')$ is given in (12). The fixed radiance value over $\Xi_-(x)$ has been cancelled from the integrals. The denominator of (27) is clearly π, that is:

$$\int_{\phi'=0}^{2\pi}\int_{\phi'=0}^{\pi/2} \cos\theta'\sin\theta'\,d\theta'\,d\phi' = \pi \qquad (28)$$

and the numerator is reducible to:

$$2\pi\int_{\theta'=0}^{\pi/2} r(\theta')\cos\theta'\sin\theta'\,d\theta' \quad ,$$

so that: (29)

$$r_- = 2\int_{\theta'=0}^{\pi/2} r(\theta')\cos\theta'\sin\theta'\,d\theta'$$

in which all reference to x has been dropped. Evaluating the integral in (29) for relative index of refraction $m(\geq 1)$, we have:

$$\boxed{\begin{aligned} r_- = \tfrac{1}{2} &+ [(m-1)(3m+1)/6(m+1)^2] - [2m^3(m^2+2m-1)/(m^2+1)(m^4-1)] \\ &+ [8m^4(m^4+1)/(m^2+1)(m^4-1)^2]\ \ln m \\ &+ [m^2(m^2-1)^2/(m^2+1)^3]\ \ln[(m-1)/(m+1)] \end{aligned}}$$

(30)

This representation of r_- was first worked out by Walsh [310] and applied in his studies of reflectances of polished glass surfaces. For glass with $m = 1.5$, it follows from (30) that $r_- = 0.092$. In the present studies, the relative index of refraction $m = 4/3 = 1.33$ for water is of central interest and for this, the associated r_-, as given by (30), is 0.066. Thus under a uniformly overcast sky, approximately 6.6 percent of the incident radiant flux on a static air-water surface is reflected from the surface. A corresponding exact algebraic formula for the *internal reflectance* r_+ apparently has never been worked out. Numerical integrations by Judd [131] indicate that $r_+ = 0.596$ for $m = 1.5$ and $r_+ = 0.472$ for $m = 4/3$. These values are listed, along with others, in Table 3.

TABLE 3

Reflectance of unpolarized light at a plane boundary between two media as a function of their relative index of refraction, m.

m	Reflectance for Perpendicular Incidence	Reflectance for Completely Diffuse Incidence	
		External Reflection	Internal Reflection
1.00	0.00000	0.0000	0.000
1.01	0.00002	0.0028	0.022
1.02	0.00010	0.0055	0.044
1.03	0.00022	0.0082	0.064
1.04	0.00038	0.0108	0.084
1.05	0.00059	0.0134	0.103
1.06	0.00085	0.0158	0.122
1.07	0.00114	0.0183	0.140
1.08	0.00148	0.0206	0.158
1.09	0.00185	0.0230	0.175
1.10	0.00227	0.0252	0.192
1.11	0.00272	0.0274	0.208
1.12	0.00320	0.0294	0.224
1.13	0.00372	0.0314	0.240
1.14	0.00428	0.0334	0.254
1.15	0.00487	0.0353	0.269
1.16	0.00549	0.0371	0.283
1.17	0.00614	0.0389	0.296
1.18	0.00682	0.0407	0.309
1.19	0.00753	0.0425	0.322
1.20	0.00826	0.0443	0.335
1.21	0.00903	0.0461	0.347
1.22	0.00982	0.0478	0.359
1.23	0.1064	0.0496	0.371
1.24	0.01148	0.0513	0.382
1.25	0.01235	0.0530	0.393

TABLE 3

Reflectance of unpolarized light at a plane boundary between two media as a function of their relative index of refraction, m.--Continued.

m	Reflectance for Perpendicular Incidence	Reflectance for Completely Diffuse Incidence	
		External Reflection	Internal Reflection
1.26	0.012]2	0.0546	0.404
1.27	0.01415	0.0563	0.404
1.28	0.01508	0.0579	0.424
1.29	0.01604	0.0596	0.434
1.30	0.01701	0.0612	0.444
1.31	0.01801	0.0628	0.454
1.32	0.01902	0.0644	0.463
1.33	0.02006	0.0660	0.472
1.34	0.02111	0.0676	0.480
1.35	0.02218	0.0692	0.489
1.36	0.02327	0.0707	0.497
1.37	0.02437	0.0723	0.505
1.38	0.02549	0.0738	0.513
1.39	0.02663	0.0754	0.520
1.40	0.02778	0.0769	0.528
1.41	0.02894	0.0784	0.536
1.42	0.03012	0.0800	0.543
1.43	0.03131	0.0815	0.550
1.44	0.03252	0.0830	0.557
1.45	0.0337	0.0845	0.564
1.46	0.03497	0.0860	0.571
1.47	0.03621	0.0875	0.577
1.48	0.03746	0.0890	0.584
1.49	0.03873	0.0904	0.590
1.50	0.4000	0.0919	0.596
1.51	0.04129	0.0934	0.602
1.52	0.04258	0.0948	0.608
1.53	0.04389	0.0963	0.614
1.54	0.04520	0.0977	0.619
1.55	0.04652	0.0992	0.624
1.56	0.04785	0.1006	0.630
1.57	0.04919	0.1020	0.635
1.58	0.05054	0.1035	0.640
1.59	0.05189	0.1049	0.645
1.60	0.05325	0.1063	0.650

Example 2. Reflectance Under Cardioidal Radiance Distributions

What would be the reflectance of the sea surface if it were absolutely calm and exposed to a heavily overcast sky? This is the problem we pose and solve in this example. Now, the actual form of the radiance distribution under a heavily overcast sky is not uniform as that discussed in Example 1, but more nearly of a *cardioidal form:*

$$N(x,\xi') = N(x,\xi_0)(1 - 2\xi' \cdot \mathbf{k}) \qquad (31)$$

where ξ_0 is any fixed horizontal direction, i.e., $\xi_0 \cdot \mathbf{k} = 0$, and where $\mathbf{k}$ is a unit vector directed toward the zenith. (Recall that ξ' is the direction of flow of the photons comprising $N(x,\xi')$.) Thus if in Fig. 12.3(b), $\mathbf{k}$ is directed toward the zenith and ξ' is any downward radiance, then (31) gives the (unpolarized) radiance $N(x,\xi')$. Equation (31) is an empirical law, found by Moon and Spencer [186]. Further empirical confirmation of (31) was made by Hopkinson [112]. Equation (31) is of the same general family as that in (14) of Sec. 6.6. That is, (31) is closely related to the solutions of the classical diffusion theory for plane-parallel media. In the case of (14) of Sec. 6.6, which holds for practical situations such as the present one, the radiance represented there is at a relatively great depth in a plane-parallel medium (assuming Fick's law for photons holds in that medium). A closely related form to (31) was predicted theoretically by Schwarzschild [282] and later by Chandrasekhar [43]. Fig. 12.4

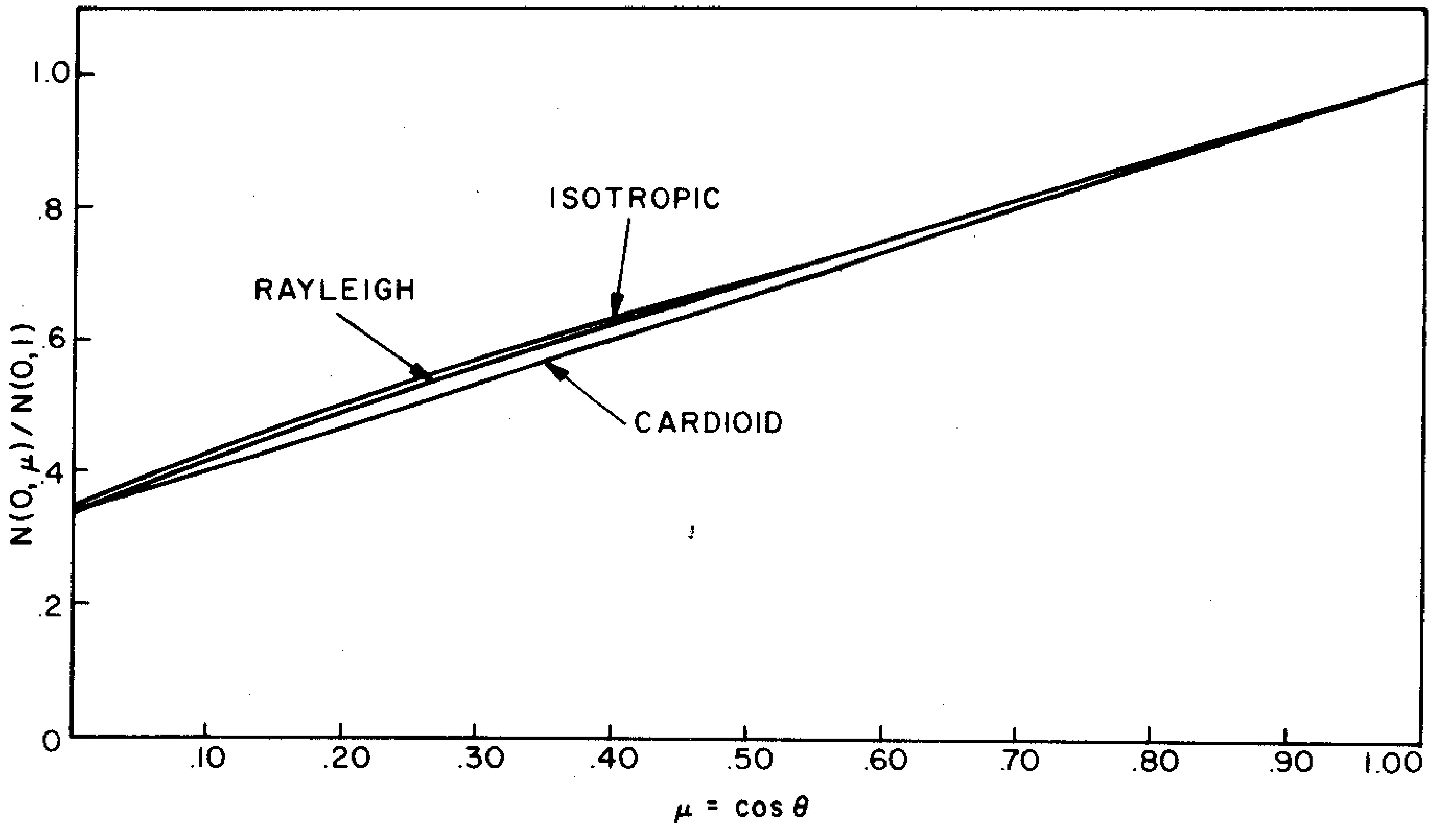

FIG. 12.4 Emergent radiance distributions for an atmosphere with no appreciable absorption. For the cases of Rayleigh and isotropic scattering functions, as compared with an emergent cardioidal radiance distribution.

compares the empirical cardioidal radiance law (31) with two theoretical radiance distributions based on isotropic scattering and Rayleigh scattering forms for σ. These computed distributions are partly based on those on page 135 in [43]. In the present example we shall find an exact representation of $r_-(x)$ for the case of a cardioidal radiance distribution of the form (31) and for the same setting as in Example 1. We shall in fact use a general form of (31) in which 2 is replaced by an arbitrary real number n. In this way we shall generalize (30), which is the case for n = 0.

Thus, using the coordinate frame of Fig. 12.3 (b), the general form of (31) may be written:

$$N(x,\theta',\phi') = N(x,\pi/2,\phi')(1 + n \cos \theta') \quad . \qquad (32)$$

Furthermore (25), with $N(x,\xi')$ given by (32), now becomes:

$$r_-(m,n) = \frac{\int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} [1+n \cos \theta']r(\theta') \cos \theta' \sin \theta' \, d\theta' \, d\phi'}{\int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} [1 + n \cos \theta'] \cos \theta' \sin \theta' \, d\theta' \, d\phi'} \qquad (33)$$

where we have written "$r_-(m,n)$" for $r_-(x)$ to point up the dependence of the reflectance on the two parameters $m \geq 1$; (the relative index of refraction) and n (the shape index of the radiance distribution). Clearly $r_-(m,0)$ is the r_- of (30), so that $r_-(m,n)$, when evaluated, will be a proper generalization of Walsh's formula for external reflectance. Once again, irradiation is from the side with index of refraction 1. We now outline the manner in which the exact form of $r_-(m,n)$ may be obtained.

We begin by making a preliminary simplification of (33) by performing the integrations over the azimuth angles ϕ':

$$r_-(m,n) = \frac{\int_{\theta'=0}^{\pi/2} [1 + n \cos \theta']r(\theta') \cos \theta' \sin \theta' \, d\theta'}{\int_{\theta'=0}^{\pi/2} [1 + n \cos \theta'] \cos \theta' \sin \theta' \, d\theta'} \quad . \qquad (34)$$

The denominator of this fraction, which we shall denote by "H'(n)" , is easily evaluated:

$$H'(n) = \frac{1}{2} + \frac{n}{3} \quad . \qquad (35)$$

The numerator of $r_-(m,n)$, which we shall designate by "W'(m,n)" , is relatively difficult to evaluate because of the presence of the factor $r(\theta')$ in the integrand. It is found that by a suitable pair of transformations of variables, done in tandem, the relatively complex numerator W'(m,n) of $r_-(m,n)$ may be systematically disassembled into manageable pieces. Thus, first we write "ϕ'" for the difference $\theta'-\theta$ occurring in the representation (13a) of $r(\theta')$, and using trigonometry with Snell's law (5), we eventually arrive at the following representation of Fresnel's reflectance law:

$$r_-(\phi') = [2m^2/(m^2-1)][(\cos\phi'-a)^2 + (\sec\phi'-a)^2] \qquad (36)$$

in which we have now written "$r_-(\phi')$" for $r_-(\theta')$ and:

$$\text{"a"} \quad \text{for} \quad (m^2+1)/2m \qquad (37)$$

and where $0 \le \phi' < \arccos(1/m)$. The term $\cos\phi'$ now plays the prominent role in $r(\phi')$, and we may thus simplify $W'(m,n)$ by writing "x" for $\cos\phi'$ and "s" for $1/m$, so that with this second transformation of variables, we eventually obtain:

$$W'(m,n) =$$

$$= \frac{1}{2}\int_1^s [1+n(x-s)/(1+s^2-2sx)^{1/2}]\left\{[(x-a)^2 + (\tfrac{1}{x}-a)^2]/(1-s^2)^2\right\} \times [(sx-1)(x-s)/(x-a)^2]\,dx \quad . \qquad (38)$$

Clearly $W'(m,n)$ can be written in the form:

$$W'(m,n) = A(m) + nB(m) \qquad (39)$$

where we write:

"A(m)" for

$$\frac{1}{2}\int_1^s \left\{\left[(x-a)^2 + \left(\frac{1}{x}-a\right)^2\right]\left[(sx-1)(x-s)\right]/(x-a)^2(1-s^2)^2\right\}dx \qquad (40)$$

and

$$\text{"B(m)"} \quad \text{for} \quad \frac{1}{2}\int_1^s \frac{\left[(x-a)^2 + \left(\frac{1}{x}-a\right)^2\right](sx-1)(x-s)^2\,dx}{(x-a)^2(1-s^2)^2(1+s^2-2sx)^{1/2}} \qquad (41)$$

Hence:

$$\boxed{r_-(m,n) = 6[A(m) + n\,B(m)]/(3+2n)} \qquad (42)$$

The classical expression (30) found by Walsh is readily forthcoming from (42) by setting $n = 0$:

$$r_{-}(m,0) = 2A(m) \quad . \tag{43}$$

Thus the task of evaluating A(m) has already been done. It remains to find B(m). Some algebraic experimentation on the integrated form of B(m) suggests that a natural representation of B(m) is the following:

$$B(m) = \left[m/2(m^2-1)^2\right]\left\{\sum_{i=1}^{4} B_{1i}(m) + m\left[\sum_{i=1}^{4} B_{2i}(m) - (a^2-1)\sum_{i=1}^{4} B_{3i}(m)\right]\right\} \tag{44}$$

where we have written:

$$\text{"}B_{11}\text{" for } (1/8m)\left[r^3\left[\left(p-q^{1/2}\right) - r^2\left(p^3 - q^{3/2}\right) + \left(3r/5\right)\left(p^5 - q^{5/2}\right)\right] - \left(1/7\right)\left(p^7 - q^{7/2}\right)\right] \tag{45}$$

$$\text{"}B_{12}\text{" for } \left[\left(m^2+2\right)/2m\right]\left[\left(-r^2/2\right)\left(p-q^{1/2}\right) + \left(r/3\right)\left(p^3-q^{3/2}\right) - \left(1/10\right)\left(p^5-q^{5/2}\right)\right] \tag{46}$$

$$\text{"}B_{13}\text{" for } \left[\left(2m^2+1\right)/2m\right]\left[r\left(p-q^{1/2}\right) - \left(1/3\right)\left(p^3-q^{3/2}\right)\right] \tag{47}$$

$$\text{"}B_{14}\text{" for } -m\left(p-q^{1/2}\right) \tag{48}$$

$$\text{"}B_{21}\text{" for } \left(r^2/8m^2\right)\left[r\left(p-q^{1/2}\right) - \left(1/3\right)\left(p^3-q^{3/2}\right)\right] \tag{49}$$

$$\text{"}B_{22}\text{" for } \left[-\,r\left(5m^2+1\right)/4m^2\right]\left(p-q^{1/2}\right) \tag{50}$$

$$\text{"}B_{23}\text{" for } \left[\left(2m^2+1\right)/r^{1/2}\right] \ln A \quad ,$$

and

$$\text{"}A\text{" for } \left(q^{1/2} - r^{1/2}\right)\left(p+r^{1/2}\right)\Big/\left(q^{1/2}+r^{1/2}\right)\left(p-r^{1/2}\right) \tag{51}$$

$$\text{"}B_{24}\text{" for } \left(-m/r\right)\left(p-mq^{1/2}\right) - \left(m^2/r^{3/2}\right) \ln A \tag{52}$$

$$\text{"}B_{31}\text{" for } \left(r^2/2\right)\left[\left(p^{-1} - q^{-1/2}\right) - \left(r/3\right)\left(p^{-3} - q^{-3/2}\right)\right] \tag{53}$$

$$\text{"}B_{32}\text{" for } \left[r(5m^2+1)/3\right]\left(p^{-3} - q^{-3/2}\right) \tag{54}$$

$$\text{"}B_{33}\text{" for } \left[4m^2\left(2m^2+1\right)/r\right]\left[\left(-2/r\right)\left(p^{-1}-q^{-1/2}\right) - \left(2/3\right)\left(p^{-3}-q^{-3/2}\right) + \left(1/r^{3/2}\right)\ln A\right] \tag{55}$$

$$\text{"}B_{34}\text{" for } \left(40m^3/r^3\right)\left(p^{-1}-q^{-1/2}\right) + \left(40m^4/3r^2\right)\left(p^{-3}-q^{-3/2}\right) - \left(4m^3/r\right)\left(p^{-3}-mq^{-3/2}\right) - \left(20m^4/r^{7/2}\right)\ln A \quad , \tag{56}$$

and, finally, where we have written:

$$\text{"}p\text{" for } m-1 \tag{57}$$

$$\text{"}q\text{" for } m^2-1 \tag{58}$$

$$\text{"}r\text{" for } m^2+1 \quad . \tag{59}$$

Equation (44), when used in (42) along with $A(m)$, as given by (30) (recall (43)), yields an exact expression for $r_-(m,n)$. Table 4 lists some values of $r_-(m,n)$ as computed from the exact formula for (42), for the indicated ranges of m and n. The help of Mr. James Bates, Mrs. Alma Schaules, Mrs. Margaret Rethwish, Mrs. Margaret Church, and Mrs. Dolores Reinbold is acknowledged in performing and checking the calculations, at various stages of the work, leading to Table 4.

Figure 12.5 summarizes the information of Table 4 in a way that reveals the m-dependence of $r_-(\cdot,n)$ as essentially a linear function of m for $1.2 \le m \le 1.9$ and with $1/2 \le n \le \infty$. Hence over these ranges $r_-(m,n)$ can be represented very nearly in the form:

$$r_-(m,n) = a(n)m + b(n) \quad . \tag{60}$$

The coefficients $a(n)$ and $b(n)$ have been evaluated for each of the n values in the region of linearity. The following is a form which, pertaining to heavy overcasts, is perhaps of greatest immediate interest:

$$r_-(m,2) = 0.141965\ m - 0.137709 \quad . \tag{61}$$

When plotted, this function has a maximum deviation of 3.2 percent from the exact function for $r_-(m,2)$ as given by (42). For example, with $m = 4/3$ in (61), we have $r_-(4/3,2) = 0.051518$. Rounded to four figures this gives 0.0515 compared to 0.0513 for the corresponding exact value in Table 4. In this case the deviation is merely 0.6 percent.

In general a good rule of thumb for finding $r_-(m_2,n)$ when $r_-(m_1,n)$ is known, is given by:

$$r_-(m_2,n) = r_-(m_1,n) + 0.142(m_2-m_1) \tag{62}$$

TABLE 4.

The Reflectance $r_{-}(m,n)$

				n			
m	-1.0	-0.5	0	0.5	1.0	1.5	2.0
1.1	0.054315	0.032448	0.025159	0.021515	0.019328	0.017870	0.016829
1.2	0.088540	0.055346	0.044281	0.038749	0.035429	0.033216	0.031636
1.3	0.112699	0.074024	0.061132	0.054686	0.050819	0.048240	0.046399
4/3	0.119641	0.079754	0.066458	0.059811	0.055822	0.053163	0.051264
1.4	0.132357	0.090698	0.076811	0.069858	0.065702	0.062925	0.060941
1.5	0.149237	0.106143	0.091778	0.084595	0.080286	0.077413	0.075361
1.6	0.163601	0.120584	0.106245	0.099076	0.094774	0.091907	0.089858
1.7	0.177942	0.134726	0.120320	0.113117	0.108796	0.105915	0.103857
1.8	0.190639	0.148201	0.134055	0.126982	0.122738	0.119909	0.117888
1.9	0.202559	0.161246	0.147476	0.140590	0.136459	0.133705	0.131737
m	2.5	3.0	3.5	4.0	4.5	5.0	∞
1.1	0.016048	0.015440	0.014955	0.014556	0.014226	0.013945	0.010581
1.2	0.030450	0.029528	0.028790	0.028187	0.027684	0.027258	0.022152
1.3	0.045017	0.043943	0.043084	0.042380	0.041794	0.041299	0.035349
4/3	0.049839	0.048731	0.047845	0.047119	0.046515	0.046004	0.039867
1.4	0.059453	0.058296	0.057370	0.056613	0.055982	0.055448	0.049039
1.5	0.073822	0.072625	0.071667	0.070883	0.070230	0.069678	0.063048
1.6	0.088322	0.087127	0.086171	0.085389	0.084737	0.084186	0.077568
1.7	0.102313	0.101113	0.100152	0.099367	0.098712	0.098158	0.091509
1.8	0.116372	0.115193	0.114250	0.113479	0.112836	0.112292	0.105763
1.9	0.130262	0.129114	0.128196	0.127445	0.126819	0.126290	0.119934

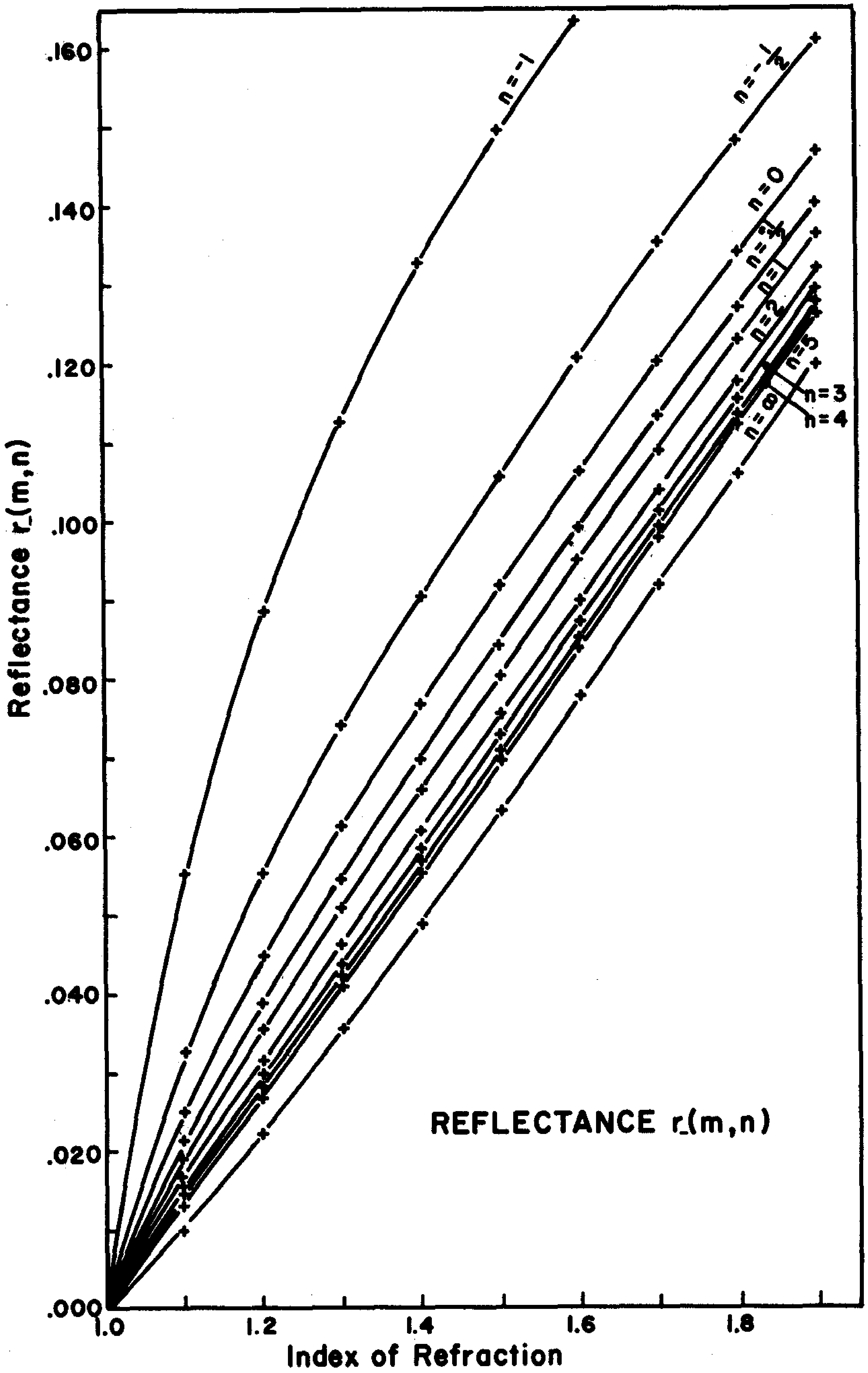

FIG. 12.5 The reflectance $r_-(m,n)$ of a plane surface between two media of relative indices of refraction 1 and *m*.

for $1/2 \leq n < \infty$, $1.2 \leq m \leq 1.9$.

The maximum errors of deviation of $r_-(m_2,n)$ in (62) from the exact values when m varies over the range $1.2 \leq m \leq 1.9$ for each given n, is summarized in Table 5.

TABLE 5

Maximum Percent Deviation of (62) from Exact Values (for: $1.2 \leq m \leq 1.9$)	n Value
6.9	0.5
5.4	1.0
3.2	2.0
2.4	2.5
1.7	3.0
1.2.	3.5
0.7	4.0
0.3	4.5
0.1	5.0
6.0	∞

Example 3. Reflectance Under Zonal Radiance Distributions

We next illustrate the use of (12) for the case where a radiance distribution is unpolarized and of uniform nonzero magnitude over some spherical zone of $\Xi_-(x)$ and zero outside of $\Xi_-(x)$. For the present example we shall continue to use the setting of Fig. 12.3(b) to depict the general orientations of the unit inward normal **k'** to the surface S at x. However, by going over the the local frame of reference based on **k'** we can write (25), for the present lighting conditions, in the form:

$$r_-(x) = \frac{\int_{\phi'=0}^{2\pi} \int_{\theta'=\theta_1}^{\theta_2} r(\theta') \cos\theta' \sin\theta' \, d\theta' \, d\phi'}{\int_{\phi'=0}^{2\pi} \int_{\theta'=\theta_1}^{\theta_2} \cos\theta' \sin\theta' \, d\theta' \, d\phi'} \qquad (63)$$

and where Fig. 12.6 conveniently summarizes the present geometrical details. Thus radiance of uniform magnitude N is incident on x from all directions in the spherical zone bounded by latitude circles of colatitude θ_2 and θ_1.

The denominator integral is readily evaluated, as usual, and shall be denoted by "$H'(\theta_2,\theta_2)$". Clearly:

$$H'(\theta_1,\theta_2) = \pi(\sin^2\theta_2 - \sin^2\theta_1) \qquad (64)$$

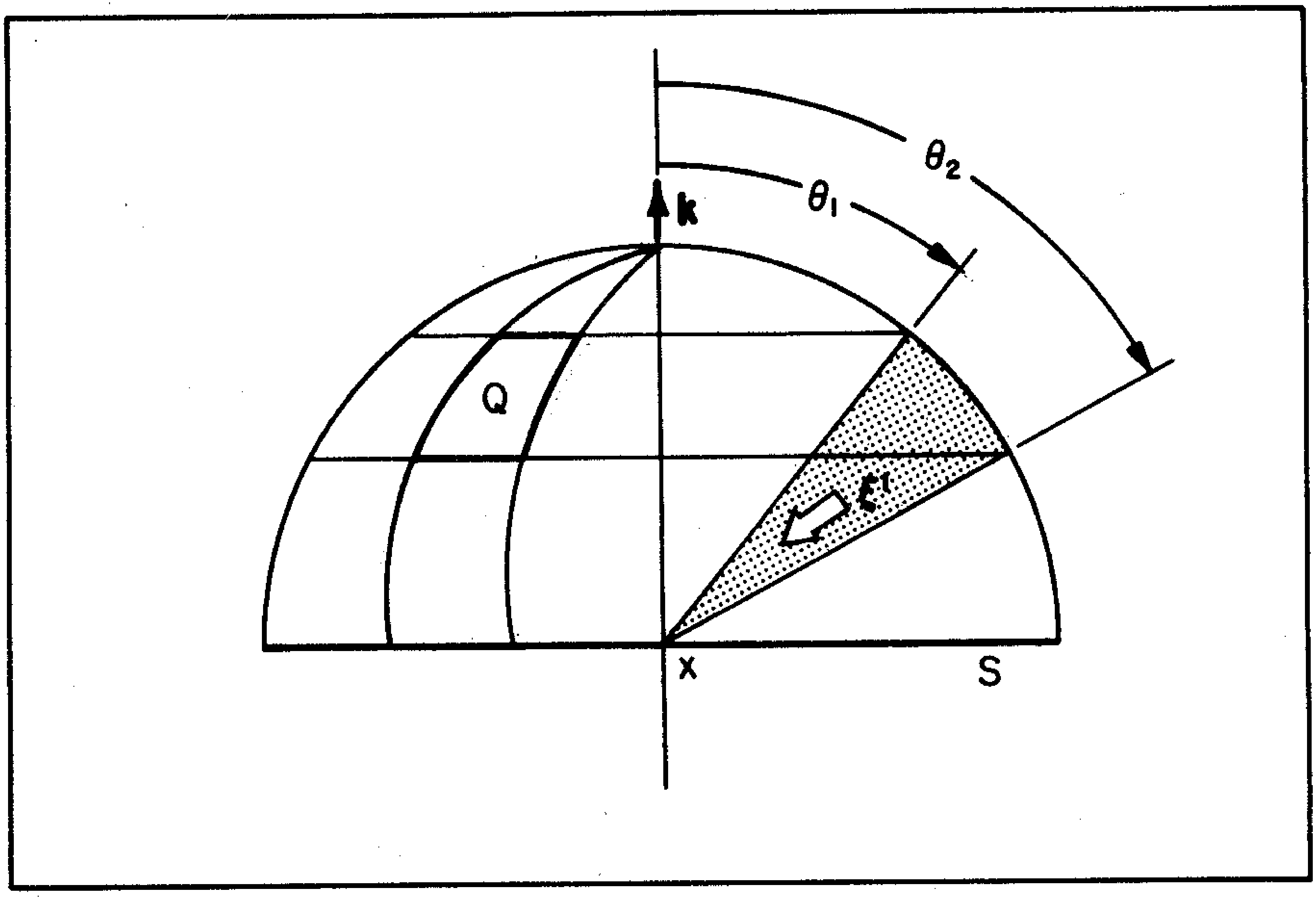

FIG. 12.6 Reflectance calculations under zonal radiance distributions.

Using in (63) the representation of $r_-(\phi')$ as given in (36) and using the *integrand* of (38) over the range of x corresponding to θ_1 and θ_2, it is possible to represent the radiant emmitance of S at x, as induced by the present irradiation, in the form:

$$W'(m,\theta_1,\theta_2) = \pi[2F(m,\theta_2) - 2F(m,\theta_1)] \tag{65}$$

where we write:

$$\text{"}2F(m,\theta)\text{" for } \left[\frac{m(4m^2+(m^2+1)^2)}{4(m^2-1)^2}\right] y - \left[\frac{m^2(m^2+1)}{2(m^2-1)^2}\right] y^2 + \left[\frac{m^3}{3(m^2-1)^2}\right] y$$

$$+ \frac{m^3[(m^4-6m^2+1)y+2m(m^2+1)]}{(m^4-1)^2 y(y-a)} + \frac{(m^2-1)^2-4m^2}{16m(y-a)}$$

$$- \frac{m^2(m^2-1)^2}{(m^2+1)^3} \ln\left|\frac{y-a}{y}\right| - \frac{(m^2+1)m^2}{(m^2-1)^2} \ln y \tag{66}$$

and where we have written:

$$\text{"}y\text{" for } \cos\theta \left(1 - \frac{\sin^2\theta}{m^2}\right)^{1/2} + \frac{\sin^2\theta}{m} \tag{67}$$

and where a is as defined in (37).

The reflectance of surface S at x is then given by

$$\boxed{r_-(m,\theta_1,\theta_2) = \frac{2F(m,\theta_2) - 2F(m,\theta_1)}{\sin^2\theta_2 - \sin^2\theta_1}} \qquad (68)$$

where $r_-(x)$ is now denoted more suggestively by "$r_-(m,\theta_1,\theta_2)$". Table 6 summarizes a computation, based on (66), for the important case m = 4/3. The help of Mrs. Alma Schaules and Mrs. Margaret Church is acknowledged in performing the calculations for Tables 6 and 7.

TABLE 6

$2F(m,\theta)(m = 4/3)$

θ	$2F(m,\theta)$		$2F(m,\theta)$	θ	$2F(m,\theta)$
0°	1.889496935				
1°	1.889503118	31°	1.895007317	61°	1.911456485
2°	1.889521867	32°	1.895344842	62°	1.912431465
3°	1.889552865	33°	1.895694082	63°	1.913455802
4°	1.889596192	34°	1.896048948	64°	1.914534945
5°	1.889651883	35°	1.896413868	65°	1.915671266
6°	1.889719962	36°	1.896790537	66°	1.916867621
7°	1.889800048	37°	1.897173922	67°	1.918126834
8°	1.889892216	38°	1.897568685	68°	1.919451648
9°	1.889996370	39°	1.897974889	69°	1.920844976
10°	1.890112275	40°	1.898391380	70°	1.922373944
11°	1.890240169	41°	1.898817873	71°	1.923841799
12°	1.890379490	42°	1.899258033	72°	1.925448748
13°	1.890530058	43°	1.899727463	73°	1.927128187
14°	1.890692023	44°	1.900175460	74°	1.928879724
15°	1.890865197	45°	1.900656981	75°	1.930699948
16°	1.891049161	46°	1.901151663	76°	1.932586309
17°	1.891243987	47°	1.901273426	77°	1.934533416
18°	1.891449230	48°	1.902192704	78°	1.936529340
19°	1.891665022	49°	1.902741668	79°	1.938572699
20°	1.891891030	50°	1.903310015	80°	1.940641672
21°	1.892126955	51°	1.903900952	81°	1.942721259
22°	1.892372888	52°	1.904515057	82°	1.944789071
23°	1.892628634	53°	1.905154936	83°	1.946820213
24°	1.892893806	54°	1.905823165	84°	1.948780998
25°	1.893167387	55°	1.906518807	85°	1.950615712
26°	1.893452248	56°	1.907247737	86°	1.952293194
27°	1.893745341	57°	1.908010594	87°	1.953745854
28°	1.894047273	58°	1.908810014	88°	1.954901148
29°	1.894358213	59°	1.909649269	89°	1.955672203
30°	1.894679063	60°	1.910530325	90°	1.955955399

Observe that the case $r_-(m,0,\pi/2)$ is the classical Walsh case (30). Table 6 readily yields this case. However Table 6 may now be used to obtain external reflectances for arbitrary zonal radiance distributions. In particular, *observe that a radiance distribution over the zone such that the radiance is constant with respect to* θ *for each* ϕ' *may vary arbitrarily with respect to azimuth* ϕ' *and still have the same reflectance as one that is uniform over the entire zone.* An immediate consequence of this is that (68) may be used to find the externally reflected flux from static air-water surfaces under arbitrarily overcast skies, or arbitrary radiance distributions which are partitioned into parts which are essentially uniform over quadrilaterals Q in each zone (Fig. 12.6). To facilitate computations of $r_-(m,\theta_1,\theta_2)$, Table 7 is appended,* which lists values of $\sin^2\theta$ in increments of 1°.

TABLE 7

θ	$\cos\theta$	$\sin^2\theta$
0°	1.0	0.
1°	0.9998476952	0.0003045864
2°	0.9993908270	0.0012179748
3°	0.9986295348	0.0027390523
4°	0.9975640503	0.0048659656
5°	0.9961946981	0.0075961234
6°	0.9945218954	0.0109261996
7°	0.9925461516	0.0148521368
8°	0.9902680687	0.0193691520
9°	0.9876883406	0.0244717418
10°	0.9848077530	0.0301536896
11°	0.9816271834	0.0364080727
12°	0.9781476007	0.0432272711
13°	0.9743700648	0.0506029768
14°	0.9702957263	0.0585262035
15°	0.9659258263	0.0669872981
16°	0.9621616959	0.0759759519
17°	0.9563047560	0.0854812137
18°	0.9510565163	0.0954915028
19°	0.9455185756	0.1059946232
20°	0.9396926208	0.1169777784

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*At the time this chapter was written (1964), the microcircuit pocket computer was still a gleam in a solid-state engineer's eye. Table 7 is now carried in every 10 decimal computer, and in this sense is superfluous. Spot checks of the table with the author's pocket computer show its general accuracy. The table is retained now only for sentimental reasons, and especially since the typist got to it before the author realized it.

TABLE 7--Continued.

θ	$\cos\theta$	$\sin^2\theta$
21°	.9335804265	0.1284275872
22°	.9271838546	0.1403300998
23°	.9205048538	0.1526708147
24°	.9135454576	0.1654346968
25°	.9063077870	0.1786061951
26°	.8987940463	0.1921692623
27°	.8910065242	0.2061073738
28°	.8829475929	0.2204035482
29°	.8746197071	0.2350403678
30°	.8660254038	0.2500000000
31°	.8571673007	0.2652642185
32°	.8480480962	0.2808144265
33°	.83867-5679	0.2966316784
34°	.8290375725	0.3126967033
35°	.8191520443	0.3289899283
36°	.8090169944	0.3454915028
37°	.7986355100	0.3621813221
38°	.7880107536	0.3790390521
39°	.7771459615	0.3960441545
40°	.7660444431	0.4131759111
41°	.7547095802	0.4304134495
42°	.7431448255	0.4477357684
43°	.7313537016	0.4651217631
44°	.7193398003	0.4825502517
45°	.7071067812	0.5000000000
46°	.6946583705	0.5174497483
47°	.6819981601	0.5348782368
48°	.6691306060	0.5522642317
49°	.6560590290	0.5695865504
50°	.6427876097	0.5868240888
51°	.6293203910	0.6039558455
52°	.6156614753	0.6209609478
53°	.6018150232	0.6378186778
54°	.5877852523	0.6545084972
55°	.5735764364	0.6710100717
56°	.5591929035	0.6873032960
57°	.5446390350	0.7033683215
58°	.5299192642	0.7191855735
59°	.5150380749	0.7347357814
60°	.5000000000	0.7500000000
61°	.4848096202	0.7649596320
62°	.4694715628	0.7795964518
63°	.4539904997	0.7938926262
64°	.4383711468	0.8078307377
65°	.4226182617	0.8213938048

TABLE 7--Continued

θ	$\cos\theta$	$\sin^2\theta$
66°	.4067366431	0.834565303
67°	.3907311285	0.847329185
68°	.3746065934	0.859669900
69°	.3583679495	0.871572413
70°	.3420201433	0.883022222
71°	.3255681545	0.894005377
72°	.3090167944	0.904508497
73°	.2923717047	0.914518786
74°	.2756373558	0.924024048
75°	.2588190451	0.933012702
76°	.2419218956	0.941473796
77°	.2249510543	0.949397023
78°	.2079116908	0.956772729
79°	.1908089954	0.963591927
80°	.1736481777	0.969846310
81°	.1564344650	0.975528258
82°	.1391731010	0.980630848
83°	.1218693434	0.985147863
84°	.1045284633	0.989073800
85°	.0871557427	0.992403877
86°	.0697564737	0.995134034
87°	.0523359562	0.997260948
88°	.0348994967	0.998782025
89°	.0174524064	0.999695414
90°	0.0	1.0

Thus, together, Tables 6 and 7 carry enough information to manufacture $(91 \times 90)/2 = 4095$ distinct reflectances of the form $r_-(4/3,\theta_1,\theta_2)$.

Still further reflectances can be written down for general partitions of $\Xi_-(x)$ into fixed numbers of zones. For example, if $\Xi_-(x)$ is divided into two zones: the *cap* from $\theta = 0$ to $\theta = \theta_1$ and the *lower zone* from $\theta = \theta_1$ to $\theta = \pi/2$, and if N_1 is the uniform radiance over the cap and N_2 the radiance over the lower zone, then from (65);

$$N_1\pi[2F(m,\theta_1) - 2F(m,0)] + N_2\pi[2F(m,\pi/2) - 2F(m,\theta_1)] \qquad (69)$$

is the radiant emittance associated with their composite distribution. The associated irradiance is:

$$N_1\pi(\sin^2\theta_1) + N_2\pi(1 - \sin^2\theta_1) \qquad (70)$$

The reflectance for this particular one-step radiance distribution is then found by dividing (69) by (70). Similar weighted averages of $W'(m,\theta_1,\theta_2)$ can be manufactured at will for larger numbers of partitions, and convenient algebraic formulas devised with the help of Tables 6 and 7, for the more frequently used combinations. These matters are best left to the interested reader and his individual needs.

As a further observation on (68), the reader may verify (either directly through (66), or numerically through Table 6) that:

$$\lim_{\theta_2 \to \theta_1} r_-(m,\theta_1,\theta_2) = r(\theta_1) \tag{71}$$

where $r(\theta_1)$ is the Fresnel reflectance for the angle of incidence θ_1, as given generally in (13a).

Finally, we observe that the internal reflectance $r_+(1/m,\theta_1,\theta_2)$ analogous to $r_-(m,\theta_1,\theta_2)$ may be obtained by a corresponding integration of the kind that yielded (68). Alternatively, a numerical integration, of the kind that produced Table 3, may be performed. When performing computations for $r_+(1/m,\theta_1,\theta_2)$ care must be taken in making fine enough integration intervals near the angle of total reflectance. Thus, by (5) and (6) of Sec. 12.1

$$\sin\theta' = m\sin\theta \quad ,$$

so that when $\theta' = \pi/2$, the corresponding θ is given by

$$1 = m\sin\theta$$

which, for $m = 4/3$, requires $\theta = 48°35'$ (Table 1 above). From this we expect, as one looks up at the air-water surface from below, that there is marked compression of the refracted images near 48°, and that there should be total reflection for angles of sight greater than 48°35'. Thus numerical integrations will be primarily concerned with the range of incident angles from 0° to 48°35' in computing $r_+(1/m,\theta_1,\theta_2)$. No tabulations of the kind in Table 6 appear to be currently available for the purpose of computing $r_+(1/m,\theta_1,\theta_2)$. It would be of interest to the subject to eventually have a closed-form integration of $r_+(1/m,\theta_1,\theta_2)$ analogous to that given in (68) for $r_-(m,\theta_1,\theta_2)$.

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12.2 Radiative Transfer and the Static Surface

We now formulate and solve the problem of radiative transfer across the static air-water surface. We shall apply the interaction principle to a three-part medium consisting of a portion of the atmosphere, the hydrosphere and the air-water surface between them. We shall formulate the interaction equations first for the case of irradiance and thereby obtain the essential algebraic idea of the formulation without the analytic complications arising in the radiance case. The irradiance case is analyzed into two parts: first, the equations governing the interaction between the air-water surface and the body of the hydrosol will be obtained; second, the equations for the full interaction between the hydrosol, the static air-water surface, and aerosol (i.e., the part of the atmosphere above the hydrosol) will be obtained with the aid of the first part of the solution.