for larger numbers of partitions, and convenient algebraic formulas devised with the help of Tables 6 and 7, for the more frequently used combinations. These matters are best left to the interested reader and his individual needs.

As a further observation on (68), the reader may verify (either directly through (66), or numerically through Table 6) that:

$$\lim_{\theta_2 \to \theta_1} r_-(m,\theta_1,\theta_2) = r(\theta_1) \qquad (71)$$

where $r(\theta_1)$ is the Fresnel reflectance for the angle of incidence θ_1, as given generally in (13a).

Finally, we observe that the internal reflectance $r_+(1/m,\theta_1,\theta_2)$ analogous to $r_-(m,\theta_1,\theta_2)$ may be obtained by a corresponding integration of the kind that yielded (68). Alternatively, a numerical integration, of the kind that produced Table 3, may be performed. When performing computations for $r_+(1/m,\theta_1,\theta_2)$ care must be taken in making fine enough integration intervals near the angle of total reflectance. Thus, by (5) and (6) of Sec. 12.1

$$\sin \theta' = m \sin \theta \quad ,$$

so that when $\theta' = \pi/2$, the corresponding θ is given by

$$1 = m \sin \theta$$

which, for $m = 4/3$, requires $\theta = 48°35'$ (Table 1 above). From this we expect, as one looks up at the air-water surface from below, that there is marked compression of the refracted images near 48°, and that there should be total reflection for angles of sight greater than 48°35'. Thus numerical integrations will be primarily concerned with the range of incident angles from 0° to 48°35' in computing $r_+(1/m,\theta_1,\theta_2)$. No tabulations of the kind in Table 6 appear to be currently available for the purpose of computing $r_+(1/m,\theta_1,\theta_2)$. It would be of interest to the subject to eventually have a closed-form integration of $r_+(1/m,\theta_1,\theta_2)$ analogous to that given in (68) for $r_-(m,\theta_1,\theta_2)$.

12.2 Radiative Transfer and the Static Surface

We now formulate and solve the problem of radiative transfer across the static air-water surface. We shall apply the interaction principle to a three-part medium consisting of a portion of the atmosphere, the hydrosphere and the air-water surface between them. We shall formulate the interaction equations first for the case of irradiance and thereby obtain the essential algebraic idea of the formulation without the analytic complications arising in the radiance case. The irradiance case is analyzed into two parts: first, the equations governing the interaction between the air-water surface and the body of the hydrosol will be obtained; second, the equations for the full interaction between the hydrosol, the static air-water surface, and aerosol (i.e., the part of the atmosphere above the hydrosol) will be obtained with the aid of the first part of the solution.

Irradiance Interaction Between the Surface and the Hydrosol

The setting for the present discussion is Fig. 12.7(a), which depicts the hydrosol $X(0,z_1)$ and its boundary X_o, the static air-water surface. The medium $X(0,z_1)$ is plane-parallel, and may be arbitrarily stratified, and source free. Irradiance of an amount $H_-(0)$ from sky light is steadily incident on X_o, is transmitted through X_o, penetrates $X(0,z_1)$ and interacts generally with the material comprising $X_o \cup X(0,z_1)$ and eventually brings about a steady light field throughout $X_o \cup X(0,z_1)$. $H_-(0)$ is the only source of flux on $X(0,z_1)$. Our main goal at present is the determination of the upward radiant emittance $W_+(0)$ and the downward radiant emittance $W_-(0)$ of X_o and to study the effect of X_o on these magnitudes.

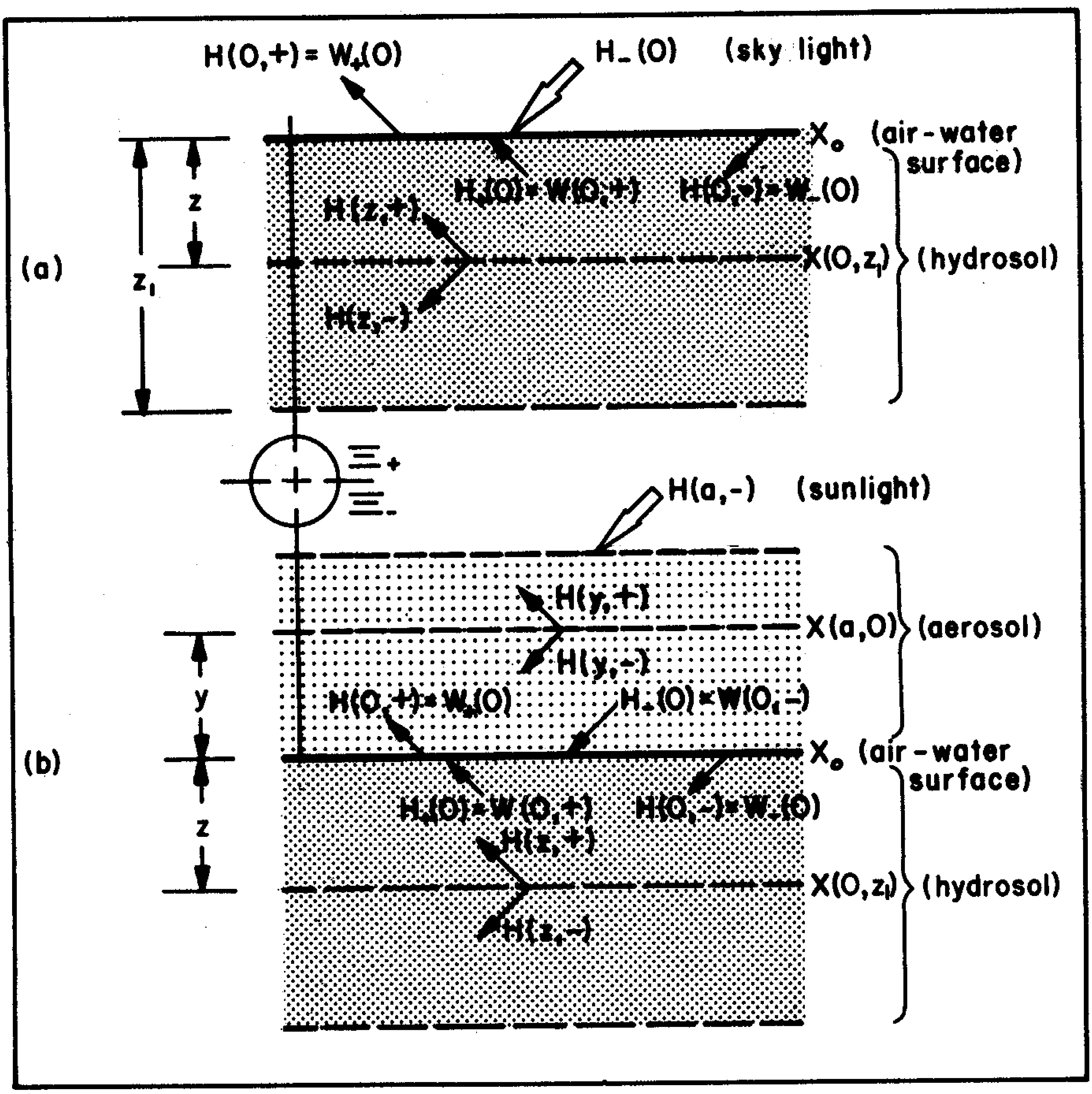

FIG. 12.7 Geometry for irradiance interactions between air-water surface and the hydrosol.

The theory leading to $W_\pm(0)$ (along with all notational conventions) has been developed in Sec. 8.5 and is summarized in (85) and (87) of that section. For convenience, we repeat the solutions below:

$$W_-(0) = \frac{H_-(0)t_-(0)}{1-R(0,z_1)r_+(0)} = H_-(0)\mathcal{T}(-1,0,z_1) \tag{1}$$

$$W_+(0) = H_-(0)\left[r_-(0) + \frac{t_-(0)R(0,z_1)t_+(0)}{1-R(0,z_1)r_+(0)}\right] \tag{2}$$

$$= H_-(0)\mathcal{R}(-1,0,z_1)$$

These equations show that the radiant emittance of X_o is jointly determined by the transmittances and reflectances of X_o and the reflectance property of $X(0,z_1)$. (Recall that H and W with subscripts ± refer to surfaces; H,W of the form $H(0,+)$, $W(0,-)$ refer to appropriate slabs.) If the general angular structures of the radiance distributions giving rise to $H_-(0)$ and $H_+(0)$ are known, then $r_\pm(0)$ and $t_\pm(0)$ are determinable using the techniques outlined in Example 3 of Sec. 12.1.

As an example of the magnitudes of $W_\pm(0)$, or more generally, of the complete reflectance $\mathcal{R}(-1,0,z_1)$ and the complete transmittance $\mathcal{T}(-1,0,z_1)$ of X_o, as given by (1), (2), we consider two wavelengths in the green and blue part of the spectrum. In the green light, $R(0,z_1)$ for large z_1 is on the order of 0.02 in oceanic water, and for blue light $R(0,z_1)$ for large z_1 is on the order of 0.08 for some clear volcanic lakes, such as Crater Lake. Such numbers must be taken as *approximately* representative. At any rate, for green light we obtain from (1):

$$\frac{W_\pm(0)}{H_-(0)} = \mathcal{T}(-1,0,z_1) = \frac{t_-(0)}{1-0.02r_+(0)} \cong t_-(0)\left[1+0.02r_+(0)\right]$$

and once more, for blue light we obtain from (1):

$$\frac{W_-(0)}{H_-(0)} = \mathcal{T}(-1,0,z_1) \cong t_-(0)\left[1+0.08r_+(0)\right] \quad .$$

Now the internal reflectance $r_+(0)$ of the air-water surface is very sensitive to the directional structure of the upward radiance distribution because of the total internal reflection taking place near 48° and also for greater angles. For natural lighting conditions $r_+(0)$ can range from 0.40 to 0.90 and perhaps even further. (For typical diffused-light values see Table 3, Sec. 12.1.) The complete transmittance $\mathcal{T}(-1,0,z_1)$ of the air-water surface for green light can be usefully bracketed between the two extremes arising from

Taking $r_+(0) = 0$ and then $r_+(0) = 1$. The variation this gives is summarized by:

$$t_-(0) \le \mathcal{T}(-1,0,z_1) \le t_-(0)\left[1 + 0.02\right] \quad ,$$

which shows that $\mathcal{T}(-1,0, z_1)$ can vary on the order of 2 percent of $t_-(0)$ for green light. Further,

$$t_-(0) \le \mathcal{T}(-1,0,z_1) \le t_-(0)\left[1 + 0.08\right]$$

which brackets the variation of $\mathcal{T}(-1,0,z_1)$ as being within 8 percent of $t_-(0)$ for blue light, an amount which could be of critical importance in biological and other flux dependent activity in the sea, lakes, or other natural media. In general, to first order terms:

$$t_-(0) \le \mathcal{T}(-1,0,z_1) \le t_-(0)\left[1 + R(0,z_1)\right] \qquad (3)$$

In a similar manner, we can bracket the effect of the variations of $r_+(0)$ on the complete reflectance of the air-water surface by means of (2):

$$r_-(0) \le \mathcal{R}(-1,0,z_1) \le r_-(0) + t_-(0)t_+(0)R(0,z_1) \qquad (4)$$

to first order terms in $R(0,z_1)$. The exact bracketing equations are:

$$t_-(0) \le \mathcal{T}(-1,0,z_1) \le \frac{t_-(0)}{1-R(0,z_1)r_+(0)} \qquad (5)$$

$$r_-(0) \le \mathcal{R}(-1,0,z_1) \le r_-(0) + \frac{t_-(0)t_+(0)R(0,z_1)}{1 - R(0,z_1)r_+(0)} \qquad (6)$$

All references to the upward flux just below the boundary X_0 may be removed by setting $t_+(0) = 1$ in (4) and (6) for still further rough bracketing estimates of the variations of $\mathcal{R}(-1,0,z_1)$ and $\mathcal{T}(-1,0,z_1)$.

To summarize, we have shown that the contribution of the interaction between the static air-water surface and the body of a hydrosol at the surface can vary from negligible contributions to the light field, to amounts which could be easily detected by modern measuring apparatus and very well be critical to biological and other flux-dependent activity in natural hydrosols. The effects of the presence of the air-water boundary at general depths z in the hydrosol may be obtained by using (88) and (89) of Sec. 8.5. Furthermore, if the bottom boundary of $X(0,z_1)$ has incident sources, then (94) of Sec. 8.5 may be used to find $H(z,\pm)$.

The Threefold Irradiance Interaction: Aerosol, Air-Water Surface, and Hydrosol

The setting for the present discusion is Fig. 12.7(b) which schematically depicts the hydrosol $X(0,z_1)$, the aerosol $X(a,0)$ and the air-water surface X_0 sandwiched between them. An amount $H(a,-)$ of irradiance is incident on the aerosol (produced by sunlight at the "top" of the atmosphere) which initiates and sustains a radiative transfer process in the three-component system. (Hence $H(z_1,+) = 0$.) In particular, irradiances $H_-(0)$ and $H_+(0)$ on the air-water surface are established, and give rise to its radiant emittances $W_\pm(0)$. It was the irradiance $H_-(0)$ which we used as the basic initiator of the radiative transfer process in the discussions of (1) and (2). Now we wish to relate $H_-(0)$ to the optical properties of the aerosol and to examine the general effects of the presence of $X(a,0)$ and $X(0,z_1)$ on $W_\pm(0)$, and as mediated by the presence of the air-water boundary X_0.

The requisite equations for $W_\pm(0)$ are obtained by finding the connection between $H(a,-)$ and $H_-(0)$ and then using this connection in (1),(2). This connection may be obtained by conceptually isolating the aerosol $X(a,0)$ and applying the interaction principle to it, which then states that:

$$H_-(0) = H(a,-)T(a,0) + H(0,+)R(0,a) \tag{7}$$

where $T(a,0)$ and $R(0,a)$ are two of the standard reflectances and transmittances of $X(a,0)$, and may be obtained empirically (as in Chapter 13) or theoretically (as in Chapter 8). It remains to relate the irradiance on $X(a,0)$ namely $H(0,+)$, to $H_-(0)$, the irradiance on X_0. But this relation is given by (2):

$$H(0,+) = H_-(0)\mathcal{R}(-1,0,z_1) \tag{8}$$

where $\mathcal{R}(-1,0,z_1)$ is the complete reflectance of X_0 and is given in (2). Solving (7), (8) for $H_-(0)$ we have:

$$H_-(0) = \frac{H(a,-)T(a,0)}{1-\mathcal{R}(-1,0,z_1)R(0,a)}$$

$$= H(a,-)\mathcal{T}(a,-1,z_1) \tag{9}$$

By (1) and (9) we then have:

$$W_-(0) = H(a,-)\mathcal{T}(a,-1,z_1)\mathcal{T}(-1,0,z_1)$$

$$= H(a,-)\mathcal{T}(a,0,z_1) \quad . \tag{10}$$

The latter equality follows from the semigroup property (52) of Sec. 3.7. Finally, by (2) and (9) we have:

$$W_+(0) = H(a,-)\mathcal{T}(a,-1,z_1)\mathcal{R}(-1,0,z_1)$$

$$= H(a,-)\mathcal{R}(a,0,z_1) \quad . \qquad (11)$$

The latter equality follows from the semigroup property (53) of Sec. 3.7. For our present purposes, the explicit representations of $W_\pm(0)$ as products of the various complete reflectance and transmittance factors are most useful. Thus, e.g., (11) shows that the radiant emittance $W_+(0)$ of X_0 is generally governed by the product of the complete transmittance $\mathcal{T}(a,-1,z_1)$ of the aerosol (as an integral part of the three-component system presently under study) and the complete reflectance of the hydrosol-surface system. In the case of the complete transmittance of the aerosol, the reflectance $R(0,a)$ plays the same general role that $r_+(0)$ did in the interactions considered above between the hydrosol and the air-water boundary. It follows that we can obtain a bracketed estimate of $\mathcal{T}(a,-1,z_1)$ by varying $R(0,a)$ between 0 and 1:

$$T(a,0) \leq \mathcal{T}(a,-1,z_1) \leq \frac{T(a,0)}{1-\mathcal{R}(-1,0,z_1)} \qquad (12)$$

These inequalities in conjunction with (5), (6) can serve as means of bracketing in turn, the complete reflectance $\mathcal{R}(a,-1,z_1)$ of the aerosol and complete transmittance $\mathcal{T}(a,0,z_1)$ associated with the aerosol-surface system considered as an integral part of the three-component system.

For solutions of the irradiances $H(y,\pm)$ and $H(z,\pm)$ *within* $X(a,0)$ and *within* $X(0,z_1)$, respectively, when the three-component system is studied as an interacting whole, one may use the results of Example 6 of Sec. 8.7 in which $X(b,c)$ in Example 6 is taken to be $X_0 \cup X(0,z_1)$ if the hydrospheric irradiances $H(z,\pm)$ are of interest; or in which $X(a,b)$ in Example 6 is taken to be $X_0 \cup X(a,0)$ if the atmospheric irradiances $H(y,\pm)$ are of interest.

The Threefold Radiance Interaction: For the Static Surface

We can now derive with relatively little effort the expressions for the surface radiance of the static interface X_0 in the three-component system depicted in Fig. 12.7(b). We shall be able to obtain the requisite expressions directly from (10) and (11) by simply changing over from H-notation to N-notation, and observing the correct order of application of the integral operators in the radiance context.* This simple

*A further generalization to the polarized radiance context will also increase the descriptive powers of the following theory. To effect this generalization, we simply replace N by the radiance vector **N** (cf. Sec. 2.10) in standard observable form. Furthermore we replace the Fresnel reflectance function by a corresponding matrix, readily evaluated using (8), (9) of Sec. 12.1.

transition is justified on the grounds that the algebraic form of the interaction principle is the same for both the irradiance and radiance contexts, as was demonstrated in detail throughout Chapter 8. Therefore, if "$N_\pm(0)$" denotes the surface radiance *distributions* of the interface X_0 in the upper (+) and lower (-) direction hemispheres, and "$N_-(a)$" denotes the downward radiance distribution incident on the aerosol, then (10) becomes:

$$\left[\frac{N_-(0)}{n_2^2}\right] = \left[\frac{N_-(a)}{n_1^2}\right] \mathcal{T}(a,-1,z_1)\mathcal{T}(-1,0,z_1)$$

Hence:

$$\boxed{\left[\frac{N_-(0)}{n_2^2}\right] = \left[\frac{N_-(a)}{n_1^2}\right] \mathcal{T}(a,0,z_1)} \quad , \qquad (13)$$

and (11) becomes:

$$N_+(0) = N_-(a)\mathcal{T}(a,-1,z_1)\mathcal{R}(-1,0,z_1) \quad .$$

That is:

$$\boxed{N_+(0) = N_-(a)\mathcal{R}(a,0,z_1)} \qquad (14)$$

where n_1, n_2 are indices of refraction in $X(a,0)$ and $X(0,z_1)$, respectively (see (4) of Sec. 2.6) and where:

$$\mathcal{T}(a,-1,z_1) = T(a,0)[I-\mathcal{R}(-1,0,z_1)R(0,a)]^{-1} \qquad (15)$$

$$\mathcal{T}(-1,0,z_1) = t_-(0)\ [I-R(0,z_1)r_+(0)]^{-1} \qquad (16)$$

$$\mathcal{R}(-1,0,z_1) = r_-(0) + t_-(0)[I-R(0,z_1)r_+(0)]^{-1}R(0,z_1)t_+(0) \quad . \qquad (17)$$

All terms in (15)-(17) are now *integral operators* which act on radiance distributions, and are of the kinds discussed in (10), (11) of Sec. 3.3 (for surfaces) and (8)-(11) of Sec. 3.6 (for slabs). Theoretical means of finding the operators $T(a,0)$, $R(0,a)$, and especially the operator $R(0,z_1)$ are discussed in Chapter 7 and empirical means are discussed in Chapter 13; $r_\pm(0)$ and $t_\pm(0)$ may have as integrands the Fresnel reflectance and an appropriate Dirac-delta function. For computational purposes, (13)-(17) can be converted into matrix form using the general manner of conversion discussed in Sec. 7.7. A generalization of operators (16) and (17) for the dynamic air-water surface is given in (14), (15) of Sec. 12.13.*

Contrast Transmittance Formulas for the Static Surface

We take up next the derivation of the contrast transmittance of a path of sight which has one endpoint in the aerosol and the other in the hydrosol. Two cases, depicted in (a) and (b) of Fig. 12.8, are to be distinguished in practice. We consider first the case where the observer is in the aerosol and directs his line of sight into the hydrosol as in Fig. 12.8(a). The endpoints are initially to be at small distances from X_o on either side of the static interface X_o, so that we shall in effect find the contrast transmittance of the path of sight as it is solely affected by its passage through the interface. The general case where the path extends arbitrarily far from X_o on either side of the interface X_o will be considered later in this section.

Let "$N^o(x,\xi')$" denote the upward radiance in the direction ξ' at point x just below X_o. As the radiant flux crosses X_o, $N^o(x,\xi')$ changes to $N^o(x,\xi)$ in accordance with the n^2-law (4) of Sec. 2.6 and in accordance with the Fresnel transmittance $t(\xi',\xi)$ for the direction pair (ξ',ξ) governed by (4) of Sec. 12.1. That is, according to (18) of Sec. 12.1 we should have, in the absence of any path radiance:

$$\frac{N_r^o(x,\xi)}{n_1^2} = \frac{N^o(x,\xi')}{n_2^2}\, t(\xi',\xi) \tag{18}$$

where n_1 and n_2 are now respectively the indices of refraction of air and water for the given wavelength implicitly associated with the radiance. The contribution to the path radiance over the very small length of the present path of sight can come only from reflection in X_o of radiance along direction ξ'' which is related to ξ by (1) of Sec. 12.1. Hence in the present singular case, by means of (12) of Sec. 12.1:

$$N_r^*(x,\xi) = N_o(x,\xi'')r(\xi'',\xi)$$

for very small r and s, in particular r = 0 and s = 0, where $N_o(x,\xi'')$ is the sky radiance at x just above X_o. Thus the apparent radiance $N_r(x,\xi)$ for the present upward directed very short path is:

---

*The further generalization to the polarized case is also readily effected by using the standard observable radiance vector of Sec. 2.10, and by using the matricial forms of the Fresnel reflectance function and the volume scattering function. The former follows from the discussion of Sec. 12.1; the latter from Sec. 13.6.

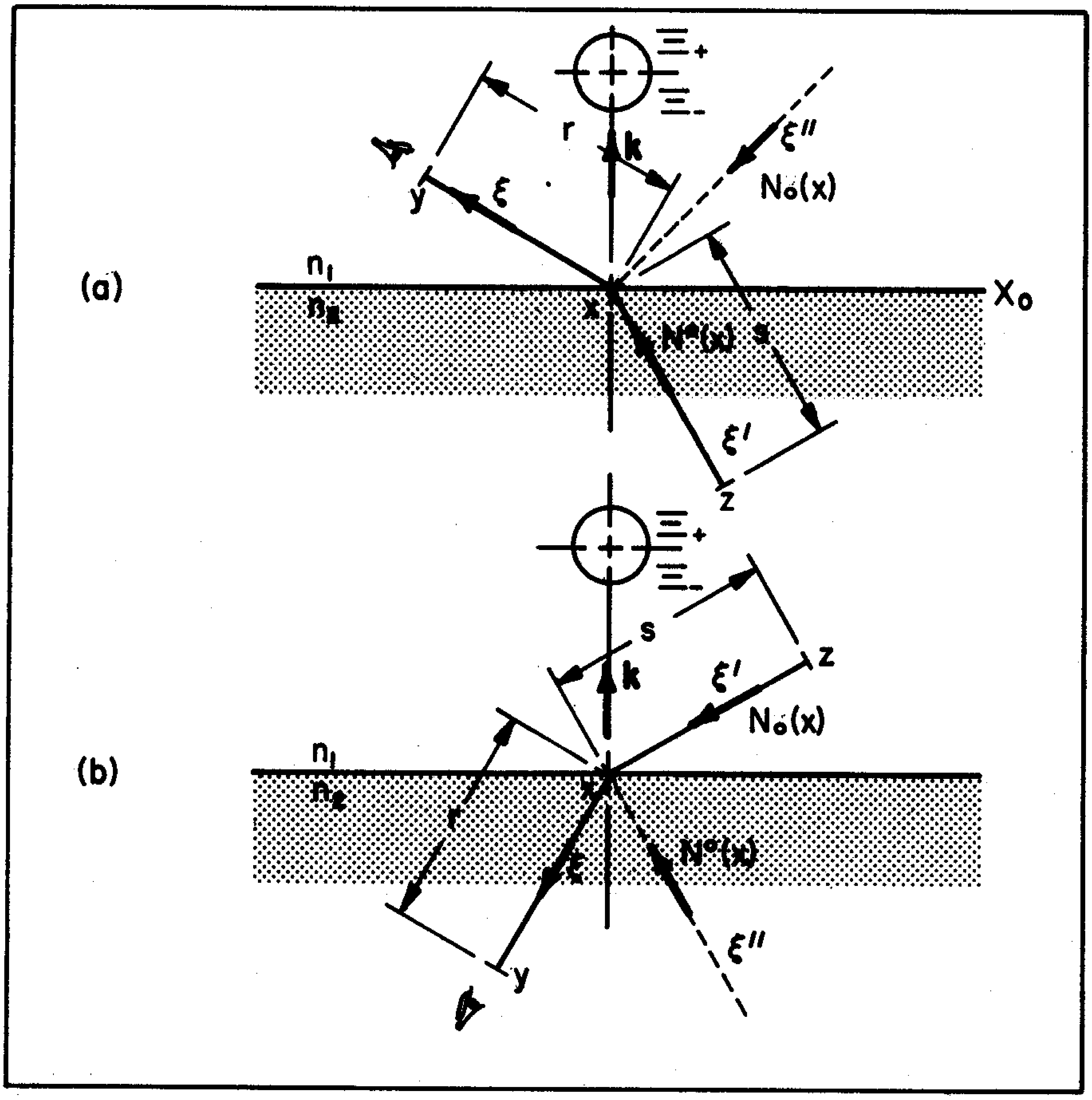

FIG. 12.8 Calculating the contrast transmittance of composite paths across the air-water surface.

$$N_r(x,\xi) = N_r^o(x,\xi) + N_r^*(x,\xi)$$

$$= N^o(x,\xi')\left(\frac{n_1}{n_2}\right)^2 t(\xi',\xi) + N_o(x,\xi'')r(\xi'',\xi) \qquad (19)$$

The contrast transmittance $\mathcal{T}_r(x,\xi')$ of the path as given in (21) of Sec. 9.5 now may take the form:

$$\mathcal{T}_o(x,\xi') = \frac{1}{1 + \left[\dfrac{N_o(x,\xi'')}{N^o(x,\xi')} \cdot \dfrac{r(\xi'',\xi)}{t(\xi',\xi)} \cdot \left(\dfrac{n_2}{n_1}\right)^2\right]} \qquad (20)$$

Fig. 12.8(a)

in which case we have gone to the limit of zero path length across the air-water surface. This is the required form of $\mathcal{T}_0(x,\xi')$ for the path of zero length which cuts across the static interface X_0.

Errata

It is possible to relate $N^o(x,\xi')$ to $N_o(x,\xi'')$ in (2) using (13) and the appropriate principle of invariance. To see this, note first that $N_-(a)\mathcal{T}(a,-1,z_1)$ is the radiance distribution of which $N_o(x,\xi'')$ is the value at x in the direction ξ''. Next, observe that $N_-(0)$, as given in (13), is reflected in $X(0,z_1)$ to give $N^o(x,\xi')$ at x for the direction ξ'. Thus (assuming once again no incident radiance at level z_1) we have:

$$N_-(0)R(0,z_1)$$

as the radiance distribution whose value at x in the direction ξ' is $N^o(x,\xi')$. Using (84) of Sec. 8.7, we write:

$$\text{"}\mathcal{R}(-1,+0,z_1)\text{"} \quad \text{for} \quad \mathcal{T}(-1,0,z_1)R(0,z_1)$$

The notation "$\mathcal{R}(-1,+0,z_1)$" (especially the "+0" term) shows that we are interested in the upward flux at level 0 *just below* X_0. It follows from (13) that:

$$\left(\frac{N^o(x)}{n_2^2}\right) = \left(\frac{N_o(x)}{n_1^2}\right)\mathcal{R}(-1,+0,z_1)$$

where $N^o(x)$ is the radiance distribution whose value for ξ' is $N^o(x,\xi')$ and $N_o(x)$ is the radiance distribution whose value at ξ'' is $N_o(x,\xi'')$. Hence (20) may be written in the form (cf., Fig. 12.8(a)):

$$\mathcal{T}_0(x,\xi') = \frac{1}{1 + \left[\dfrac{N_o(x,\xi'')}{N_o(x)\mathcal{R}(-1,+0,z_1)(\xi')} \cdot \dfrac{r(\xi'',\xi)}{t(\xi',\xi)}\right]} \tag{21}$$

which shows that the contrast transmittance $\mathcal{T}_0(x,\xi')$ depends not only on the Fresnel reflectance and transmittance of the interface X_0 but also on its complete reflectance operator $\mathcal{R}(-1,+0,z_1)$, which in turn involves $\mathcal{T}(-1,0,z_1)$ and the standard reflectance operator $R(0,z_1)$ of the hydrosol. Note also how the index of refraction does not explicitly enter when using only the incident radiance $N_o(x)$.

The derivation of the contrast transmittance $\mathcal{T}_0(x,\xi')$ for the case depicted in Fig. 12.8(b) proceeds in a completely analogous manner, to that leading to (20) and (21). Using the radiance distributions $N_o(x)$ and $N^o(x)$ introduced in the preceding case we have for the apparent radiance $N_r(x,\xi)$ at x just below X_0:

$$N_r(x,\xi) = N_o(x,\xi')\left(\frac{n_2}{n_1}\right)^2 t(\xi',\xi) + N^o(x,\xi'')r(\xi'',\xi). \tag{22}$$

From this we see, naturally enough, that N_o and N^o (along with n_1 and n_2) change places relative to (19), and ξ, ξ', ξ'' are once again appropriately related by means of the reflection and refraction laws. We assume of course that ξ is such that $\xi' \cdot \mathbf{k} < 0$, otherwise only a simple total internal reflection takes place. Hence (20) now has the companion formula:

$$\mathcal{T}_o(x,\xi') = \frac{1}{1 + \left[\frac{N^o(x,\xi'')}{N_o(x,\xi')} \cdot \frac{r(\xi'',\xi)}{t(\xi',\xi)} \cdot \left(\frac{n_1}{n_2}\right)^2\right]} \quad . \quad (23)$$

(Fig. 12.8(b))

The radiances $N^o(x)$ and $N_o(x)$ can be related in (23) after the manner of (21); thus (cf. Fig. 12.8(b)):

$$\mathcal{T}_o(x,\xi') = \frac{1}{1 + \left[\frac{N_o(x)\mathcal{R}(-1,+0,z_1)(\xi'')}{N_o(x,\xi')} \cdot \frac{r(\xi'',\xi)}{t(\xi',\xi)}\right]} \quad (24)$$

Contrast Transmittance Formulas for Extended Paths Across the Static Air-Water Surface

Let $\mathcal{P}_t(z,\xi')$ be a path of sight of length t which has its initial point z in an hydrosol, its terminal point y in an aerosol and which pierces the static air-water interface at point x, as shown in Fig. 12.8(a). We shall now derive the formula for the contrast transmittance of $\mathcal{P}_t(z,\xi')$. In analyzing the present problem it is at once clear that the contrast transmittances of the two subpaths $\mathcal{P}_r(x,\xi)$ and $\mathcal{P}_s(z,\xi')$, where $t = r + s$, are readily determinable, being in regions of continuous index of refraction. Furthermore, our work just concluded yielded expression (20) for the contrast transmittance of the singular path $\mathcal{P}_o(x,\xi')$ which has initial point x and is of length zero and which is directed along ξ'. In view of Sec. 9.5, the next step after finding these three contrast transmittances is simply to multiply them together to obtain:

$$\mathcal{T}_{r+s}(z,\xi') = \mathcal{T}_s(z,\xi')\mathcal{T}_o(x,\xi')\mathcal{T}_r(x,\xi) \quad . \quad (25)$$

Fig. 12.8(a),(b)

Observation (4) of Sec. 9.5, on which (25) is based, may readily be shown to hold for singular paths. The demonstration may be based directly on the definition of $\mathcal{T}_r$ given in Definition 3 of Sec. 9.5, and we now outline the major steps.
- We observe first that the semigroup property of radiance transmittance T_r holds for *any* pair of contiguous paths, a fact which follows trivially from the definition of radiance transmittance in (6) of Sec. 9.5. Observe in particular that

one of the paths may be singular, or contain a singular path within itself. Next we observe that the beam transmittance T_r also enjoys the semigroup property for both ordinary and singular paths. This may be seen by noting first that the beam transmittance of the singular path $\mathcal{P}_o(x,\xi')$ in Fig. 12.8(a) is, by virtue of (18):

$$T_o^o(x,\xi') = \left(\frac{n_1}{n_2}\right)^2 t(\xi',\xi) \tag{26}$$

and then recalling the definition of residual radiance ((4) of Sec. 3.10). The radiance transmittance of the singular path $\mathcal{P}_o(x,\xi')$ in Fig. 12.8(a) is simply:

$$T_o(x,\xi') = \frac{N^o(x,\xi') \left(\frac{n_1}{n_2}\right)^2 t(\xi',\xi) + N_o(x,\xi'')r(\xi'',\xi)}{N^o(x,\xi')} \tag{27}$$

It now follows from the semigroup properties of beam and radiance transmittances, that the beam transmittance for the extended path $\mathcal{P}_t(z,\xi')$ is:

$$T_{r+s}^o(z,\xi') = T_s^o(z,\xi')T_o^o(x,\xi')T_r^o(x,\xi) \tag{28}$$

and that the radiance transmittance of $\mathcal{P}_t(z,\xi')$ is:

$$T_{r+s}(z,\xi') = T_s(z,\xi')T_o(x,\xi')T_r(x,\xi) \quad . \tag{29}$$

The contrast transmittance of $\mathcal{P}_t(z,\xi')$ is then found by four applications of Definition 3 of Sec. 9.5:

$$\mathcal{T}_{r+s}(z,\xi') = \frac{T_{r+s}^o(z,\xi')}{T_{r+s}(a,\xi')} = \frac{T_s^o(z,\xi')}{T_s(z,\xi')} \cdot \frac{T_o^o(z,\xi')}{T_o(z,\xi')} \cdot \frac{T_r^o(x,\xi)}{T_r(x,\xi)}$$

$$= \mathcal{T}_s(z,\xi')\mathcal{T}_o(x,\xi')\mathcal{T}_r(x,\xi) \quad ,$$

which establishes (25). It is now readily seen that the general form of (25) holds also for the path depicted in Fig. 12.8(b); so that (25) is the general contrast transmittance formula for both downward and upward paths of sight through the air-water surface.

<< Table of Contents << Title Page >> Next Page >>