12.3 Elementary Hydrodynamics of the Air-Water Surface

In order to reach this chapter's goal of a useful, quantitative description of the optical properties of the dynamic surface, especially the statistical properties of the reflected and transmitted radiances over the surface, it is necessary that we first lay a foundation of certain hydrodynamical concepts. It is the purpose of this section to provide such a simple basis for the discussions in the remainder of the chapter. In particular we shall conduct an exposition of the classical theory of gravity and capillary waves and one of our main results will be the Kelvin-Helmholtz model for the air-water surface. This model predicts the critical wind speeds which are associated with an unstable air-water surface. It is this simple model which forms the main point of demarcation in hydrodynamics between the classical and modern theories of wind-generated waves, to be discussed in Sec. 12.9. Another principal goal of this section is to develop the concept of the spectral density function of the air-water surface. The hydrodynamical developments below are quite standard. However by developing the simple models below from first principles, it is hoped that we will be able to keep the developments of this chapter self-contained and also tailor the associated concepts to the special needs of hydrologic optics.

The Fluid Transfer Process

Suppose that a moving fluid occupies a region of a space as prescribed with respect to some frame of reference such as that depicted in Fig. 12.9. A small volume X of fluid at point x (considered as an ordered triple (x_1, x_2, x_3)) at time t may be followed, in principle, as it moves, subject to various natural forces, through the body of the fluid along with its neighbor fluid volumes. We may imagine that the motion of the fluid packed at x is such that it is subject to some general law which can uniquely predict the location of the fluid packet at some later time t, t > s. Suppose $T_{s,t}$ is the transformation which assigns to each point x at time s the unique point x(s) at which a fluid packet is subsequently located at time t. Thus,

Errata

$$x(t) = x(s)T_{s,t} \quad . \tag{1}$$

Whatever the analytical form this transformation has, it is clear that it must satisfy the following simple properties:

$$T_{r,s}T_{s,t} = T_{r,t} \quad , \tag{2}$$

$$T_{t,t} = I \quad . \tag{3}$$

The first of these properties is the *semigroup property* for the family $\{T_{s,t}: s < t\}$ of such fluid transformations, and is quite analogous to the semigroup properties encountered several times earlier in this work, especially in Chapters 3, 7, and 8, within the radiative transfer context. A particularly deep analogy exists between the set of transformations

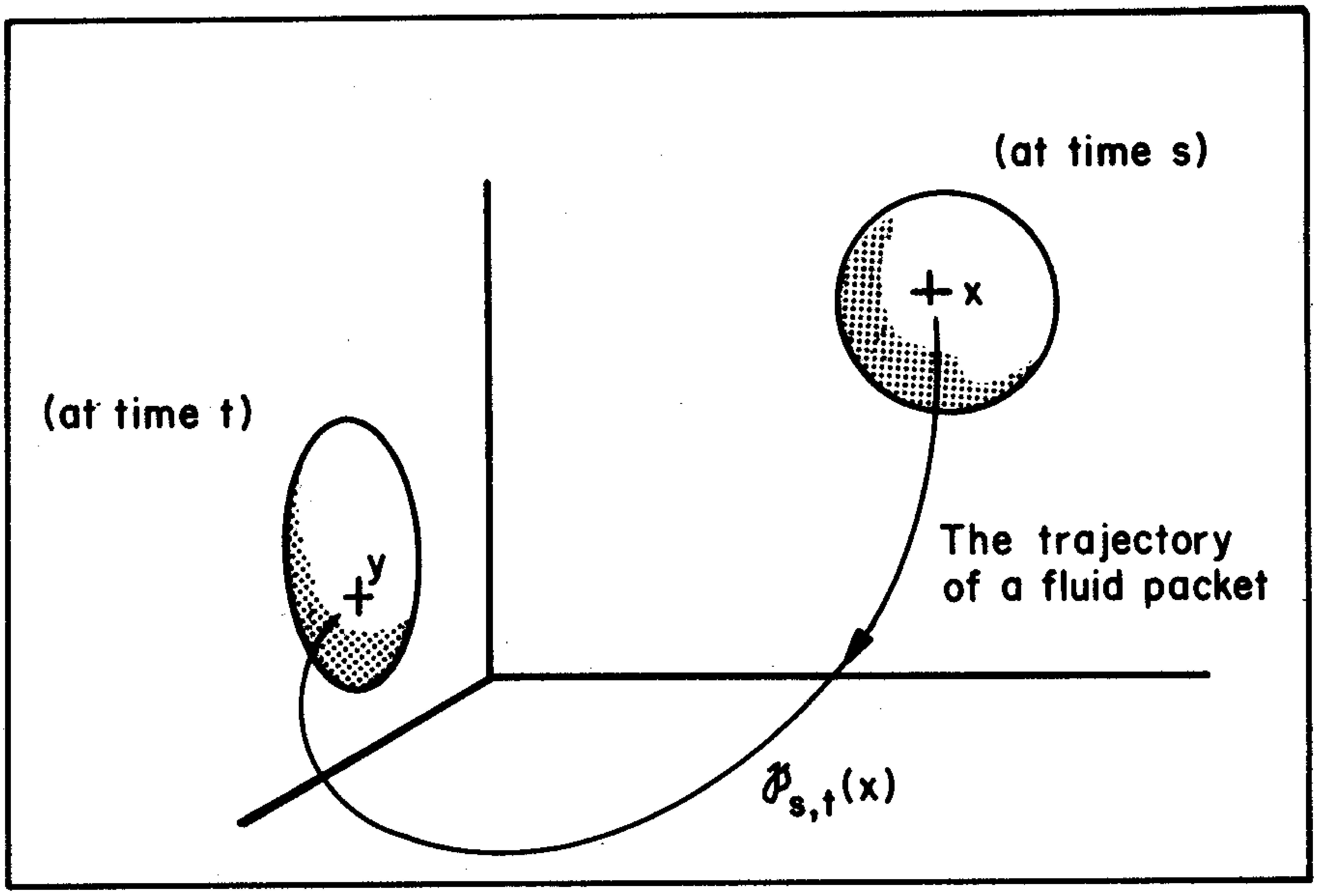

FIG. 12.9 The general trajectory of a fluid packet considered in deriving the equations of hydrodynamic theory.

$\{T_{s,t}: s < t\}$ as defined above, and the transfer process for radiative transfer theory (see Chapter III of Ref. [216]), and it is on this latter analogy that an invariant imbedding (or general group-theoretic) approach to hydrodynamics may possibly be built. The second of the preceding relations, namely (3), asserts the identity property of the family of transformations $\{T_{s,t}: s < t\}$. Hence $x(s) = x$. We shall call the set $\{T_{s,t}: s < t\}$ the *fluid transfer process*. The semigroup property (2) is an analytic embodiment of the uniqueness of the fluid motion under appropriate boundary and initial conditions. The identity property (and certain regularity properties of $T_{s,t}$ not required here) will allow temporal derivatives to exist. The preceding group theoretic formulation of the fluid transfer process is based directly on one of two classical modes of representation of the process, namely the Lagrangian mode of representation (the other is the Eulerian mode).

Physics of the Fluid Transfer Process

The analytic form of the transformations $T_{s,t}$ of the fluid transfer process may be determined by invoking two basic laws of physics. Classical hydrodynamics adopts these two laws in the form of Newton's second law of motion ($F = ma$) for a fluid packet and the law of conservation of mass of a fluid packet. The part of Newton's law which denotes the acceleration of the fluid packet may be represented by means of the fluid transfer process. In fact the transformations $T_{s,t}$ are ideally suited to describe this kinematic part of Newton's law of motion. Thus:

$$\frac{xT_{s,t} - x}{t - s} = \frac{x(t) - x(s)}{t - s}$$

is the average velocity of x over the set $\mathcal{P}_{s,t}(x)$ of points in X comprising the path with initial point x and internal points $xT_{s,t'}$, $s \leq t' \leq t$. Let us write:

$$\text{"}\mathbf{v}(x(t),t)\text{"} \quad \text{for} \quad \lim_{s\to t} \frac{x(t) - x(s)}{t - s} \tag{4}$$

Thus $\mathbf{v}(x,t)$ is the *velocity* of the fluid packet at x at time t. Furthermore,

$$\frac{\mathbf{v}(xT_{s,t},\, t) - \mathbf{v}(x,s)}{t - s} = \frac{\mathbf{v}(x(t),t) - \mathbf{v}(x(s),s)}{t - s}$$

is the average acceleration of x over the same set $\mathcal{P}_{s,t}(x)$ of points as before. Let us write:

$$\text{"}\mathbf{a}(x(t),t)\text{"} \quad \text{for} \quad \lim_{s\to t} \frac{\mathbf{v}(x(t),t) - \mathbf{v}(x(s),s)}{t - s} \tag{5}$$

the vector $\mathbf{a}(x,t)$ is the *acceleration* of the fluid packet at x at time t.

The two fundamental equations of motion of a general fluid packet may now be written down. Let the mass of the fluid packet at time t at point x(t) (= $xT_{s,t}$) on the path $\mathcal{P}_{s,t}(x)$ be denoted by "M(x(t),t)". Then

$$M(x(t),t)\mathbf{a}(x(t)) = \mathbf{F}(x(t),t) \tag{6a}$$

$$M(x(t),t) = M(x(s),s) \tag{6b}$$

The first of these equations is Newton's second law of motion: the mass of the fluid packet times its acceleration equals the net force on the packet. The second equation expresses the invariance of the fluid packet's mass along the packet's path, $\mathcal{P}_{s,t}(x)$. Implicit in the first of these equations is the assumption that the packet remains small as it travels over the path $\mathcal{P}_{s,t}(x)$, so that it may be treated as a moving point mass at x(t) with mass M(x(t),t). This can always be arranged by choosing the initial volume occupied by the packet at x(s) small enough so that its volume at x(t) is still within the acceptable size range. Otherwise, Newton's second law for a point mass must be replaced by more complex mathematical laws--the laws, in fact, which hydrodynamics is primarily formulated to find!

General Equations of Motion of a Fluid

The laws of motion (6) of the fluid packet are the *global forms* of motion, in that they do not expressly use spatial derivatives of the principal quantities involved (M,**F** and **a**). However, the classical approach to the determination of $T_{s,t}$ is to cast (6) into local form, that is differential equations for $x(t)$, the point occupied by the fluid packet. These equations are then to be solved for $x(t)$ over $\mathcal{P}_{s,t}(x)$ given $M(x(s),s)$, $\mathbf{F}(x(s),s)$ and certain boundary conditions. We now undertake this transition to the differential equations of motion of a general fluid packet.

Observe first of all that if the initial position $x(s)$ and the velocity function **v** of the fluid packet are known for all times $t \geq s$, then the requisite location $x(t)$ of that fluid packet is obtainable by a simple integration:

$$x(t) = x(s) + \int_s^t \mathbf{v}(x(t'),t')\, dt' \tag{7}$$

For this reason it is customary in classical hydrodynamics to consider an equation of motion solved if the velocity field **v** can be determined for a given fluid. The velocity function **v** also plays an important role in the further reduction of the equations of motion to differential form. Thus the acceleration $\mathbf{a}(x(t),t)$ may be rendered into a form which is a total derivative of **v** along $\mathcal{P}_{s,t}(x)$. That is, from (5) we see that $\mathbf{a}(x(t),t)$ is the time derivative of the composite function of velocity **v** and position x. This derivative is evaluated using the concepts of vector analysis. The result is:

$$\mathbf{a}(x(t),t) = \frac{D\mathbf{v}(x(t),t)}{Dt} \tag{8}$$

where we have written:

$$\frac{\text{"}D\text{"}}{Dt} \quad \text{for} \quad \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{9}$$

and where "∇" denotes the gradient operator. (See (5) and (6) of Sec. 3.15 for an earlier use of D/Dt.) This operator known as the *mobile-derivative*, or *Lagrangian derivative* operator, is used quite generally to find the rate of change of some function describing a small aggregate of things moving along some path in space, whether the aggregate be comprised of photons or fluid particles or other substances. Thus if f is a general function which assigns a quantity $f(x(t),t)$ to each time t and associated position $x(t)$ of the packet along the path $\mathcal{P}_{s,t}(x)$, then:

$$\frac{Df(x(t),t)}{Dt} = \lim_{s \to t} \frac{f(x(t),t) - f(x(s),s)}{t - s}$$

$$= \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla \qquad (10)$$

In particular, if f is the mass function for the fluid packet we see from (6) that:

$$\frac{DM(x(t),t)}{Dt} = 0 \quad . \qquad (11)$$

Equation (11) is the *equation of continuity* for the mass of the fluid packet. Now when the fluid packet is quite small, as we intend it to be in these discussions, then the mass M(x,(t),t) is expressible as:

$$M(x,(t),t) = \rho(x(t),t)V(x(t),t) \qquad (12)$$

where ρ and V are respectively the mass density and volume functions for the fluid packet. Now it is easy to verify by (10) that the mobile derivative operator acting on a product of functions works exactly analogously to the ordinary derivative operator. Hence from (11) and (12):

$$\frac{DM}{Dt} = \frac{D(\rho V)}{Dt} = \rho\frac{DV}{Dt} + V\,\frac{D\rho}{Dt} = 0 \quad ;$$

whence:

$$\frac{1}{\rho}\,\frac{D\rho}{Dt} + \frac{1}{V}\,\frac{Dv}{Dt} = 0 \quad . \qquad (13)$$

In this way we have split the variation of M into two parts: a purely geometric part involving the volume V of the packet and an ideal physical part involving the density of the packet. It is easy to see that the purely geometric part of (13) is represented by:

$$\frac{1}{V}\,\frac{DV}{Dt} = \nabla \cdot \mathbf{v} \quad , \qquad (14)$$

which is quite plausible intuitively, and incidentally an excellent way of picturing the geometric significance of the divergence of the velocity field.

With these preliminaries established we can cast the basic equations of motion (6) into the more familiar differential (i.e., local) form:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{f} \qquad (15)$$

$$\frac{1}{V}\frac{DM}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0 \qquad (16)$$

where we have written:

"$\mathbf{f}$" for $\mathbf{F}/M$,

i.e., $\mathbf{f}$ is the force per unit mass on the fluid packet.

An outstanding feature of the equations of motion, namely the nonlinearity of the hydrodynamic equations, is evident in (15) wherein the velocity function $\mathbf{v}$ is multiplied by its derivatives in the term $\mathbf{v} \cdot \nabla\mathbf{v}$. It is this particular nonlinearity which has launched the search for countless linear and simplified nonlinear hydrodynamic models of fluids. For, by suitably choosing the components of $\mathbf{f}$, the behavior of ρ, and the behavior of $\mathbf{v}$ itself, various simplifications of (15) and (16) can be effected which lead to tractable equations of motion. We shall now adopt those assumptions which lead us to the hydrodynamic models of the air-sea surface that are of interest in the optical studies of the present chapter.

Special Equations of Motion for the Air and Water Masses

The model we shall adopt for the equations of motion describing the air-water surface and the masses it bounds, rests on two main sets of assumptions: one set is about the fluids on each side of the air-water surface, and the other set is about the form of the surface itself and the forces in its immediate neighborhood. We first consider the movements of a packet X in either the air or water mass. We shall limit the forces on the packet X to consist only of gravity and normal surface pressures arising from contact with other packets. (Thus viscosity forces are assumed negligible along with tidal and coriolis forces.) The force of gravity on a unit mass of matter at or near the surface of the earth is of magnitude g, the gravitational acceleration constant (980 cm/sec^2 ≅ 32 ft/sec^2) and of direction $-\mathbf{k}$ in a terrestrially based coordinate frame (Fig. 12.10). If $\mathbf{n}(y)$ is the unit inward normal to the packet X at a point y on its surface S and p(y) is the associated (scalar) normal pressure then the net force on the small packet is:

$$M\mathbf{f} = \int_S p(y)\mathbf{n}(y)\,dA(y) - \mathbf{k}\int_X \rho(x) g\; dV(x)$$

$$= -\int_X \nabla p(x)\, d\,V(x) - \mathbf{k}\int_X p(x)\, g\, d\,V(x) \qquad (17)$$

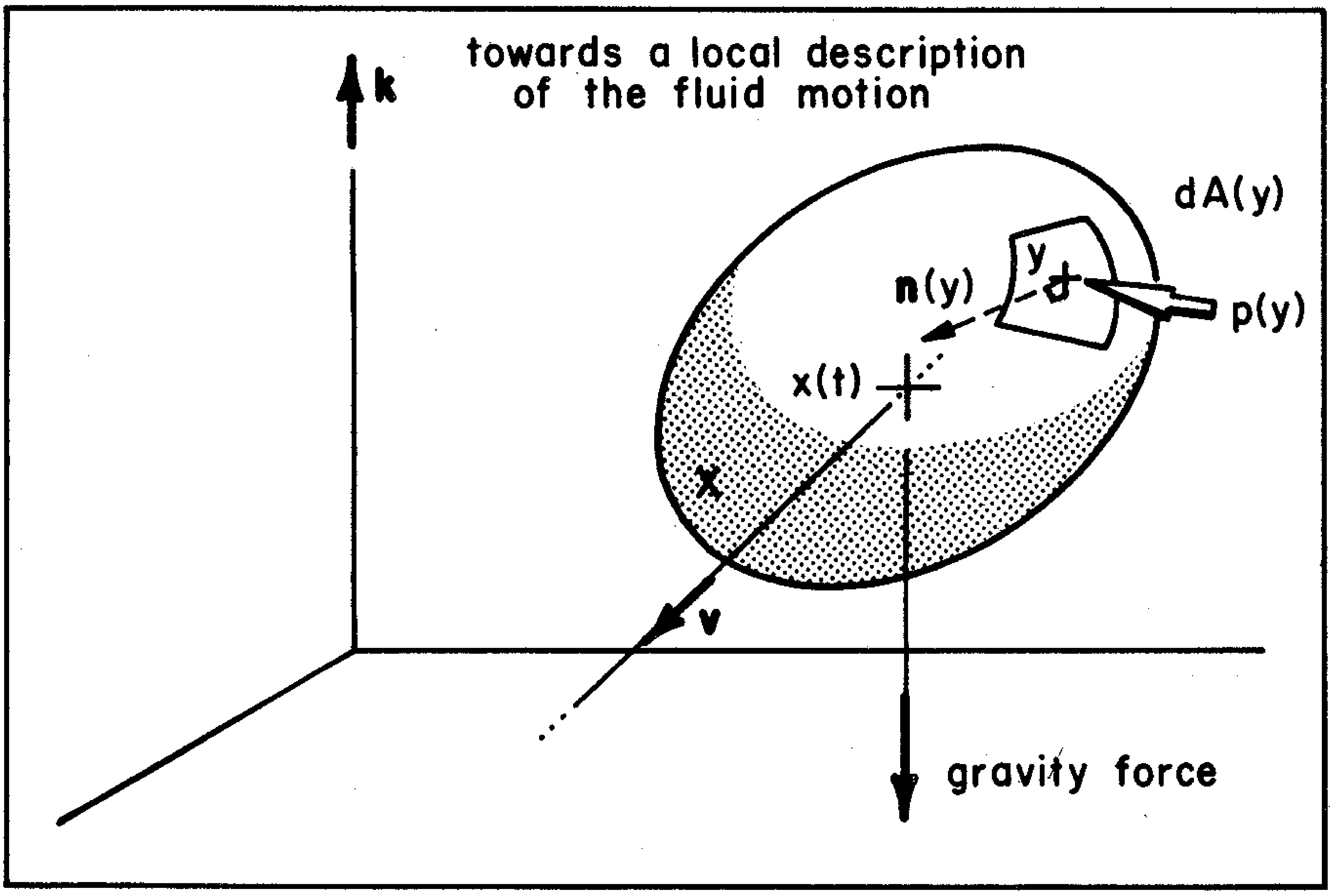

FIG. 12.10 Considering the forces on a fluid packet in the derivation of its equations of motion.

where the transition from the surface integral to the volume integral is by means of Gauss' theorem. From (15) we have:

$$M \frac{D\mathbf{v}}{Dt} = M\mathbf{f} = - \int_X [\mathbf{k}\, \rho(x)\, g + \nabla p(x)]\, dV(x)$$

which over the small volume of X may be written as:

$$\int_X \left[\rho \frac{D\mathbf{v}}{Dt} + \mathbf{k}\, \rho\, g + \nabla p\right] dV = 0 \quad .$$

Since X is arbitrary, it follows that (15) can be cast into the form:

$$\frac{D\mathbf{v}}{Dt} = - \left[g\,\mathbf{k} + \frac{1}{\rho}\nabla p\right] \quad , \qquad (18)$$

Continuing to work on this version of (15), we observe that:

$$\nabla(g\ z) = g\,\mathbf{k} \quad . \qquad (19)$$

In other words there is a function χ, namely that whose value at point (x,y,z) is $\chi(x,y,z) = g\,z$, and whose gradient is $g\,\mathbf{k}$. This allows us to write (18) as:

$$\frac{D\mathbf{v}}{Dt} = - \left[\nabla g z + \frac{1}{\rho} \nabla p\right] \tag{20}$$

where z is now the z-coordinate of the fluid packet. It is tempting to remove the gradient operator outside the brackets in (20). Thus (20) supplies a strong motivation for requiring ρ to be some nonzero constant independent of location at all times; in other words for requiring the fluid to be incompressible. Therefore we shall assume:

$$\nabla \rho = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial t} = 0 \tag{21}$$

and see where it leads us. This assumption permits, first of all, (20) to be written as:

$$\frac{D\mathbf{v}}{Dt} = - \nabla \left[g z + \frac{p}{\rho}\right] \tag{22}$$

and secondly allows (16) to be simplified to:

$$\nabla \cdot \mathbf{v} = 0 \quad . \tag{23}$$

Hence the equations of motion of the fluid are now of the form (22), (23) as a result of assumptions (17) and (21).

We next concentrate on the term $\mathbf{v} \cdot \nabla\mathbf{v}$ in $D\mathbf{v}/Dt$. This is a nonlinear term and a traditional mathematical trouble spot for classical hydrodynamics. Its presence is eased out of the present picture by noting that it is the vectorial counterpart to the simpler scalar case $v(dv/dx)$. This latter term may be written as $(1/2)\, d v^2/dx$. A search for a vectorial counterpart to this scalar situation uncovers the identity:

$$\frac{1}{2} \nabla v^2 = \mathbf{v} \cdot \nabla\mathbf{v} + 2\mathbf{v} \times \nabla \times \mathbf{v} \quad . \tag{24}$$

where we have written

" v " for $|\mathbf{v}|$

i.e., v is the magnitude of the velocity vector $\mathbf{v}$. Using this identity (24) in (22) we have:

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\frac{1}{2} \nabla v^2 - 2\mathbf{v} \times \nabla \times \mathbf{v}\right) = - \nabla \left[g z + \frac{p}{\rho}\right] \quad . \tag{25}$$

It seems that we have traded one complication (namely $\mathbf{v} \cdot \nabla\mathbf{v}$) for another (namely $\mathbf{v} \times \nabla \times \mathbf{v}$). However a further simplification follows if we observe that the term $\nabla \times \mathbf{v}$ describes the local rotation of the fluid motion, that is, by Stokes theorem:

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dA = \int_C \mathbf{v} \cdot \mathbf{t}\, ds$$

where t is a unit tangent to any closed curve C bounding a surface S in the fluid, and **n** is a unit outward normal to S. The line integral evidentally describes the circulation of the fluid around the curve C. Now in the present fluids (air water) and for our present purposes, this circulation turns out to be a relatively unimportant motion of the fluid as compared to its translatory motion. This observation permits us to make one more assumption, namely:

$$\nabla \times \mathbf{v} = 0 \tag{26}$$

in addition to (17) and (21). As a result (25) becomes:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left[\frac{1}{2} v^2 + g z + \frac{p}{\rho} \right] = 0 \quad . \tag{27}$$

By a theorem in vector analysis, we now can assert that, by virtue of (26), there exists a scalar valued function ϕ defined in the domain of the fluid (either air or water) such that:

$$\boxed{\mathbf{v} = - \nabla \phi} \quad . \tag{28}$$

We call ϕ the *velocity potential* for the appropriate fluid. The minus sign in (28) is conventional, though it can be justified using simple physical interpretations of ϕ. Using (28), equation (27) can now be written as:

$$\nabla \left[- \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g z + \frac{p}{\rho} \right] = 0 \quad .$$

This means that all three spatial derivatives of the bracketed quantity are zero, so that, at most, the bracketed quantity can be an arbitrary function of time, say C. Hence:

$$\boxed{- \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + g z + \frac{p}{\rho} = C(t)} \quad . \tag{29}$$

Further, the continuity equation (23) now becomes (using (28)):

$$\nabla \cdot (\nabla \phi) = 0 \quad ,$$

that is:

$$\boxed{\nabla^2 \phi = 0} \quad . \tag{30}$$

In this way we finally arrive at the required forms of the equations of motion for the air-water masses. Thus, we have recast Newton's law of motion (6a) in the guise of (29) (*the pressure equation*) and the law of conservation of mass (6b) in the guise of (30) (*the potential equation*) under the following assumptions (repeated here for reference):

(i) $\nabla \rho = 0$, $\partial\rho/\partial t = 0$ (incompressible fluid)

(ii) $\nabla \times \mathbf{v} = 0$ (irrotational motion)

(iii) $\mathbf{f}$ consists only of gravitational and scalar pressure forces.

Our main task henceforce will be to solve, with the help of (29), the potential equation (30), subject to suitable boundary conditions, so that $\mathbf{v}$ can be determined and hence also the form of the fluid transfer operators $T_{s,t}$, via (7) and (1). The principal boundary conditions required for this task are those for the air-water surface, to which we now turn.

Surface Kinematic Condition

The first of the principal boundary conditions to which the equations of motion (29) and (30) are to be subjected will now be considered. This condition ties together the movement of the air-water film with the motion of the bodies of air and water on either side of it.

Suppose that ζ is the function which assigns to each pair of spatial variables x and y and each time t the elevation $\zeta(x,y,t)$ of the air-water surface above (or below) point (x,y) in some datum plane at time t. This datum plane may be a mean sea surface, an average bottom surface, or some other hypothetical surface. The function ζ is the function of principal interest in the study of the air-water surface. It is one of the principal problems of hydrodynamics to describe ζ as a function of x, y, and t, given appropriate mathematical constraints based on geophysical conditions. Indeed, for the remainder of this chapter we shall be concerned with ways and means of describing the spatial and temporal behavior of ζ in order that the problem of radiative transfer at the dynamic air-water surface can be solved on various levels of detail.

Now suppose that at time s the center of a small water packet is located at point $(x',y',\zeta(x',y',x))$ $(= x(s))$. Thus, for all practical purposes the center of this particular packet describes the location of the air-water surface at instant s. (See Fig. 12.11.) Suppose that at a little later time t, the packet still comprises part of the air-water boundary but that its center's location is now at $(x,y,\zeta(x,y,t))$ $(= x(t))$. Then, on the one hand, since the packet journeys with the air-water film over the time interval (s,t):

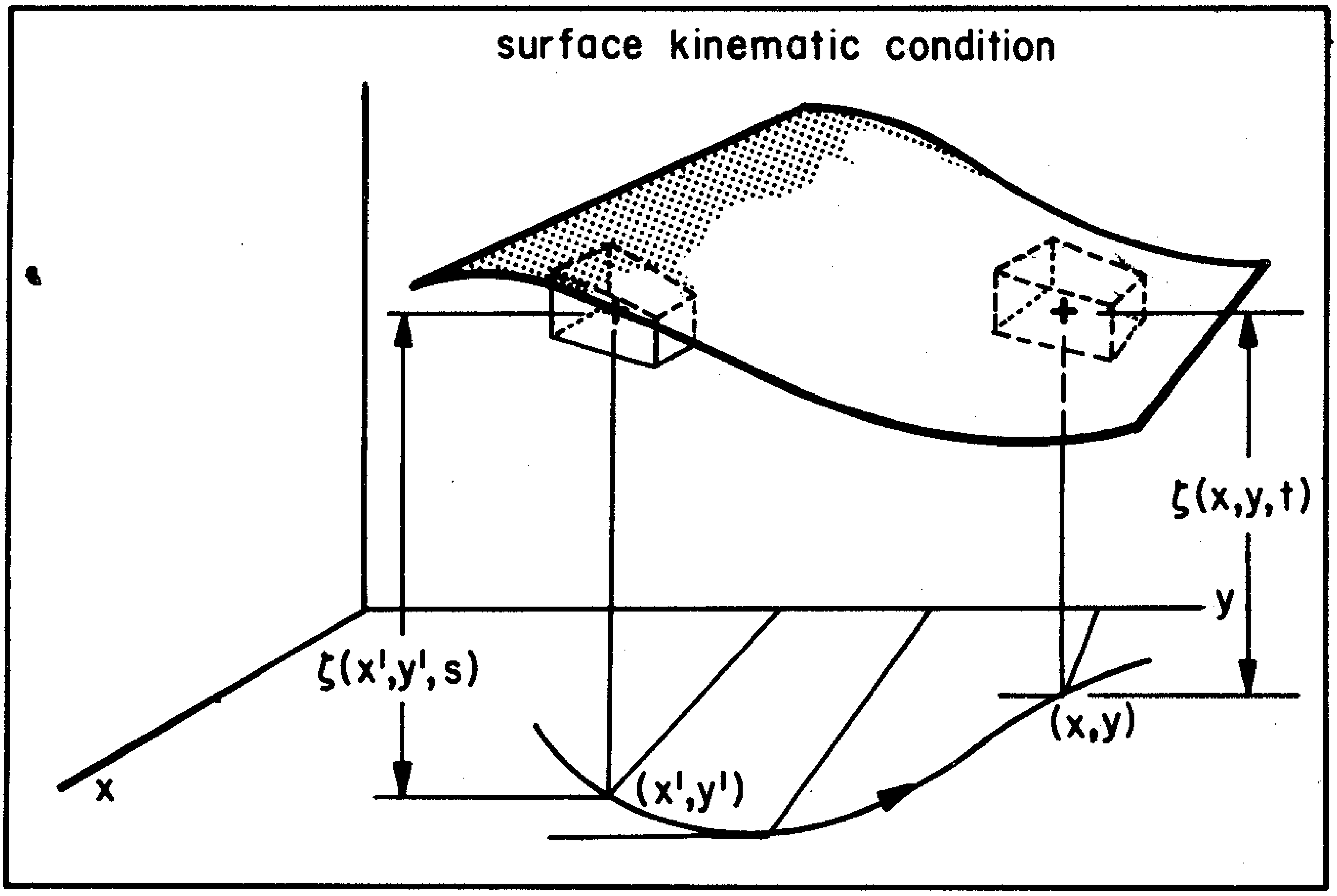

FIG. 12.11 Following the motion of a fluid packet at the air-water surface yields an important boundary condition (the surface kinematic condition) for the hydrodynamic equations governing water waves.

$$\lim_{s\to t} \frac{x(s)T_{s,t}-x(s)}{t-s} = \lim_{s\to t} \frac{x(t)-x(s)}{t-s}$$

$$= \lim_{s\to t} \frac{(x,y,\zeta(x,y,t) - (x',y',\zeta(x',y',s))}{t-s}$$

$$= \lim_{s\to t} \frac{(x-x',y-y',\zeta(x,y,t) - \zeta(x',y',s))}{t-s}$$

$$= \lim_{s\to t} \left(\frac{x-x'}{t-s}, \frac{y-y'}{t-s}, \frac{\zeta(x,y,t) - \zeta(x',y',s)}{t-s}\right) . \qquad (31)$$

On the other hand, since the packet is part of the whole fluid mass, we have:

$$\lim_{s\to t} \frac{x(s)T_{s,t} - x(s)}{t-s} = \mathbf{v}(x(s),s) \qquad (32)$$

by (4).

By equating the z-components of each of the vectors in (31) and (32), we have the desired condition on ζ. Thus by (10), (30), and (32), we can write:

$$\boxed{W = \frac{D\zeta}{Dt}} \tag{33}$$

where "W" denotes the z-component v_z of **v**, the other two components being denoted by "U" and "V". Equation (33) is the desired connection. An alternate form of (33) may be obtained as follows. By (28), W may be represented as:

$$W = -\frac{\partial\phi}{\partial z} \quad . \tag{34}$$

Using (34) in the following expanded version of (33):

$$-\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x} + V\frac{\partial\zeta}{\partial y} \quad ,$$

and using (28) once again to replace U and V by ϕ-derivatives we finally arrive at:

$$\boxed{\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y} - \frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t}} \tag{35}$$

which is called the *surface kinematic condition*. It serves to tie together the surface elevation function ζ with the velocity potential ϕ at the surface.

Surface Pressure Condition

The second and last of the conditions required for the air-water surface in the present study concerns an analytic connection between the pressures within the air and within the water media in the immediate neighborhood of the air-water surface.

To see the nature of this connection first imagine the air-water surface to be flat calm and the air and water masses to be at rest. Then each small patch of air-water surface is in static equilibrium, so that all forces on it add up to zero. The forces on the patch are those of the pressing downward of air on its upper side, the resistive pushing upward of the water below, and tensive forces acting within the plane of the film and arising from the molecular forces of the fluids on either side of the film.

If the surface tension forces were to be disturbed, as may be accomplished, for example, by placing a chemical wetting agent in the water, then there is an abrupt tearing motion of the air-water film, very much like the tearing motion

of a rubber balloon that has been punctured. (This experiment with water surfaces can be performed using droplets of certain household detergents or toothpastes. By this means, the tension can actually be measured. It turns out that the surface tension force is on the order of 74 dynes per centimeter at room temperature.) Such an experiment serves to point up vividly the important role that surface tension plays in the configuration of the air-water surface. Moreover, if one now blows gently on a clean air-water surface, the breath of air pushes a roundish concave dimple in the water and as long as the gentle air stream is maintained the dimple will persist and the small patch of curved surface will remain in equilibrium with the three principal forces: air pressure, water pressure, and surface tension. At each instant these forces are adding up to zero. The same phenomenon occurs when one blows up a toy balloon: there is, in the resultant configuration, a well-defined relation between the air pressures inside and outside the balloon, the curvature of the balloon, and the tensile forces within the balloon's surface. Figure 12.12(a) depicts the common essence of these two situations. A small rectangular patch of surface S is in equilibrium with pressures p_a and p_w (force per unit area, respectively, induced say by air and water masses)
Errata and surface tension T (force per unit boundary length) acting over it. For simplicity, the surface is assumed for the moment to take the shape of a circular cylinder of radius R and that the dimensions of the patch of surface are $2R\theta$ by a (measured into the plane of the figure). Hence the area of the patch is $2R\theta a$, and the net downward force on the patch induced by the pressures is:

$$(p_a - p_w)\,2R\theta a \quad .$$

This force is exactly balanced by the upward component of the tension acting over the two edges of the patch indicated end-on in the figure. This upward component is $-2T_1 a\theta$. Here $T_1 a$
Errata is force per unit length so that $T a$ is total force acting in the tangent planes to the surface on each side of the patch. θ is the approximant to $\tan\theta$, the actual number required to find the upward component of the tension. Since the patch is in equilibrium we have:

$$-\,2T_1 a\theta = (p_a - p_w)\,2R\theta a \quad ,$$

whence:

$$T_1 = (p_w - p_a)R \quad .$$

or alternately:

$$\boxed{p_w - p_a = \frac{T_1}{R}} \qquad (36)$$

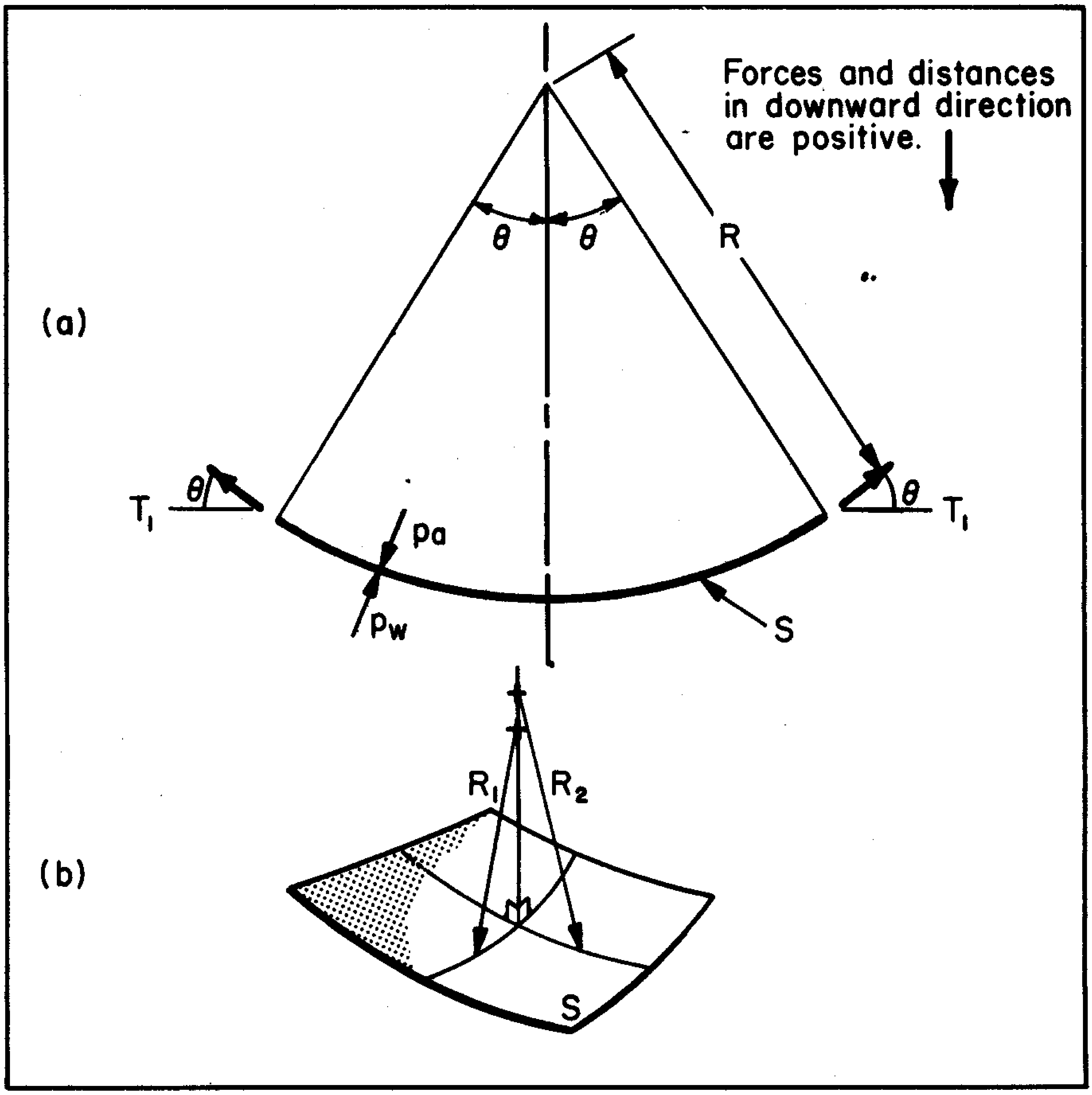

FIG. 12.12 The surface pressure condition introduces surface tension forces into the equations of motion for water waves.

which is the required *surface pressure condition.*

Observe how the assumption that θ is small is built into the derivation. Our use of (36) below will remain within the domain of this approximation.

The reader may now readily show, using the same principles, that if the patch were not of cylindrical form but of double curvature with principal radii of curvature R_1 and R_2 ((b) of Fig. 12.12) then

$$p_w - p_a = T_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{37}$$

which reduces to (36) when R_2, say, is infinite. Further generalizations may be made, such as having T_1 depend on direction of extension within the surface, but such generality will not be needed here; indeed, (36) will suffice for our purposes.

Sinusoidal Wave Forms

Our present object is to see if the equations of motion (29) and (30) along with the surface conditions (35) and (36) yield solutions of the surface function ζ which are recognizable as the periodic waves and ripples we see on the surfaces of ponds, lakes, seas and other natural hydrosols. That is, we are looking for functions ζ such that:

$$\zeta(x,y,t) = a \cos(kx - \sigma t + \varepsilon) \quad . \tag{38}$$

Clearly the graph of ζ as given in (38) is sinusoidal and of amplitude a. The whole sinusoidal wave form moves to the right, as in Fig. 12.13, with a speed such that the argument of cos in (38) is constant. In particular, if the constant value of the argument of cos is zero (so that we move with a crest of the wave) then

$$kx - \sigma t = 0$$

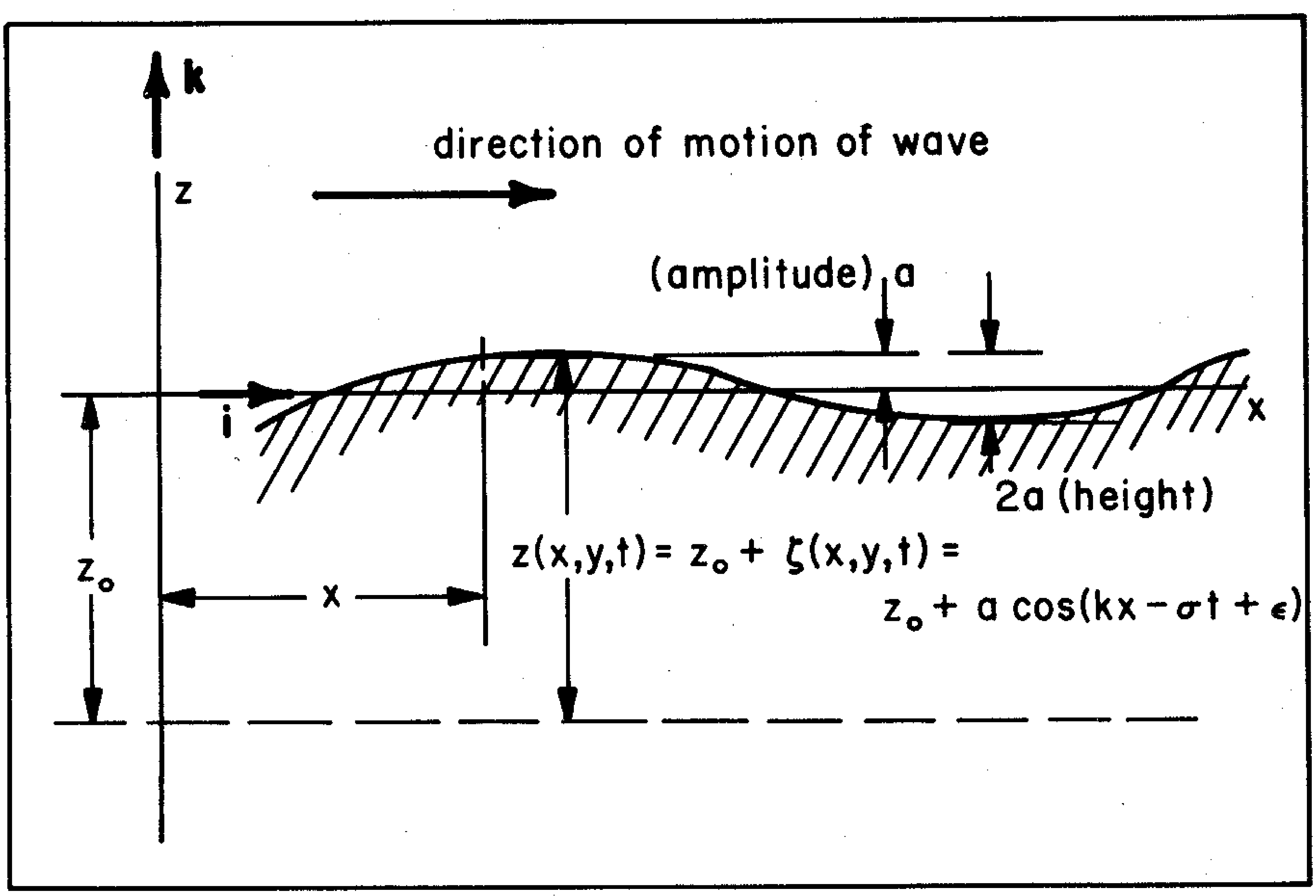

FIG. 12.13 A small-amplitude sinusoidal water-surface wave as predicted by the linearized equations of hydrodynamics (the contour lines are normal to the plane of the diagram).

implies

$$x = \frac{\sigma}{k} t$$

so that the speed c of the wave form is σ/k. We shall write

$$\text{"c" for } \sigma/k \quad . \tag{39}$$

Thus the elevation function ζ in (38) constitutes a model of a long-crested (i.e., essentially cylindrical) sinusoidal wave, of *amplitude* a and *phase speed* (*celerity*) c, and *phase* ε. Observe in particular that the wave surface, as given in (38), is cylindrical with the cylinder generators perpendicular to the plane of Fig. 12.13. We shall need to consider only such cylindrical (one-dimensional) waves in order to develop a workable model of dynamic air-water surfaces. Quite complex seas can be synthesized by suitably superimposing wave forms of the type (38). Thus in general a one-dimensional sinusoidal wave train moving at an angle θ with respect to the x-axis (Fig. 12.14) may be represented as:

$$\zeta(x,y,t) = a \cos (\mathbf{k} \cdot \mathbf{r} - \sigma t + \varepsilon) \tag{40}$$

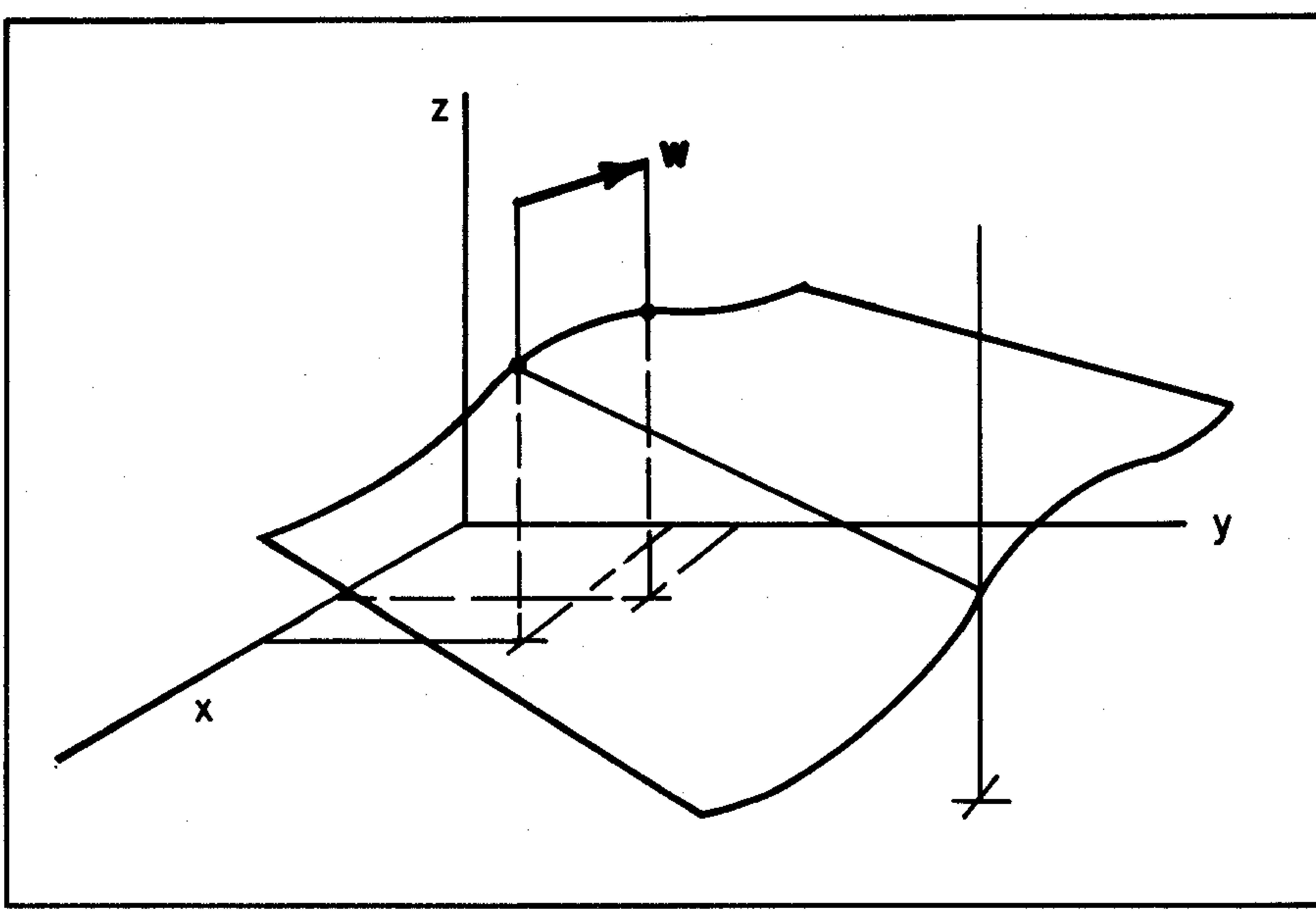

FIG. 12.14 The small-amplitude sinusoidal water wave of Fig. 12.13 now traveling in a general direction over the xy direction.

Errata

where we have written:

$$\text{"}\mathbf{k}\text{"} \quad \text{for} \quad (u,v)$$

$$\text{"}\mathbf{r}\text{"} \quad \text{for} \quad (x,y) \quad .$$

and so

$$\mathbf{k} \cdot \mathbf{r} = ux + vy = kx \cos\theta + ky \sin\theta$$

where we write

$$\text{"k"} \quad \text{for} \quad |\mathbf{k}| \quad .$$

The length k of the vector $\mathbf{k}$ is the *wave number* of the wave form in (40). The wavelength λ and the celerity of the wave form are clearly:

$$\lambda = 2\pi/k \quad , \quad c = \sigma/k \quad .$$

For the most part we can simplify the exposition by setting $v = 0$ in (40), so that the waves progress parallel to the x-axis. Henceforth this simplification will be in force. There will be no essential loss in generality by adopting this simplification.

Linearized Equations of Motion

We now turn to the actual details of the search for sinusoidal solutions of the form (38) of the equations of motion (29) and (30), subject to the conditions (35) and (36). Some preliminary experimentation with these equations shows that the desired solutions are forthcoming if we assume that the sinusoidal wave forms have small slope $d\zeta/dx$ (Fig. 12.13) and furthermore that the speed v of the fluid packets (air and water) are small so that v^2 may be set to zero in (29). Hence we in effect must make the assumptions:

(iv) $v^2 < < 1$

(v) $(d\zeta/dx)^2 < < 1$

(vi) product terms negligible in (35)

(vii) $\varepsilon(t) = 0$

where the numbering continues the list of assumptions begun below (30). Assumption (vii) makes use of the arbitrariness of the function C in (29). As a result we have from (29), (iv) and (vii):

$$\boxed{-\frac{\partial\phi}{\partial t} + gz + \frac{p}{\rho} = 0} \qquad (41)$$

and from (30) and (38) (i.e., wave forms independent of y):

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0} \tag{42}$$

and from (35) and (vi):

$$\boxed{-\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t}} \quad . \tag{43}$$

Finally, the curvature 1/R of the sinusoidal wave form is such that, by (v), 1/R is essentially $\partial^2\zeta/\partial x^2$, so that from (36):

$$\boxed{p_w - p_a = T_1 \frac{\partial^2\zeta}{\partial x^2}} \quad . \tag{44}$$

Equations (41) and (42), are the *linearized equations* of fluid motion and (43) and (44) the associated *linearized surface conditions*. Equations (41) and (42) describe ϕ in either the air domain or the water domain. To distinguish between these domains when using ϕ (or other concepts) we shall append to "ϕ" (or other symbols) a subscript "a" or "w", as the case may be.

Classical Wave Model

We now show how (41)-(44) yield sinusoidal wave forms in a case of extreme simplicity and of surprisingly wide applicability. We assume that the waves are so small and mildly curved that $\partial^2\zeta/\partial x^2$ is negligible so that, by (44) $p_a = p_w$ at the surface. Since the pressures play no further role in the fluid flows, we may reset our pressure scales so that $p_a = p_w = 0$, so that (41) becomes:

$$\zeta = \frac{1}{g}\frac{\partial\phi}{\partial t} \tag{45}$$

at the surface. This shows how to find ζ once ϕ is determined for either the air or water mass in the neighborhood of the surface. From (43) and (45) we see that:

$$-\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} = \frac{1}{g}\frac{\partial^2\phi}{\partial t^2} \quad .$$

In other words, we have:

$$\boxed{\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial z} = 0} \tag{46}$$

at the air-water boundary. It is interesting to note in passing that this equation has the general form of a diffusion equation with respect to depth z, At the boundary we choose $\phi(x,z,t)$ to be a sinusoidal wave form of the kind:

$$\phi(x,0,t) = be^{i(kx - \sigma t + \varepsilon)} \tag{47}$$

where b is an arbitrary constant and k, and σ are parameters to be suitably determined. Using this form of ϕ in (46) we reduce (46) to:

$$\frac{\partial\phi}{\partial z} = \left[\sigma^2/g\right]\phi \tag{48}$$

for z at the air-water surface.

Taking the hint from (47) let us assume that

$$\phi(x,z,t) = \Phi(z)\phi(x,o,t) \tag{49}$$

in other words, that at every depth z, $\phi(x,z,t)$ may be written as the product of the values of function Φ of z and $\phi(x,0,t)$. In this way we can set into motion a standard separation of variables technique in the solution of the partial differential equation (42). Using (49) in (42) we arrive at:

$$\frac{d^2\Phi}{dz^2} - k^2\Phi = 0 \quad . \tag{50}$$

We have now used all four equations (41) through (44) to arrive at (50).

The general solution of (50) is of the form

$$\Phi(z) = d_+ e^{kz} + d_- e^{-kz} \quad . \tag{51}$$

The constants $d_\pm$ may be evaluated for each hydrosol of constant depth h by noting that

$$\Phi(0) = 1 \tag{52}$$

as required by (49), and that:

$$\frac{\partial\phi}{\partial z} = 0 \tag{53}$$

at depth h. Here we are using the fact that there is no vertical motion of the fluid at the lower boundary, and translating it into mathematical form via (28). By (49), condition (53) may be expressed as:

$$\frac{d\Phi}{dz} = 0 \quad , \tag{54}$$

at depth h. From (51) and (52) we have:

$$d_+ + d_- = 1 \quad , \tag{55}$$

and from (54) we have:

$$d_+ \, e^{kh} - d_- \, e^{-kh} = 0 \quad , \tag{56}$$

whence:

$$d_+ = \frac{e^{-kh}}{e^{kh} + e^{-kh}} \quad , \tag{57}$$

and

$$d_- = \frac{e^{kh}}{e^{kh} + e^{-kh}} \quad . \tag{58}$$

We may then express $\Phi(z)$ as:

$$\Phi(z) = \frac{\cosh k(z-h)}{\cosh kh} \quad . \tag{59}$$

In the case of infinitely deep media, we have for every fixed depth z:

$$\lim_{h\to\infty} \frac{\cosh k(z-h)}{\cosh(kh)} = e^{-kz} \quad , \tag{60}$$

so that in such media:

$$\Phi(z) = e^{-kz} \tag{61}$$

In this way we arrive at the following representations for the velocity potential for finitely deep hydrosols:

$$\phi(x,z,t) = b \, \frac{\cosh k(z-h)}{\cosh(kh)} \, e^{i(kx - \sigma t + \varepsilon)} \quad \text{(finite depth)} \tag{62}$$

and

$$\phi(x,z,t) = b \, e^{-kz + i(kx - \sigma t + \varepsilon)} \quad \text{(infinite depth)} \tag{63}$$

for infinitely deep hydrosols.

Having found ϕ, we can now deduce from it a multitude of physical results. For example, using ϕ, as given in (62), in (28), we can find the velocity of the water packets at any depth z, and any time t. Using (7) the trajectories of the packets are determinable in detail (they turn out to be ellipses which in deep media decrease exponentially with size as depth increases). However, the most important result for the present studies is the determination of the celerity c of the wave form as given in (39). According to (39), to find c, we must know σ, and this in turn is characterized by (48). Using (62) in (48) and setting z = 0, we have:

$$c = \frac{\sigma}{k} = \left[\frac{g}{k} \tan h(kh)\right]^{1/2} \tag{64}$$

For very deep waters ($h = \infty$), we have:

$$c = \sqrt{g/k} = \sigma/k \tag{65}$$

Oceanographers and other geophysicists occasionally prefer to have c in terms of the wave length λ. Since, as a perusal of (40) has shown,

$$\lambda = 2\pi/k \tag{66}$$

(64) may be written:

$$c = \left[\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}\right]^{1/2} \tag{67}$$

and (65) becomes:

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \frac{\sigma}{k} \tag{68}$$

which is the classical equation for the celerity c of deep water gravity waves in terms of their wavelengths λ. Equation (66), incidentally, shows the reciprocal relation between the wave number k and wavelength λ associated with any sinusoidal wave.

From (45) and (62) we can now obtain an explicit formula for the form of the air-water surface. Retaining only real parts (which is tantamount to using only the real part of ϕ throughout this discussion) we have (to within some arbitrary phase angle ε of cos):

$$\zeta(x,y,t) = a \cos(kx - \sigma t + \varepsilon) \tag{69}$$

where:

$$a = - b\,\sigma/g \quad .$$

Kelvin-Hlemholtz Model

We finally arrive at the heart of the present discussion of hydrodynamics with a derivation of the Kelvin-Helmholtz model for waves on the air-water surface. The Kelvin-Helmholtz theory builds on the classical wave model just completed by assuming two further physical features of the air-water masses: First, the air and water masses are no longer at rest, but rather set into motion with speeds U_a and U_w, respectively, along the x axis. This simulates the movement of wind over a hydrosol which itself may be drifting along at some speed U_w. Second, the surface tension forces are allowed to act within the air-water boundary, so that capillary waves--or *ripples* as they were called by Kelvin--can be explicitly incorporated in the theory of surface waves. The first of these additional features was studied about a hundred years ago (1868) by Helmholtz.* The second additional feature was combined with Helmholtz's hypothesis several years later (1871) by Kelvin. Both men used their respective theories to carry out some of the first studies of the phenomenon of wind-generated waves. Each model, as crude as it was, showed that there were critical wind speeds, relative to the speed of the hydrosol, at which waves of either gravity or capillary type would begin to grow exponentially in amplitude. Below these critical wind speeds, the wave forms are stable and the celerity c is a well-defined number depending on the speeds U_a and U_w, the densities ρ_a, ρ_w, and surface tension T_1.

In accordnace with the introductory remarks above we imagine the air and water masses set into translatory motion along the x axis with speeds U_a and U_w, respectively, and that a sinusoidal wave motion is superimposed on these motions at the interface. Hence we assume that the velocity potentials ϕ_a and ϕ_w for these media are of the form:

$$\phi_a = - U_a x + \phi_a' \tag{70}$$

$$\phi_w = - U_w x + \phi_w' \quad . \tag{71}$$

The potentials ϕ_a' and ϕ_w' are to be viewed as small oscillatory perturbations of $U_a x$ and $U_w x$ and such that ϕ_a' and ϕ_w' and their associated motions are subject to the conditions (iv) through (vii) leading to the classical linearized equations (41) through (44). Hence all our results for the classical wave model are applicable to those components of the motions generated by ϕ_a' and ϕ_w'. In particular for the present model we shall assume infinitely deep media, so that ϕ_a' and ϕ_w' are given by (63) as follows:

$$\phi_a' = b_a \, e^{+kz + i(kx - \sigma t)} \tag{72}$$

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*For references to these models, see pp. 22, 374, 459 of [149].

$$\phi'_w = b_w e^{-kz + i(kx - \sigma t)} \tag{73}$$

where, as agreed, distance is measured positive downward into the water. Furthermore since a translated sinusoid is still sinusoidal, the form of the air-water surface will be given by:

$$\zeta = a\, e^{i(kz - \sigma t)} \tag{74}$$

which is the complex form of (69). To return to the physically meaningful setting we shall need only take the real parts of all complex expressions, as usual.

Having fixed the form of the velocity potential functions and the desired sinusoidal form of the air-water surface, it remains to see what conditions are imposed on the celerity c of the required sinusoidal wave form by the additional wind speed and surface tension conditions. Toward this end, we return to the surface kinematic condition (35) and note its present forms for the air and water masses:

$$\frac{\partial \zeta}{\partial t} + U_a \frac{\partial \zeta}{\partial x} = - \frac{\partial \phi_a}{\partial z} \tag{75}$$

$$\frac{\partial \zeta}{\partial t} + U_w \frac{\partial \zeta}{\partial x} = - \frac{\partial \phi_w}{\partial z} \tag{76}$$

which follow by recalling that $V = 0$ for either fluid, i.e., that $V_a = V_w = 0$, by hypothesis. Furthermore, the pressure equation (29) for each fluid now takes the form (let $((t) = 0)$:

$$\frac{p_a}{\rho_a} = \frac{\partial \phi_a}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial \phi_a}{\partial x}\right)^2 + \left(\frac{\partial \phi_a}{\partial z}\right)^2\right] - g\,z \tag{77}$$

$$\frac{p_w}{\rho_w} = \frac{\partial \phi_w}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial \phi_w}{\partial x}\right)^2 + \left(\frac{\partial \phi_w}{\partial z}\right)^2\right] - g\,z \tag{78}$$

which follows by recalling that, for either fluid,

$$v^2 = U^2 + W^2$$

and that:

$$U = - \frac{\partial \phi}{\partial x}$$

$$W = - \frac{\partial \phi}{\partial z} \quad .$$

Finally, the surface pressure condition (36) now becomes

$$p_w - p_a = T_1 \frac{\partial^2 \zeta}{\partial x^2} \quad . \tag{79}$$

Equations (75)-(79) tie together the parameters k, σ so that the celerity c is rigidly determined for the present physical situation. Using the representations of ζ, ϕ_a, and ϕ_w in (70)-(74), the requisite c may be found as follows. First, equations (75) and (76) become:

$$a(c - U_a) = i\, b_a \tag{80}$$

$$a(c - U_w) = - i\, b_w \tag{81}$$

after substituting (70)-(74) and simplifying.

Equation (77) becomes:

$$\frac{p_a}{\rho_a} = \frac{\partial \phi_a'}{\partial t} - \frac{1}{2}\left[\left(- U_a + \frac{\partial \phi_a'}{\partial x}\right)^2 + \left(\frac{\partial \phi_a'}{\partial z}\right)^2\right] - g\, z$$

$$= \frac{\partial \phi_a'}{\partial t} + U_a \frac{\partial \phi_a'}{\partial x} - g\, z \tag{82}$$

after using the assumptions that U_a, ϕ_a' and its derivatives are small. In a similar way, from (78):

$$\frac{p_w}{\rho_w} = \frac{\partial \phi_w'}{\partial t} + U_w \frac{\partial \phi_w'}{\partial x} - g\, z \tag{83}$$

We now connect (82) and (83) by means of the surface pressure condition:

$$p_w - p_a = \rho_w \left[\frac{\partial \phi_w'}{\partial t} + U_w \frac{\partial \phi_w'}{\partial x} - g\,\zeta\right] - \rho_a \left[\frac{\partial \phi_a'}{\partial t} + U_a \frac{\partial \phi_a'}{\partial x} - g\,\zeta\right]$$

$$= T_1 \frac{\partial^2 \zeta}{\partial x^2} \tag{84}$$

Using the current forms for ϕ_a', ϕ_w', ζ in (84) and simplifying, the result, with the aid of (80), (81), and recalling (39), is:

$$\rho_a (c - U_a)^2 + \rho_w (c - U_w)^2 = T_1 k + \frac{g}{k} (\rho_w - \rho_a) \tag{85}$$

This is basically the required condition on the celerity c. We can solve for c explicitly and easily by noting that if $U_a = U_w = 0$, we determine the celerity c_o for the stationary case. Setting the values of U_a, U_w in (85) to zero, we find:

$$\boxed{c_o^2 = \left[T_1 k + \frac{g}{k} (\rho_w - \rho_a)\right] \Big/ (\rho_a + \rho_w)} \quad . \tag{86}$$

With this, (85) becomes:

$$\rho_a(c - U_a)^2 + \rho_w(x - U_w)^2 = (\rho_a + \rho_w)\ c_o^2$$

Solving this quadratic for c, and simplifying, we have:

$$c = \frac{\rho_a U_a + \rho_w U_w}{(\rho_a + \rho_w)} \pm \sqrt{c_o^2 - \frac{\rho_a \rho_w}{(\rho_a + \rho_w)^2}(U_a - U_w)^2} \tag{87}$$

which is the requisite expression for the celerity c of the air-water wave form.

Kelvin-Helmholtz Instability

The first thing one usually does when a quadratic equation is solved is to look under the radical sign to see when the radicand takes on positive, zero, or negative values. In the latter case, one would then expect the roots to be complex numbers and usually some interesting physical insight is forthcoming in the associated physical phenomenon (cf., e.g., (13)-(16) of Sec. 8.5). In the present case we note that the celerity c in the Kelvin-Helmholtz model, as given in (87), becomes complex when

$$c_o^2 - \frac{\rho_a \rho_w}{(\rho_a + \rho_w)^2}(U_a - U_w)^2 < 0 \quad . \tag{88}$$

that is, when

$$(U_a - U_w)^2 > \frac{(\rho_a + \rho_w)^2}{\rho_a \rho_w}\, c_o^2 \quad . \tag{89}$$

This indicates that there exists a relative speed $|U_a - U_w|$ between the air and water masses at which instabilities in the wave forms may occur and grow. In short, if the wind blows fast enough over the water surface, waves of any given amplitude and length will build up in time. This may be seen by rewriting (74) as:

$$\zeta = a\, e^{ik(z - ct)}$$

with $c = \alpha + i\beta$, where $\beta > 0$. Then:

$$\zeta = a\, e^{ik(z - \alpha t - i\beta t)}$$

$$= \left[a\, e^{ik(z - \alpha t)}\right] \cdot e^{+k\beta t} \quad .$$

The first factor (in brackets) is a complex number with finite magnitude $|a|$. The second factor is an exponential with positive exponent and is the one that attests to the instability of the wave forms, which has been called the *Kelvin-Helmholtz* instability.

In this way, a relatively simple model of the sea surface is developed with the property that it predicts the growth of wind generated waves. All the terms on the right of (89) are computable for the air-water case, and it turns out that when $|U_a - U_w| > 6.6$ m/sec for waves of $\lambda = 1.7$ cm, instabilities, according to this model, should occur. This predicted speed is somewhat higher than the observed wave generating wind speeds and lower than others predicted by other theories. We shall briefly reconsider this matter in Sec. 12.9 wherein some modern theories of wind generated waves are surveyed.

Capillary and Gravity Waves

Another dividend of the Kelvin-Helmholtz model of the dynamic air-water surface is the formula it yields for the celerity of the surface waves in otherwise still air and water. Thus, by (87) and (86); if $U_a = U_w = 0$, then:

$$\sigma/k = c = c_o = \left[\frac{T_1 k + \frac{g}{k}(\rho_w - \rho_a)}{(\rho_a + \rho_w)}\right]^{1/2} \quad . \qquad (90)$$

This equation for c may be checked by noting that if we set $T_1 = 0$ (no surface tension), and assume the hydrosol ($\rho_w = 1$) is bounded by a vacuum ($\rho_a = 0$), then the equation for c in (90) reduces to that in (65). On the other hand, if we could arrange hydrodynamic studies in gravity-free space (as we soon will be able to do) so that $g = 0$, we would be able to observe wave forms on the air-water surface which are sustained by surface tension forces alone, and with celerity given by:

$$c = \sqrt{T_1 k} = \sqrt{\frac{2\pi T_1}{\lambda}} \quad . \qquad (91)$$

This shows that the surface tension waves increase in celerity with *decreasing* wavelength. This is completely inverse to the gravity wave relation as given in (68). It follows that when both gravity and tensile forces are present over an air-water surface, the celerity of a given wave form is the result of the combination of these two causes, and that if we could vary k (or λ) in (90) from small to large values we would find a relatively complicated dependence of c on λ. We can study this relation best be rewriting (90) in the form):

$$c^2 = \frac{1}{\lambda}\left[\frac{2\pi T_1}{\rho_w + \rho_a}\right] + \lambda\left[\frac{g}{2\pi}\,\frac{\rho_w - \rho_a}{\rho_w + \rho_a}\right] \qquad (92)$$

Functions of this form increase without bound as $\lambda \to 0$ or $\lambda \to \infty$ and have a minimum for some finite value λ_m of λ which may be found by standard calculus techniques. It turns out that for the air-water case:

$$\lambda_m = 1.7 \text{ cm} \tag{93}$$

and that the associated minimum celerity c_m is:

$$c_m = 23 \text{ cm/sec} \quad . \tag{94}$$

Waves with smaller or greater wavelengths than 1.7 cm travel with speeds greater than 23 cm/sec. These speeds may generally be computed from the following rearrangement of (92):

$$\boxed{c^2 = \frac{1}{2} c_m^2 \left[\frac{\lambda m}{\lambda} + \frac{\lambda}{\lambda m}\right]} \quad . \tag{95}$$

An approximate useful form of (95) is forthcoming if we note that $\lambda_m \simeq \sqrt{3}$ cm, for then (95) may be reduced to:

$$c = 12.5\sqrt{\frac{3}{\lambda} + \lambda} \tag{96}$$

which yields c in centimeters per second when λ is in centimeters.

The minimum λ given in (93) is both mathematically so well defined and physically meaningful that it has been used to define the difference between capillary waves and gravity waves. Thus if a surface wave in a natural hydrosol has wavelength greater than λ_m, it is called a *gravity wave;* if its wavelength is equal to or smaller than λ_m, it is called a *capillary wave* (or *ripple*). By means of (92) or (95) one can see that there will always be some tensive effects in a gravity wave and some gravity effects in a ripple, but as λ departs from λ_m on either side of λ_m, one term in (92) or (95) will soon begin to markedly dominate.

Energy of Surface Waves

We now take up the matter of the energy of waves associated with the surface of a natural hydrosol, for the purpose of laying the groundwork for the concept of the power spectrum of the dynamic air-water surface.

Imagine a flat calm air-water surface with motionless air and water masses above and below the surface. Each water and air packet is motionless with respect to the terrestrial reference frame. Hence the total kinetic energy of the air-water system is zero. The potential energy of the system relative to the terrestrial reference frame is some finite number which we may take as a fiducial point and effectively set to zero. Now the system is set into motion, say in the framework of the Kelvin-Helmholtz model considered above.

As a result, each air packet or water packet is set into motion with an oscillatory motion superimposed on a translatory motion. Thus each fluid packet has a well-defined speed from which its kinetic energy is computable at each instant. Furthermore, the change of position of each fluid packet within the earth's gravitational field changes the packet's potential energy, which may now be reckoned relative to the zero fiducial potential energy fixed above. Finally, the tensile forces within the air-water surface, being brought into play by the wave motion, act exactly analogously to the forces of a thin deformed rubber sheet between the air and water masses. Thus there is potential energy built up in the air-water surface as work is done to stretch it into the shape of the passing wave form. As a result of all this motion and change of position and surface deformation, there is a continual interchange between the potential and kinetic energies of the moving air-water system. As our studies of the Kelvin-Helmholtz model have shown, the surface motion and configuration completely characterizes the motion and configuration throughout the entire air-water mass. Hence we may associate the kinetic and potential energies of the system--which in truth arises from the activity of the entire medium--solely with the waves on the air-water interface, and conveniently speak of the energy of the entire system simply as the *energy of the waves*.

Let us now consider a sample calculation of the energy of the waves on a natural hydrosol. Figure 12.14 is a setting which is of sufficient generality in which to perform the computation. A sinusoidal wave train is moving over the xy plane in the direction of the unit vector **w**. Suppose we slice through the medium along the direction of **w** with two parallel vertical planes one unit distance apart and remove the slice for examination. A portion of the excised slice, one wavelength long, is shown in Fig. 12.15. As noted earlier, the apparent progressive motion of the wave is induced by the oscillations of fluid packets in relatively small elliptical orbits in a frame of reference locked to the main body of the fluid. For simplicity we shall assume $U_a = U_w = 0$, and concentrate only on the kinetic and potential energy changes brought about by the water packets moving in these local elliptical orbits.

The kinetic energy of a small water packet of unit volume mass is $(1/2)\rho v^2$, where v is its orbital speed. The total kinetic energy in the volume of the water slice of Fig. 12.15 is:

$$\int_0^\infty \int_0^\lambda \frac{1}{2}\rho\, v^2(x,z)\, dx\, dz = \frac{1}{2}\rho \int_0^\infty \int_0^\lambda \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dx\, dz$$

$$= \frac{1}{2}\rho \int_0^\lambda \phi \frac{\partial\phi}{\partial z}\, dx$$

where the last equality is the result of an application of Green's theorem to the excised volume, and where we once again assume small wave slopes so that $\partial\phi/\partial z$ will simultate $\partial\phi/\partial n$, the normal derivative of ϕ at the air-water surface. Using (48) we have:

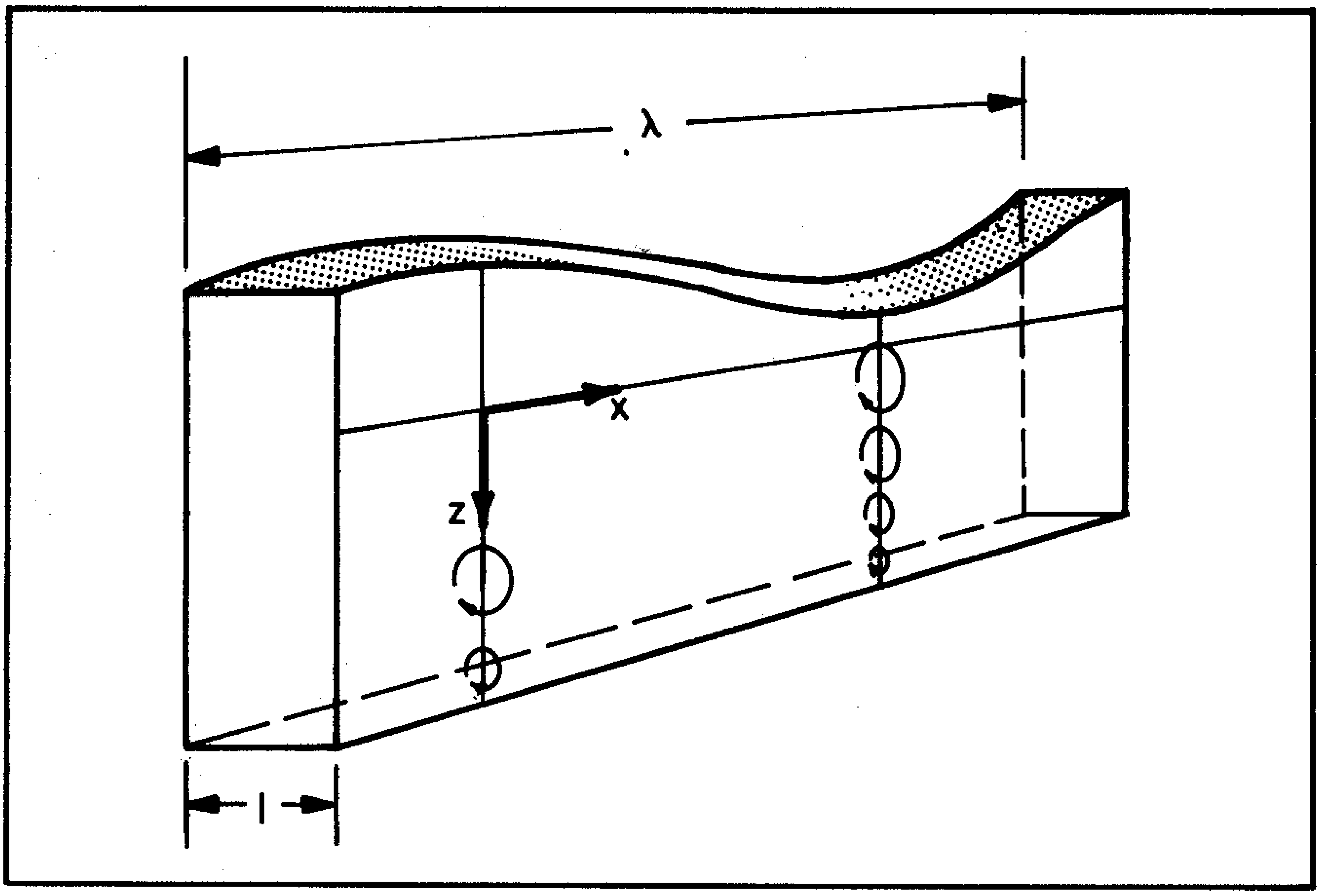

FIG. 12.15 A vertical slice of the hydrosol for study of the kinetic and potential energy of a progressing water wave.

$$\phi \frac{\partial \phi}{\partial z} = \frac{\sigma^2}{g} \phi^2$$

and working with the real part of ϕ as given in (63), this becomes:

$$\phi \frac{\partial \phi}{\partial z} = \frac{b^2 \sigma^2}{g} \cos^2(kx - \sigma t)$$

for $\varepsilon = 0$ and $z = 0$. Recall that the connection between b and the amplitude a of the surface wave is:

$$a = - b\sigma/g \quad ,$$

as given in (69). Hence:

$$\phi \frac{\partial \phi}{\partial z} = a^2 \, g \cos^2(kx - \sigma t) \quad .$$

Integrating this, as required:

$$\frac{1}{2} \rho \int_0^\lambda \phi \frac{\partial \phi}{\partial z} = \frac{1}{2} \rho \, a^2 \, g \int_0^\lambda \cos^2(kx - \sigma t) \, dx$$

where, by (66), $k = 2\pi/\lambda$; so that we have at last:

$$\frac{1}{\lambda}\int_0^{\infty}\int_0^{\lambda}\frac{1}{2}\,\rho\, v^2(x,y)\,dx\,dz = \frac{1}{4}\,\rho\, a^2 g \qquad (97)$$

which shows that the *average kinetic energy of gravity waves per unit wavelength* over the slice in Fig. 12.15 *varies directly as the square of the amplitude of the wave*, ρ and g being fixed constants. Recalling that the slice is of a unit thickness, (97) shows that, alternately, $(1/4)\rho a^2 g$ is the average area density of kinetic energy at each point in the horizontal plane over which the wave train in Fig. 12.14 is moving.

It is now easy to show, without the necessity of further detailed calculation and using only energy conservation considerations, that the the potential energy of the displaced water mass slice of Fig. 12.15 is precisely $(1/4)\rho a^2 g$ (see, e.g., Art. 174 in Ref. [149]).

Finally, the potential energy of stretching of the surface from a straight line into the sinusoidal curve, in Fig. 12.15, is obtained by simply applying the formula: work equals force times distance. Here T_1 is the force, and the distance is the difference between the wavelength of the wave and its actual length considered as a curve:

$$T_1\int_0^{\lambda}\left[1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right]^{1/2}dx - T_1\lambda \cong \frac{1}{2}\,T_1\int_0^{\lambda}\left(\frac{\partial\zeta}{\partial x}\right)^2 dx$$

$$= \frac{1}{2}\,T_1\int_0^{\lambda} a^2 k^2 \sin^2(kx-\sigma t)\,dx$$

$$= \frac{1}{4}\,T_1\, a^2 k^2 \lambda \quad .$$

Total energy $\mathcal{E}(k)$ of the wave per unit horizontal area is then:*

$$\mathcal{E}(k) = \frac{1}{2}\,\rho\, a^2 g + \frac{1}{4}\,T_1\, a^2 k^2$$

i.e.,

$$\boxed{\mathcal{E}(k) = K(k)a^2} \qquad (98)$$

*Oceanographers frequently use wave height H instead of amplitude a, where H is measured from crest to trough, so that a = H/2.

where we have written:

$$\text{"}K(k)\text{"} \quad \text{for} \quad \frac{1}{2}\rho g + \frac{1}{4} T_1 k^2$$

Thus, all other factors being fixed, the total energy density E per unit horizontal area, associated with a sinusoidal wave train of amplitude a, varies directly as a^2. Observe that the energy density $\mathcal{E}$ is independent of λ (or k) for the purely gravity component of the energy, but that it depends on k^2 (or $1/\lambda^2$) for the capillary component of the energy. Thus as a wave is imagined to shrink down to capillary size, we see that capillary waves could in principle store a sizable fraction of the energy of motion on steady state-wind-blown surfaces, especially freshly wind-blown surfaces over which the gravity waves have not yet built up. However, under fully risen seas driven by strong winds, the gravity waves take the lion share of the total energy. This matter will be discussed quantitatively in (27) of Sec. 12.8, when enough theoretical machinery will have been constructed and enough empirical knowledge will have been gained. Observe finally, that the energy density $\mathcal{E}$ of a wave train is independent of the direction of travel **k** of that wave train.

Superposition of Waves

As one of the final topics in the present development of hydrodynamics for hydrologic optics we observe an extremely important property of the classic and Helmholtz wave models studied above. This is the readily verified property that the sum of two velocity potentials ϕ_1, ϕ_2 associated with two wave trains ζ_1, ζ_2, each train being governed by the linearized equations (41)-(44), is again a solution of the set (41)-(44). At this point the reader should verify that, by a simple rotation of axes, (41)-(44) are transformed to forms which hold for (40). This means that, in view of (45), the linearized wave models can be generalized to describe air-water surfaces where functions ζ are linear combinations of arbitrary finite numbers of one-dimensional wave trains of the kind pictured in Fig. 12.14. As a result, the dynamic air-water surfaces of many types of wind-blown hydrosols can be arbitrarily closely represented by linear combinations of the kind:

$$\zeta(x,y,t) = \sum_{n=0}^{p} a_n \cos(\mathbf{k}_n \cdot \mathbf{r} - \sigma_n t + \varepsilon_n) \qquad (99)$$

where p is an integer and where $\mathbf{k}_n (=(u_n, v_n))$ is the vector wave number defined for (40). The minus sign before σ_n is chosen so that the associated wave component with wave number $\mathbf{k}_n$ travels in the direction of $\mathbf{k}_n$. Alternatively, we may write (99) as:

$$\zeta(x,y,t) = \sum_{n=0}^{p} a_n \cos(u_n x + v_n y - \sigma_n t + \varepsilon_n)$$

The nth wave train again travels in the direction of $\mathbf{k}_n$, and has celerity c_n, where:

$$c_n = \sigma_n / k_n$$

and where:

$$k_n = \sqrt{u_n^2 + v_n^2} \quad .$$

In mathematical discussions of the dynamic air-water surface, for analytical convenience, all finite stops are pulled out in (99) by setting p equal to ∞, so that (99) becomes a *Fourier series representation* of the function ζ. In contrast to this, when $p < \infty$, (99) is the *finite Fourier series (or Fourier polynomial) representation* of ζ. A convenient graphical means of picturing the wave train components of the Fourier representation of an air-water surface, as in (99), is to plot the vector wave numbers $\mathbf{k}_n$ on the uv plane, as in Fig. 12.16(b).

The vector in Fig. 12.16(b) represents the wave train depicted in (a) of that figure. Thus the direction of $\mathbf{k}$ gives the direction of travel of the train, and its magnitude $|\mathbf{k}|$ (= k) contains the means of computing the wavelength of the train ($\lambda = 2\pi/k$). This vector characterization of a wave train and other superpositions can be carried out in some detail using analogies with force vectors in mechanics. For example Fig. 12.16(c) shows the vector wave-number means of finding the resultant of two wave trains. For if

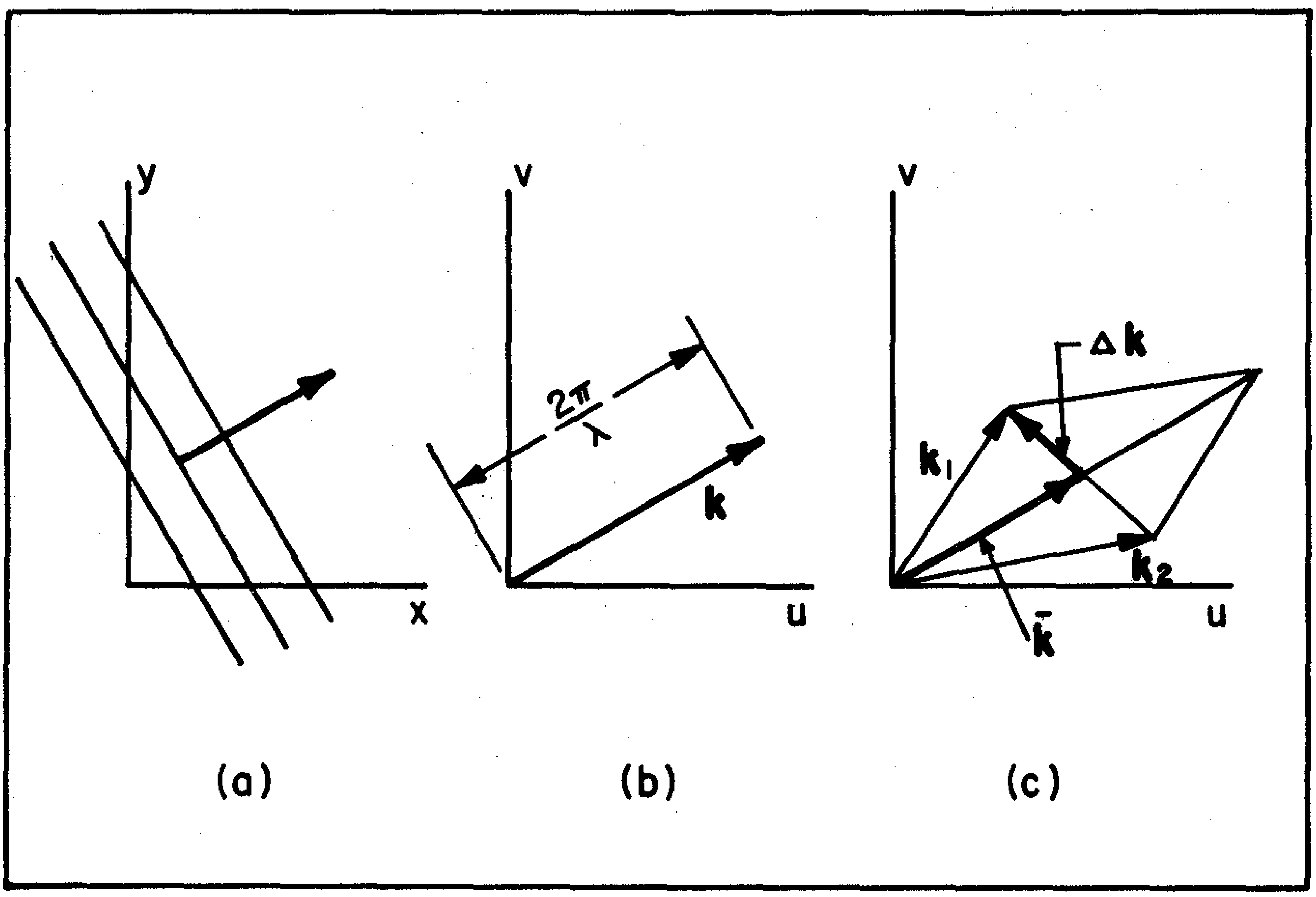

FIG. 12.16 Depicting a sinusoidal wave by means of the $\mathbf{k}$-vector.

$$a \cos(\mathbf{k}_1 \cdot \mathbf{r} - \sigma_1 t)$$

and

$$a \cos(\mathbf{k}_2 \cdot \mathbf{r} - \sigma_2 t)$$

are two wave trains, their sum is representable as:

$$2a \cos(\Delta\mathbf{k} \cdot \mathbf{r} - \Delta\sigma t) \cos(\overline{\mathbf{k}} \cdot \mathbf{r} - \overline{\sigma} t)$$

where

$$\Delta\mathbf{k} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$$

$$\overline{\mathbf{k}} = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$$

and

$$\Delta\sigma = \frac{1}{2}(\sigma_1 - \sigma_2),$$

$$\overline{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2) \quad .$$

From wave-vector diagrams one can, after some practice, tell at a glance the general appearance of a sea. Figure 12.17 illustrates some representative "seas." In Fig. 12.17(a) we have wave vectors clustered about a single direction and of relatively small magnitude. Since small k indicates large wavelength, (a) depicts a heavy swell configuration with a well-defined direction. Figure 12.17(b) is also a highly directional sea but of relatively smaller wavelengths. Figure 12.17(c) and (d) indicate jumbled seas with criss-crossing wave trains, with slightly larger wavelengths on the average in case (c) than in case (d). We shall return to the discussion of the Fourier representation of the air-water surface in Sec. 12.4.

Spectrum of the Air-Water Surface

The present discussion of the hydrodynamics of the air-water surface concludes with one of the more significant concepts to be added to the repertory of oceanographics in the past decades, namely the concept of the *spectrum* of the sea surface, or air-water surfaces for general hydrosols. As we shall see, this concept also plays an important role in the study of radiative transfer across dynamic air-water surfaces. The basis for the concept lies in the Fourier series representation of the air-water surface and this, in turn, rests on the classical wave model developed earlier in this section.

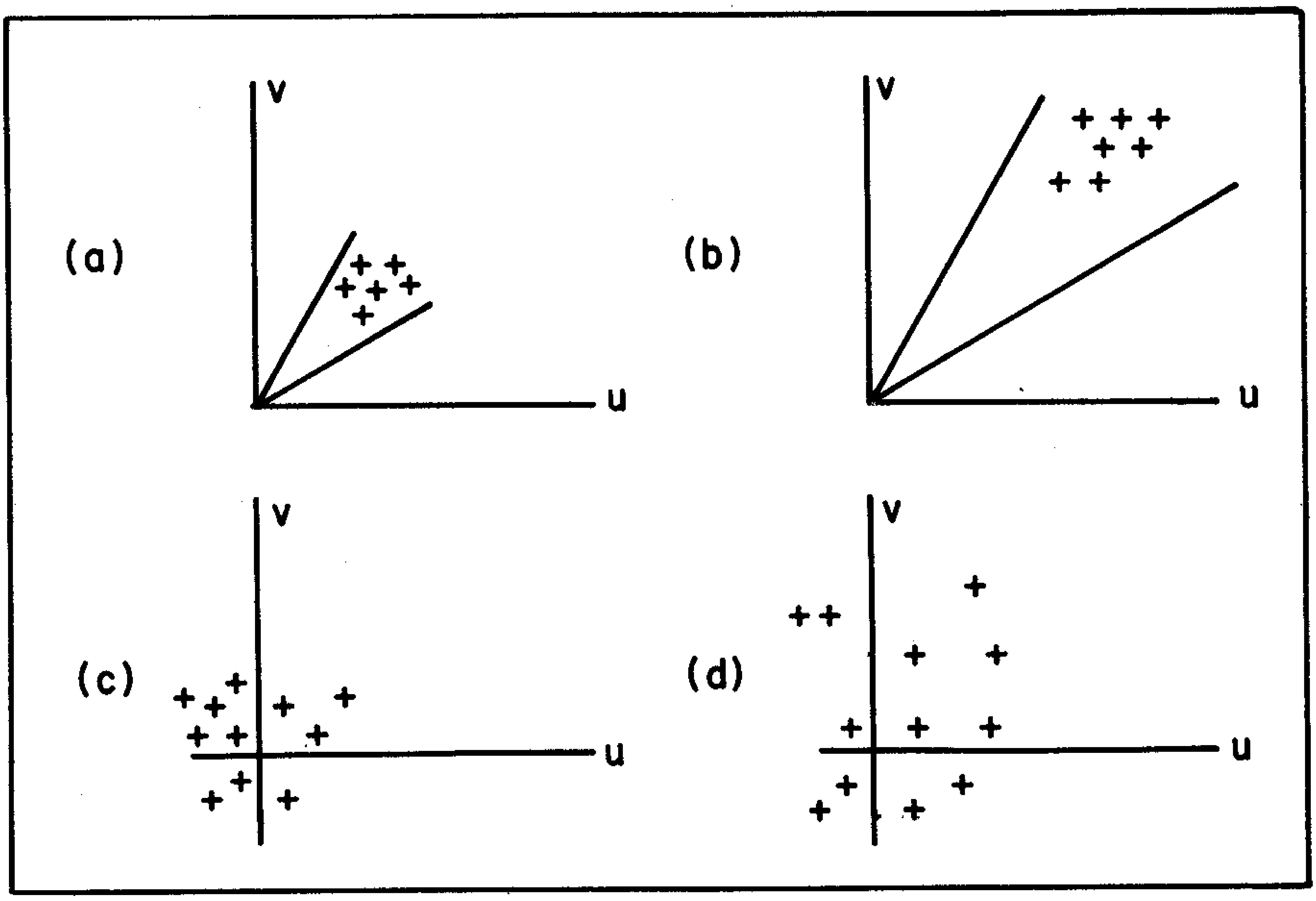

FIG. 12.17 Depicting the appearance of a sea surface using the **k**-vectors of its component sinusoidal waves.

A preliminary description of the spectrum of an air-water surface would be achieved by simply pointing to the set of all coefficients a_n in the finite Fourier representation (99) and saying that the set of all the a_n comprises the *spectrum* of the elevation function ζ. Indeed, if ζ is exactly represented by (99), then the *spectrum* of ζ is the range of a function A which assigns to each $\mathbf{k}_n$ (= (u_n, v_n)) the number a_n:

$$A(\mathbf{k}_n) = a_n \quad , \quad n = 1, \ldots, p. \tag{100}$$

The spectrum A associated with a finite Fourier series representation of ζ is described as *nondense, discrete* because of the finite number of separate wave numbers $\mathbf{k}_n$ involved in the representation. The *energy spectrum* of ζ in (98) is the set of all numbers $(1/2)\ a_n^2$, $n = 1, \ldots, p$. The factor "1/2" is included for formal reasons which will become clear later in this discussion and in (35) of Sec. 12.4. However, the reason for squaring the a_n rests in (98) of Sec. 12.3.

In the case of the air-water surface for natural hydrosols it is usually found that a great range of wave numbers is associated with the analyzed surface function. These numbers are closely packed together and found to occur virtually everywhere on extensive reaches within the wave number diagrams of the kind displayed in Fig. 12.18. We shall say that the spectrum of ζ is *dense, discrete* over a region $\mathcal{R}$ of the uv plane if every neighborhood $\mathcal{W}(\mathbf{k})$ of every point

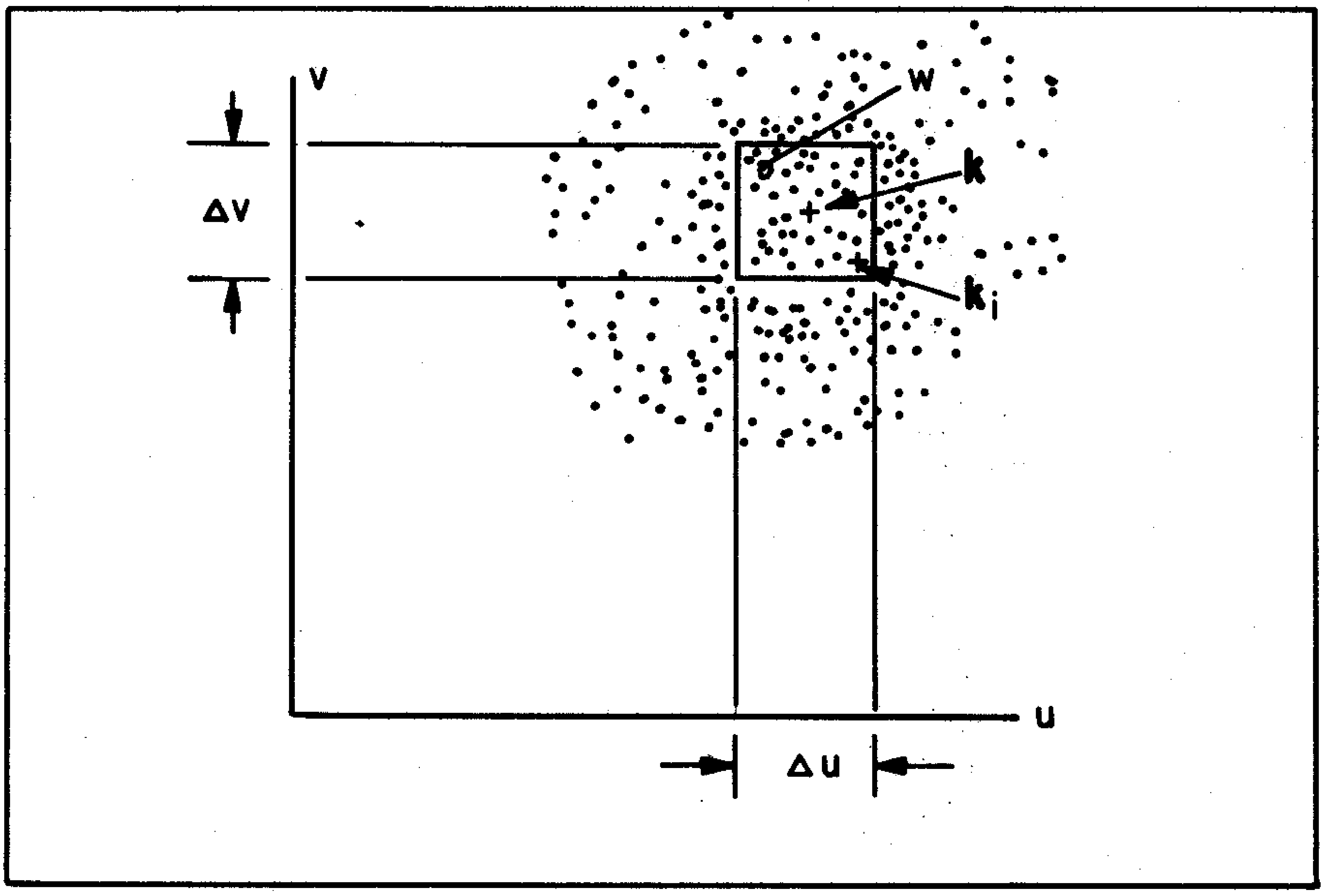

FIG. 12.18 The **k**-vectors of a sea surface with a dense discrete spectrum.

k in $\mathcal{R}$ has points $\mathbf{k}_i \neq \mathbf{k}$ in it for which the associated a_i in the Fourier Series are not zero.

A dense discrete spectrum function is still of the form (100) but now it has a countably infinite number of a_i in its range (i.e., set of values). Since the **k**'s are densely packed over various parts of the uv plane in the case of a dense discrete spectrum, it is possible in principle to define a spectral density function, much in the way we defined irradiance in Sec. 2.4 as a flux density function. Indeed, the same general properties of additivity and continuity can be used to make rigorous the heuristic discussion on which we now embark. To see how such a definition will go in outline, consider an arbitrary region $\mathcal{W}$ in the uv plane for an air-water function ζ. In this region there is a set of **k**'s whose indices run over some set $J(\mathcal{W})$ of integers. We next select the square a_i^2 of all the coefficients a_i in the Fourier series representation of ζ whose indices i are in $J(\mathcal{W})$ and form their sum:

$$\frac{1}{2} \sum_{i \in J(\mathcal{W})} a_i^2 \quad . \qquad (*)$$

Now let $\mathbf{k}(= (u,v))$ be a point in $\mathcal{W}$ and for simplicity $\mathcal{W}$ could be a rectangular region of sides $\Delta u, \Delta v$, so that its "area" $A(\mathcal{W})$ is $\Delta u \Delta v$ (Fig. 12.18). At any rate we can form the quotient:

$$\frac{\frac{1}{2} \sum_{i \in J(\mathcal{W})} a_i^2}{A(\mathcal{W})} \quad .$$

We shall assume that this quotient has a limit as $\mathcal{W} \to \{\mathbf{k}\}$ (here the additivity and continuity properties of (*) with respect to $\mathcal{W}$ would enter) and we shall write:

$$\text{"E}(\mathbf{k})\text{"} \quad \text{or} \quad \text{"E}(u,v)\text{"} \quad \text{for} \quad \lim_{\mathcal{W} \to \{\mathbf{k}\}} \frac{\frac{1}{2} \sum_{i \in J(\mathcal{W})} a_i^2}{A(\mathcal{W})} \tag{101}$$

This function E which assigns to each $\mathbf{k}$ the number $E(\mathbf{k})$ is called the *spectral (energy) density* function* (or the *energy spectrum*). The connection between the spectral density function $\mathcal{E}$ given in (98) is clearly:

$$\boxed{\frac{\mathcal{E}(\mathbf{k})}{K(k)} = 2E(\mathbf{k})A(\mathcal{W})} \tag{102}$$

for a *small* region $\mathcal{W}$ about point $\mathbf{k}$ of area $A(\mathcal{W})\,(= \Delta u \Delta v)$. Thus to find the total energy density $\mathcal{E}$ per unit horizontal area over the sea surface** we perform either the sum:

$$\sum_{i=1}^{\infty} \mathcal{E}(\mathbf{k}_i) = \sum_{i=0}^{\infty} K(k_i)\ a_i^2$$

or the integration:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u,v)K(k)\ du\,dv = \frac{1}{2} \sum_{i=1}^{\infty} K(k_i)a_i^2$$

where:

$$k^2 = u^2 + v^2 \quad .$$

The former (infinite sum) operation is useful when the a_i are known directly, the integral when E is known from prior analysis, or, when it would be expedient to use the calculus.

Finally, according to (101), the arbitrary form of $\mathcal{W}$, and the preceding equality, we deduce that:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(u,v)\ du\,dv = \frac{1}{2} \sum_{i=0}^{\infty} a_i^2 \quad . \tag{103}$$

---

*Occasionally "intensity" is used instead of "density."

**The logical basis for the summation of the a_i^2 to obtain total energy rests in the derivation of (98) now for the case of ϕ being a finite linear combination of orthogonal sinusoids. Then the passage to the infinite limit may be made.

We have now taken the exposition of classical hydrodynamics as far as we need to in the present work. To proceed any further along the present path would take us into the domain of harmonic analysis of air-water surfaces. This study is reserved for the following section. For the present it suffices to note that we have laid the groundwork for an intuitive understanding of the spectral density function associated with a dynamic air-water surface. The spectrum of the elevation and spectral density function play the role of central unifying concepts in the several important problems concerned with the dynamic air-water surfaces. Mathematically, the spectrum is equivalent to knowledge of the coefficients of the Fourier series for the elevation function ζ. Physically, the spectral density function has manifold applications. On the one hand it has been used in one of its earliest applications to explain microseisms generated by the dynamic air-water surface [167]. On the other hand it is useful in describing the reflectance properties of the sea surface with respect to irradiation by radar, sound, and light [25], [85], [56]. In the present work we shall show that the spectral density function is closely connected with the solution of radiative transfer problems at the air-water boundary of natural hydrosols (Sec. 12.9). But before we relate it to radiative transfer problems it will be of considerable help to have a battery of associated harmonic analysis concepts at hand which will facilitate the formulation and solution of these problems and which will further the general discussions of recent experimental and theoretical studies of the physical and geometric properties of the dynamic air-water surface. To this task we now turn.

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12.4 Harmonic Analysis of the Dynamic Air-Water Surface

We shall devote some attention in this section to the topics in harmonic analysis required for our present studies of radiative transfer across the dynamic air-water surface. The battery of concepts of harmonic analysis, as they are applied to the air-water surface, are relatively new, having been intensively applied during the past decade by increasing numbers of workers in mathematical and experimental oceanography. A survey of the history of the subject is out of place in this work, but it can be begun by consulting the references [320], [307], [191], and others listed during the discussion below.

Our primary aim in the discussion below is to prepare the ground for answering some of the initial basic questions raised by researchers entering this domain of ideas for the first time. The most frequently occurring questions are: What are the sources of the ideas of harmonic analysis? What is the difference between harmonic analysis and synthesis? Sometimes one sees Fourier integral representations of an analyzed function, and other times a Fourier series representation. Is there some way of deciding between these two modes of representation for a given context? Is there some special justification for choosing the tools of harmonic analysis for use in describing the sea surface and the dynamic surfaces of natural hydrosols in general? Even if such harmonic analyses of the dynamic air-water surfaces can be made, why is the energy spectrum singled out for so much