12.11 Time-Averaged Radiance Field Over a Dynamic Air-Water Surface

In order to obtain the time averaged radiance distribution over a dynamic air-water surface it appears that we need only average each side of equation (12) of Sec. 12.10 over a sufficiently long time interval. While this operation indeed seems the natural one to perform and outwardly seems straightforward, in order to achieve a useful averaged version of (12) of Sec. 12.10, some care must be taken in first of all establishing a meaningful definition of a time averaged radiance distribution over Ξ. Once this is done there is the additional task of assuming enough regularity properties of the averaged radiance, reflectance, transmittance and characteristic functions appearing in (12), to allow the averaging process to culminate in the *workable static description* of the dynamic surface. The latter feature is of course the prime virtue of the time-averaged radiance field. We shall devote most of the efforts in this section to achieve this end. But throughout the discussion of all these subsidiary details the reader should retain the single idea which prompts all the action: that the light field over an arbitrarily curved air-water surface is given for every instant by (12) of Sec. 12.10 and that by adding together such representations for each time in an interval of times, an averaged time-independent version of (12) of Sec. 12.10 can be obtained.

Direct and Indirect Radiance Averages

A study of (12) of Sec. 12.10 preparatory to a time averaging operation reveals that the radiance values $N(x,\xi,t)$ are to be taken at points x of the moving surface while the direction argument ξ is held fixed. A natural method of averaging such a radiance function is suggested when one imagines a fixed observation point somewhere off in the atmosphere from which one can view the water surface along a rigidly fixed line of sight. Figure 12.57 depicts the essential geometric arrangements for such an averaging process for three different instants in time. Let a point $\hat{x}$ on the mean water surface $\hat{S}$ be fixed and let a straight line directed along a given ξ in Ξ_+ pass through $\hat{x}$. Then as time proceeds, the point of intersection of the line of sight with S is a moving point $x(t)$ on S defined by the farthest intersection of S with the line of sight from $\hat{x}$ (or equivalently, the nearest point of intersection to the observer). We shall write:

$$"N_+(\hat{x},\xi)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(x(t),\xi,t)\,dt \tag{1}$$

whenever ξ is in Ξ_+ and call $\hat{N}(x,\xi)$ the *upward time-averaged radiance* of $\hat{S}$ at $\hat{x}$ along the direction ξ in Ξ_+. In this manner we can assign to each $\hat{x}$ of $\hat{S}$ a certain radiance in any direction ξ of Ξ_+. Errata

On the other hand, as can be seen from Fig. 12.58, downward radiances can also be observed to flow along the line through $\hat{x}$ and in directions ξ in Ξ_-. In this case we use the

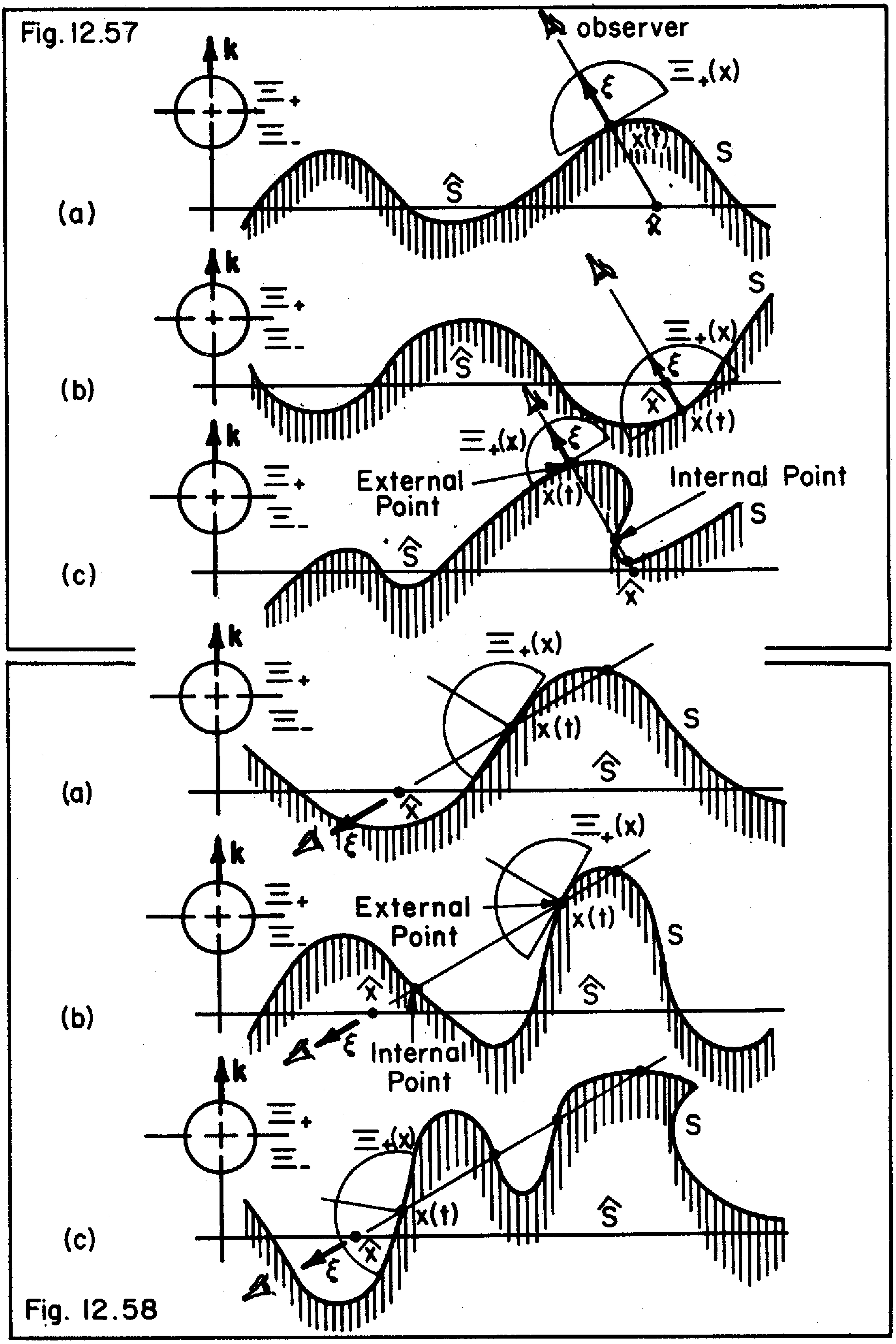

FIG. 12.57 Geometric construction convention for finding instantaneous position x(t) of observed point of dynamic air-water surface (downward line of sight).
FIG. 12.58 Further cases of the constructions in Fig. 12.57 (upward line of sight).

first external point of intersection $x(t)$ of S (again, this is the external point nearest the observer) with the line of sight extended back in the direction $-\xi$. Observe, however, that this point may not be always be defined because the surface S may never rise above the back part of the line through $\hat{x}$ (i.e., as one moves along the direction $-\xi$, from $\hat{x}$, S is never encountered). For this reason, the radiances $N(x(t),\xi,t)$ with ξ in Ξ_- must be appropriately weighted by $\chi(\xi,D(S,\hat{x},t))$ prior to averaging in order to account for the occasional absence of a point of S contributing to $N(x,(t),\xi,t)$. Therefore we shall write:

$$"\hat{N}_-(\hat{x},\xi)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(x,(t),\xi,t)\chi(\xi,D(S,\hat{x},t))dt \tag{2}$$

whenever ξ is in Ξ_- and call $\hat{N}_-(\hat{x},\xi)$ the *downward time-averaged radiance* of $\hat{S}$ at x along the direction ξ in Ξ_-. In this way, via (1) and (2), we can assign a time-averaged radiance to $\hat{x}$ in $\hat{S}$ for every direction ξ in Ξ.

Definitions (1) and (2) give a precise meaning to the time-averaged radiance for the radiances $N(x,\xi,t)$ occurring on the left side of (12) of Sec. 12.10. We shall now give the associated definitions of time averages of $N(x-r_m\xi',\xi',t)$ occurring under the integral sign on the right side of (12) of Sec. 12.10. It is at once clear that we must at least follow suit after (1) and (2) and construct definitions separately for the upward and downward directed radiances, so that we shall be able in principle to consistently relate the averages of N under the integral sign to those of N not under the integral sign. But we must go further and take cognizance of the displaced (or retarded) argument $x(t)-r_m\xi'$ of $N(x(t)-r_m\xi',\xi',t)$. Figure 12.59 shows by means of four typical cases what is entailed in taking this retardation of argument into account. (There are eight basic types of cases in all. The four depicted in Fig. 12.59 will suggest the remaining cases to the reader. For example, the companion case to case (a) is that which extends the path at $x(t)$ downward and to the right, analogously to case (c).) In each case the point $x(t)$ is first constructed from the point x on S after the manner described above for (1) and (2). Then, once ξ' is chosen, a new point $x'(t)$ on S and distinct from $x(t)$ is found by going from $x(t)$ in the direction $-\xi'$ until the first intersection $x'(t)$ with S, such that ξ' is in $\Xi_+(x'(t))$. The point $x'(t)$ so found is $x(t)-r_m\xi'$ as required in $N(x(t)-r_m\xi',\xi',t)$. In this way, to each $\hat{x}$ on $\hat{S}$ and ξ' in Ξ_+ we can assign a radiance $\hat{N}'(\hat{x},\xi')$, where we have written:

$$"\hat{N}'_+(\hat{x},\xi')" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(x(t)-r_m\xi',\xi',t)dt \tag{3}$$

and for ξ' in Ξ_- we write:

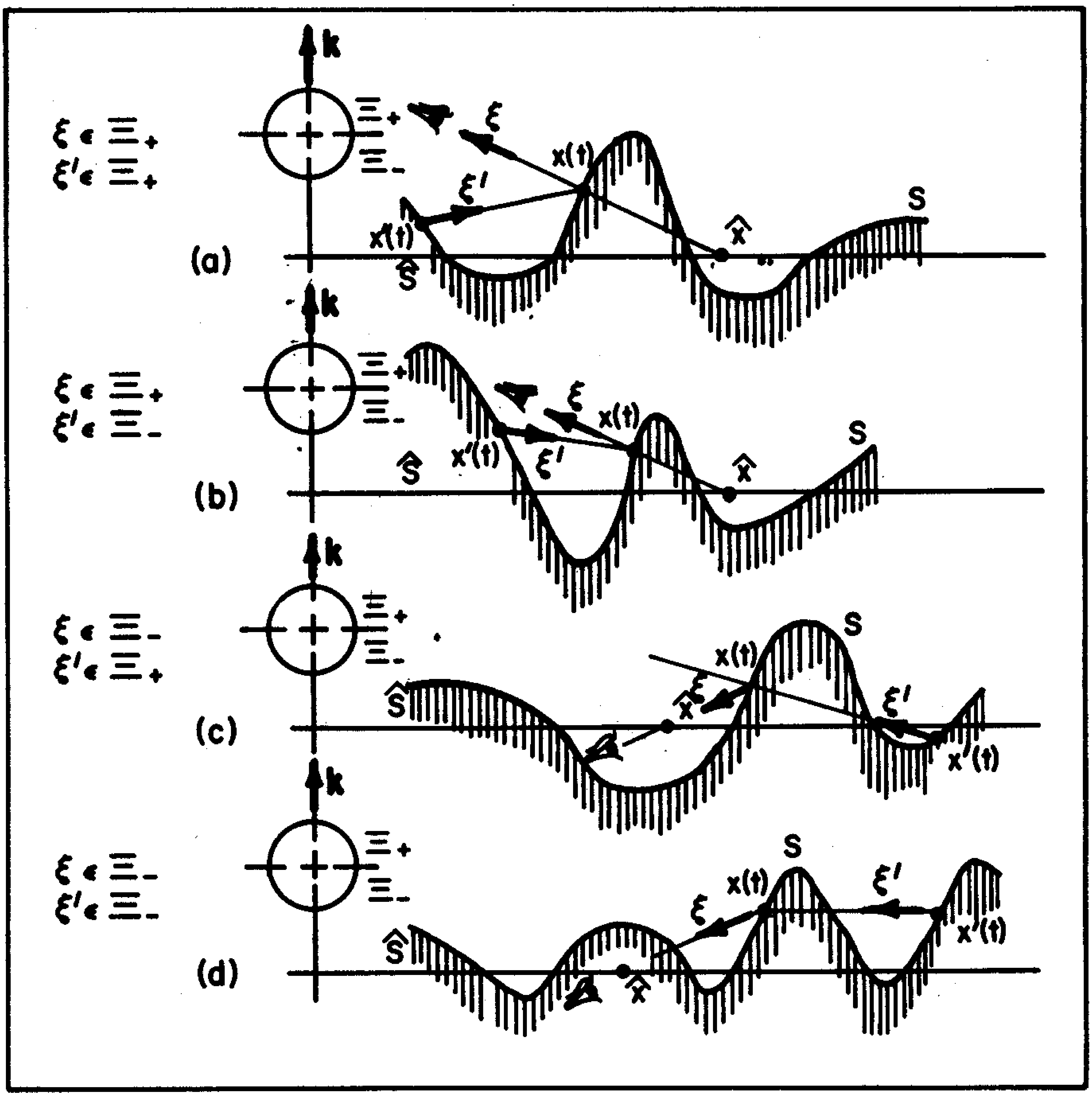

FIG. 12.59 Geometric construction convention for finding the secondary point x'(t) on dynamic air-water surface.

$$\text{"}\hat{N}'_-(\hat{x},\xi')\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(x(t)-r_m\xi',\xi',t)\chi(\xi',D(S,x(t),t)\,dt \tag{4}$$

The Stationarity Condition

In order to obtain an equation which determines a physically meaningful time-averaged radiance distribution associated with the mean surface $\hat{S}$ we shall be encouraged by the statistical stationarity property, frequently encountered over air-water surfaces (recall the discussion after (28) of Sec. 12.5), to postulate that:

$$\hat{N}_{\pm}(\hat{x},\xi) = \hat{N}'_{\pm}(\hat{x},\xi) \tag{5}$$

for every ξ in $\Xi_{\pm}$, and $\hat{x}$ in $\hat{S}$. In view of the definitions (1)-(4) and condition (5) above it is clear that the time-averaged radiance distribution to be determined by (12) of Sec. 12.10 compresses a complex time dependent radiance function over a highly convoluted random surface S into a steady state radiance distribution associated with the plane mean surface $\hat{S}$. On some reflection of the matter, condition (5) relating the four averages introduced above appears to be but one condition of several that may connect various types of averages of radiances of the moving air-water surface. The present condition relates directly observable radiances ((1), (2)) with those that are observable only indirectly, that is from within the concavities of the surface ((3),(4)). The natural random movements of the surface tend to make condition (5) occur very nearly if not exactly, so that condition (5), which is admittedly introduced as a mathematical convenience, is in the last analysis apparently closely met in the natural setting. It should be emphasized that equation (12) of Sec. 12.10 could have been averaged mechanically by applying the operator

$$\lim_{T\to\infty} \frac{1}{T} \int_{o}^{T} [\ ]\ dt \tag{6}$$

to each side. However, by introducing the averages (2) and (4) in addition to (1) and (3) and relating them via (5), we are using our intuition to add to the mathematical form of (12) of Sec. 12.10 enough detail about the radiometric structure of the sea surface to lead toward a realistic time-averaged radiance distribution. This distribution is governed by an integral equation whose specific form will be determined below.

The Independence Condition

We begin the reduction of (12) of Sec. 12.10 by choosing any ξ in Ξ and writing:

$$N(x,\xi,t) = A_{+}(x,\xi,t) + \int_{\Xi_{+}} N(x-r_m\xi',\xi',t)\chi(\xi',D(S,x,t))r_{-}(x;\xi',\xi)d\Omega(\xi')$$

$$+ \int_{\Xi_{-}} N(x-r_m\xi',\xi',t)\chi(\xi',D(S,x,t))r_{-}(x;\xi';\xi)d\Omega(\xi') \tag{7}$$

where "$A_{+}(x,\xi,t)$" denotes the integral involving N^{o} in (12) of Sec. 12.10. The second integral in (12) of Sec. 12.10 has been partitioned over Ξ into Ξ_{-} and Ξ_{+}.

Next, choosing ξ in Ξ_+ and applying the averaging operator (6) we have,

$$\hat{N}(\hat{x},\xi) = \hat{A}_+(\hat{x},\xi) + \lim_{T\to\infty} \frac{1}{T} \int_o^T \int_{\Xi_-} N(x-r_m\xi',\xi',t)\chi(\xi',D(S,x,t)) \cdot r_-(x;\xi';\xi)\,d\Omega(\xi')$$

$$+ \lim_{T\to\infty} \frac{1}{T} \int_o^T \int_{\Xi_-} N(x-r_m\xi',\xi',t)\chi(\xi',D(S,x,t)) \cdot r_-(x;\xi';\xi)\,d\Omega(\xi') \tag{8}$$

where we have written:

$$"\hat{A}_+(\hat{x},\xi)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T} \int_o^T A(x,\xi,t)\; dt \quad . \tag{9}$$

Now the dynamic air-water surface at a point is generally in completely random motion* so that functions such as N, χ, t_+, and r_- occurring in (8) and (9) are very likely to be pairwise statistically independent over every time interval in the sense that:

$$\lim_{T\to\infty} \frac{1}{T} \int_o^T f(t)g(t)dt = \lim_{T\to\infty} \left(\frac{1}{T} \int_o^T f(t)dt\right) \left(\frac{1}{T} \int_o^T g(t)dt\right) \tag{10}$$

where f and g are any pairs of time-dependent functions made up of N, χ, r_-, t_- or their products for arbitrary values of their arguments other than time. At any rate we shall adopt condition (10) (the *independence condition*) and add it to (5) in the list of regularity properties leading to the requisite integral equation below.

The Weighting Functions

The effect of adopting (10) is to permit the operator (6) to be applied individually to each function in the integrands of the integrals in (8). Recalling that "x" in (8) denotes the point x(t) on S determined according to the procedure depicted in Figs. 12.57-12.59, let us write:

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*If the motion were periodic and of prescribed form then an alternative approach is clearly indicated. By far the more realistic case is that of random motion.

$$"Q^{o}(\hat{x},\xi)" \qquad \text{for} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} \chi(\xi,D^{o}(x,t))dt \qquad (11)$$

$$"Q(\hat{x},\xi)" \qquad \text{for} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} \chi(\xi,D(S,x,t))dt \qquad (12)$$

$$"Q_{+}(\hat{x},\xi)" \qquad \text{for} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} \chi(\xi,\Xi_{+}(x,t))dt \qquad (13)$$

$$"R_{-}(\hat{x};\xi';\xi)" \qquad \text{for} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} r_{-}(x;\xi';\xi)\ dt \qquad (14)$$

$$"T_{+}(\hat{x}'\xi';\xi)" \qquad \text{for} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} t_{+}(x;\xi';\xi)\ dt \quad . \qquad (15)$$

We shall refer to the preceding five functions as the *weighting functions* for $\hat{N}$. It will be occasionally useful (in view of (9) of Sec. 12.10) to combine (11) and (12) by writing

$$"Q_{-}(\hat{x},\xi)" \qquad \text{for} \qquad Q^{o}(\hat{x},\xi) + Q(\hat{x},\xi) \quad . \qquad (13a)$$

The Time-Averaged Integral Equation for $N_{+}^{+}(S)$

With (10) in effect and adopting (11)-(15), equation (18) becomes: Errata

$$\hat{N}(\hat{x},\xi) = \int_{\Xi} \hat{N}^{o}(\hat{x},\xi')[Q^{o}(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi) + Q_{+}(\hat{x},\xi')T_{+}(\hat{x};\xi';\xi)]\cdot d\Omega(\xi')$$
$$+ \int_{\Xi_{+}} \hat{N}(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi)\ d\Omega(\xi')$$
$$+ \int_{\Xi_{-}} \hat{N}(\hat{x},\xi')Q(\hat{x};\xi')R_{-}(\hat{x};\xi';\xi)\ d\Omega(\xi') \qquad (16)$$

for ξ in Ξ_{+}. Returning to (8), choosing ξ in Ξ_{-}, multiplying each side of (8) by $\chi(\xi,D(S,x,t))$ and then applying the operator (2) with the independence condition (10) in effect, we have:

$$\hat{N}(\hat{x},\xi) = \left\{ \int_{\Xi} \hat{N}^{o}(\hat{x},\xi')[Q^{o}(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi) + Q_{+}(\hat{x},\xi')T_{+}(\hat{x};\xi';\xi)] \cdot d\Omega(\xi') + \int_{\Xi_{+}} \hat{N}(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi)\, d\Omega(\xi') + \int_{\Xi_{-}} \hat{N}(\hat{x},\xi')Q(\hat{x};\xi')R_{-}(\hat{x};\xi';\xi)\, d\Omega(\xi') \right\} Q(\hat{x},\xi) \tag{17}$$

for ξ in Ξ_{-}. By adopting the function $S(\hat{x},\xi)$ such that:

$$S(\hat{x},\xi) = \begin{cases} Q(x,\xi) & \text{if} \quad \xi \in \Xi_{-} \\ 1 & \text{if} \quad \xi \in \Xi_{+} \end{cases},$$

(16) and (17) may be written as a single integral equation for $N(\hat{x},\xi)$:

$$\hat{N}(\hat{x},\xi) = S(\hat{x},\xi) \left\{ \hat{A}_{+}(\hat{x},\xi) + \int_{\Xi} \hat{N}(\hat{x},\xi')S(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi)\,d\Omega(\xi') \right\}$$

where

$$\hat{A}_{+}(\hat{x},\xi) = \int_{\Xi} N^{o}(\hat{x},\xi')[Q^{o}(\hat{x},\xi')R_{-}(\hat{x};\xi';\xi) + Q_{+}(\hat{x},\xi')T_{+}(\hat{x},\xi';\xi)] \cdot d\Omega(\xi')$$

$\xi \in \Xi$, $\hat{x} \in \hat{S}$

(18)

This is the requisite integral equation governing the time-averaged surface radiance $\hat{N}(\hat{x},\xi)$ of the mean air-water surface $\hat{S}$ at point $\hat{x}$ with time-averaged incident sky radiance distribution $\hat{N}^{o}(\hat{x},\cdot)$ as the basic input. The term $\hat{A}_{+}(\hat{x},\xi)$ is a convenient contraction of two terms, one of which is the integral of $\hat{N}^{o}Q^{o}R_{-}$ over Ξ, and the other the integral of $\hat{N}^{o}Q_{+}T_{+}$ over Ξ, as may be seen by comparing (18) with (16) and (17). In the first integral, $\hat{N}^{o}$ is the time-averaged sky radiance defined over Ξ_{-}. In the second integral N^{o} is time-averaged upward hydrosol radiance defined over Ξ. In sum,

(18) is the time-averaged version of (12) of Sec. 12.10, which in turn is the expanded version of (5) of Sec. 12.10. When necessary for clarity, we may write "$\hat{N}(\hat{x},\xi)$" in (18) as "$N_+(x,\xi)$". (Recall the convention of dropping the "+" sign in going over to (7) of Sec. 12.10.)

Structure of the Weighting Functions

In order to solve equation (18) for the time-averaged radiance $\hat{N}(\hat{x},\xi)$, the structures of the weighting functions occurring in (18) must be known in detail and some particular model of the air-water surface must be adopted, if numerical estimates of $\hat{N}$ are desired. We shall now consider these matters, and begin with $R_-(\hat{x},\xi';\xi)$, as defined in (14).

In Fig. 12.60(a), the incident and reflected directions ξ' and ξ are related to the unit outward normal $\mathbf{n}(x,t)$ to the surface at x by means of (1) of Sec. 12.1. By that relation, we can express ξ in terms of ξ' and $n(x,t)$ as follows:

$$\xi' = \xi - 2[\xi \cdot \mathbf{n}(x,t)]\mathbf{n}(x,t) \quad . \tag{19}$$

This leads to a representation of $r_-(x;\xi';\xi)$ of the form:

$$r_-(x;\xi';\xi) = r(\xi',\xi)\delta(\xi' - [\xi - 2[\xi \cdot \mathbf{n}(x,t)]\mathbf{n}(x,t)]) \tag{20}$$

where $r(\xi',\xi)$ is the Fresnel reflectance function given in (12) or (13) of Sec. 12.1, and δ is the Dirac-delta function defined on E_3, and has dimension of *steradian*$^{-1}$. Thus, unless $\mathbf{n}(x,t)$, ξ' and ξ are related at x exactly as in (19), $r_-(x;\xi';\xi)$ will be zero, in accordance with the requirements of specular reflection. From (20):

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T r_-(x;\xi';\xi)dt = r(\xi',\xi)\lim_{T\to\infty}\frac{1}{T}\int_0^T \delta(\xi' - [\xi - 2[\xi\cdot\mathbf{n}(x,t)]\cdot \mathbf{n}(x,t)])dt \quad . \tag{21}$$

In (21), as time varies, so will $\mathbf{n}(x,t)$ and at occasional instants the fixed directions ξ' and ξ will be related as in (19). At such instants the argument of δ is the zero vector and the integration will pick up a contribution toward the average value of $r_-(x;\xi';\xi)$. This observation follows by virtue of the analogous property of δ that:

$$\int_0^T |f'(t)|\delta(f(t))dt$$

is the number of zeros of f in the interval (0,T) over which f can be written as a succession of monotonic functions (cf., e.g., [95]). Hence, in general:

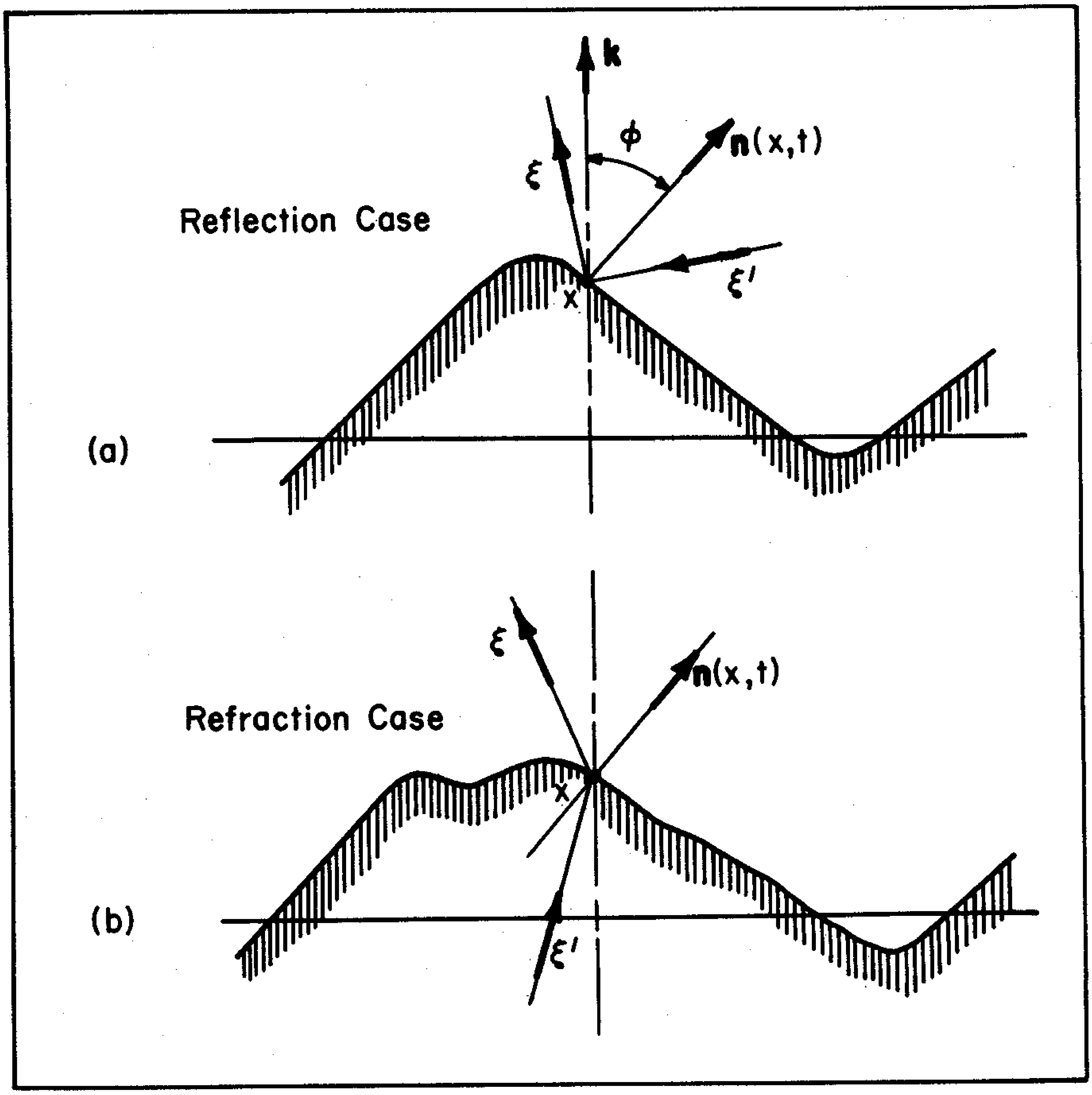

FIG. 12.60 Showing how the incident direction (ξ') and reflected (or refracted) direction (ξ) are related to the wave normal **n**(x,t) at point x on the dynamic air-water surface at an instant t. All three lie in a common plane.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f'(t)|\delta(f(t))dt$$

is the average density of zeros per unit time of f over the time interval $(0,\infty)$. For example, if $f(t) = \sin \omega t$, then

$$\int_{-\varepsilon}^{(n\pi/\omega)+\varepsilon} |\omega \cos \omega t|(\sin \omega t)dt = n + 1$$

where ε is any positive number such that $\varepsilon < \pi/\omega$. Hence if $T = n\pi/\omega$, we have:

$$\frac{1}{T}\int_{-\varepsilon}^{T+\varepsilon} |\omega \cos \omega t|\delta(\sin \omega t)dt = \frac{\omega}{\pi}\left(\frac{n+1}{n}\right)$$

Therefore, for every ε, $0 < \varepsilon < \pi/\omega$:

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} |\omega \cos \omega t|\delta(\sin \omega t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\varepsilon}^{T+\varepsilon} |\omega \cos \omega t| \cdot \delta(\sin \omega t)dt = \frac{\omega}{\pi} \quad .$$

From this we see that the average density of zeros per unit time of $\sin \omega t$ on the interval $(0,\infty)$ is $\omega/\pi = 2f$ (where $\omega = 2\pi f$, and f is the number of cycles per second). The greater ω, the greater the density of zeros of $\sin \omega t$.

With the preceding example in mind (which is a suggestive analogy only) let us assume once again that the air-water surface is statistically stationary so that time averages at a point x on S are independent of location x. In particular, let us write:

$$\text{"}p(\xi',\xi)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \delta(\xi' - [\xi - 2[\xi\cdot\mathbf{n}(x,t)]\mathbf{n}(x,t)])dt \quad . \tag{22}$$

Thus $p(\xi',\xi)$ may be viewed as the fraction of a unit of time that the wave normal at x assumes the correct orientation within a unit solid angle which is directed along $\mathbf{n}$) relative to the pair ξ', ξ, so that a reflection can take place from ξ' to ξ at x. It follows from (14) that:

$$\boxed{R_{-}(\hat{x}'\xi;\xi') = p(\xi',\xi)r(\xi',\xi)} \quad . \tag{23}$$

The probabilistic interpretation of $p(\xi',\xi)$ is straightforward: for each given ξ', ξ, $p(\xi',\xi)$ is the probability (based on temporal frequencies) that the wave normal's orientation is $\mathbf{n}(x,t)$ per unit solid angle where $\mathbf{n}(x,t)$ is determined from ξ' and ξ by means of the equation:

$$\mathbf{n}(x,t) = \frac{\xi - \xi'}{|\xi - \xi'|} \quad .$$

As an example of the general frequency distribution $p(\xi',\xi)$, we reconsider (28) of Sec. 12.5, where ζ_x and ζ_y are the slope parameters which are fixed once ξ',ξ are chosen (cf. (8)-(10) of Sec. 12.5). In the case $\sigma_c = \sigma_u$, $p(\xi',\xi)$ depends only on the angle ϕ between the normal $\mathbf{n}$ and the vertical **k**, where $\mathbf{n}\cdot\mathbf{k} = \cos\phi$. Hence if ξ' and ξ are such that $\phi \geq 90°$, then $p(\xi',\xi) = 0$ for the case of the gaussian wave slope distribution. In the case $\sigma_c = \sigma_u$ $(=\sigma)$ we have the simple representation:

$$p(\xi',\xi) = \begin{cases} \dfrac{1}{2\pi\sigma^2} e^{-\frac{\tan^2\phi}{2\sigma^2}} & \text{if} \quad \phi \leq 90° \\ 0 & \text{if} \quad \phi > 90° \end{cases} \tag{24}$$

where

$$\phi = \text{arc cos}\left[\frac{(\xi-\xi')}{|\xi-\xi'|}\cdot\mathbf{k}\right] \tag{25}$$

and the normalization constant $(1/2\pi\sigma^2)$ is for the *slope domain* (cf. (16) of Sec. 12.5). The directions ξ',ξ in (25) determine the unit normal **n** to S for a reflection operation in conjunction with (23). The representation of $p(\xi',\xi)$ in (24) is for a gaussian wave slope distribution of the air-water surface. If a particular Neumann spectrum ((1) of Sec. 12.8) is known to apply to a given region of the sea, or some other given natural hydrosol, then we know from (12) or more generally (22) of Sec. 12.8 how to estimate σ as a function of wind speed U_a, since σ^2 varies linearly with U_a. In particular, if $U_a = 0$, then $\sigma = 0$ and $p(\xi',\xi) = \delta((\xi-\xi')/|\xi-\xi'| - \mathbf{k})$ for all ξ' and ξ. As wind speed U_a builds up (24) yields the associated distributions with U_a as a parameter.

It should be noted in passing that the limit in (22) can in some models of S be dependent on x. The presence of swells and other spatially or temporally periodic phenomena on S can in principle be included in $p(\xi',\xi)$ and in the theory below. However, for simplicity of exposition we shall limit the discussion to statistically stationary surfaces S.

In a similar way we can represent $T_+(\hat{x};\xi';\xi)$, defined in (15), as:

$$\boxed{T_+(\hat{x};\xi';\xi) = p(\xi',\xi)t(\xi',\xi)} \tag{26}$$

where $t(\xi',\xi)$ is the Fresnel transmittance function defined in (19) of Sec. 12.1 and where ξ',ξ and the normal **n** to the surface are related by means of Sec. 12.1 (cf. (b) of Fig. 12.60). Thus, for a given pair ξ',ξ, $p(\xi',\xi)$ and $t(\xi',\xi)$ are evaluated analogously to the case of $R_-(x;\xi';\xi)$, but now of course ξ' and ξ are to be related through a refraction rather than a reflection operation. Once **n** is found from the refraction pair ξ',ξ, the value of $p(\xi',\xi)$ for the gaussian model, for example, is given by (24). To find **n** from ξ'

and ξ observe that by (4) of Sec. 12.1, $\mathbf{n}$ is the *unit* vector which may be associated with $n'\xi' - n\xi$ in the following way:

$$\mathbf{n} = \frac{n'\xi' - n\xi}{|n'\xi' - n\xi|} \quad .$$

Observe that if we set $n = n'$, we formally obtain the reflection case from this formula for $\mathbf{n}$. This relation is made quite clear by studying Fig. 12.1(c).

We turn next to the study of $Q^o(\hat{x},\xi)$ and $Q(\hat{x},\xi)$, defined in (11) and (12). Observe first that, by (10) of Sec. 12.10, and (13a) we have:

$$Q_+(\hat{x},\xi) + Q_-(\hat{x},\xi) = 1$$

i.e.,

$$Q_+(\hat{x},\xi) + Q^o(\hat{x},\xi) + Q(\hat{x},\xi) = 1 \tag{27}$$

where we have written (cf. (13)):

$$\text{"}Q_+(\hat{x},\xi)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T} \int_o^T \chi(\xi,\Xi_+(x,t))dt \ . \tag{28}$$

It is clear that Q, Q^o and Q_+ are so interrelated that knowledge of any two of them is sufficient to determine the third. It turns out that, for our present exposition, it is convenient to determine Q and Q_+ so that the average Q^o which occurs in (18) and (45) below, is determined indirectly through (27) and knowledge of Q and Q_+. Care must be taken in the determination of Q and Q_+ by adopting compatible statistical models for each so that Q^o turns out to be nonnegative.

The representation of Q_+ is readily obtained. From its definition above, $Q_+(\hat{x},\xi)$ is evidently the probability that a given direction ξ is in the outer hemisphere $\Xi_+(x,t)$ at x on the surface S at any time t, i.e., $Q_+(\hat{x},\xi)$ is the probability that $\xi \cdot \mathbf{n} > 0$ where $\mathbf{n}$ is the unit outer normal to S at x. This dot product characterization $\mathbf{n} \cdot \xi > 0$ permits still another way of viewing $Q_+(\hat{x},\xi)$ as the probability that $\mathbf{n}$ is in $\Xi_+(\xi)$, i.e., in the directional hemisphere determined by ξ. This characterization of $Q_+(\hat{x},\xi)$ leads directly to:

$$\boxed{Q_+(\hat{x},\xi) = \int_{\Xi_+(\xi)} p(\xi',\xi)\ d\Omega(\mathbf{n})} \tag{29}$$

in which:

$$\xi' = \xi - 2(\xi \cdot \mathbf{n})\mathbf{n} \quad . \tag{30}$$

In the gaussian model of the air-water surface, it follows that $Q_+(\hat{x},\xi)$ depends on the mean square slopes σ_u^2, σ_c^2, and hence on the wind speed U_a, after the manner discussed in Sec. 12.5 or Sec. 12.8.

For example, if the gaussian model is adopted, and we assume that the slope distribution is isotropic: $\sigma_u = \sigma_c = \sigma$ $(m_{02}^{1/2} = m_{20}^{1/2})$, then it has been shown by Keith MacAdam (in a private communication) that the integral (29) has the value:

$$Q_+(\hat{x},\xi) = \frac{1}{2}\,[1 + E(\tan\psi/\sigma)] \qquad (29a)$$

where we write:

$$\text{"}E(t)\text{"} \quad \text{for} \quad \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-u^2/2}\,du$$

and where ψ at present is the angle (between 0° and 90° in magnitude) that ξ makes with the horizontal plane $\hat{S}$, and such that $\psi > 0$ if ξ lies below $\hat{S}$. Hence, in particular, $Q_+(\hat{x},\mathbf{k}) = 1$, $Q_+(\hat{x},-\mathbf{k}) = 0$, and $Q_+(\hat{x},\xi) = 1/2$ for ξ in $\hat{S}$. Since, by (27),

$$Q_+(\hat{x},\xi) + Q_-(\hat{x},\xi) = 1$$

we have at once that:

$$Q_-(\hat{x},\xi) = \frac{1}{2}\,[1 - E(\tan\psi/\sigma)] \quad . \qquad (29b)$$

We come finally to the determination of $Q(\hat{x},\xi)$, as defined in (12). It is simpler to consider $Q(\hat{x},\xi)$ as obtained by means of a space average rather than a time average. For this purpose we shall adopt the ergodic hypothesis for S which in the case of $Q(\hat{x},\xi)$ takes the form:

$$Q(\hat{x},\xi) = \lim_{L\to\infty} \frac{1}{L}\int_{\mathcal{P}} \chi(\xi,D(S,x,t))\,d\ell \qquad (31)$$

where the integration is taken at a fixed time t over any straight path on the mean surface $\hat{S}$, and where x (i.e., x(t)) is on S and determined for each $\hat{x}$ after the manner explained in Figs. 12.57 and 12.58 above. We can now interpret $Q(\hat{x},\xi)$ as the fraction of a unit length along the path $\mathcal{P}$ which a straight line of sight from x(t) on S, and extending along $-\xi$, meets S. When phrased in this way, the interpretation of $Q(\hat{x},\xi)$ is reminiscent of the ordinate-crossing properties of random functions (cf., e.g., [168], [87]). One such property, which is pertinent to the present problem, may be stated as follows: *If S is a surface with a gaussian distribution of elevations (cf., e.g., (12) of Sec. 12.9), then the number of times $n(\zeta)$ per unit length a horizontal straight path $\mathcal{P}$ a distance ζ above $\hat{S}$ crosses S, is given by*

$$n(\xi) \doteq \frac{1}{\pi}\left[\frac{m_2(\theta)}{m_{oo}}\right]^{1/2} e^{-\frac{\zeta^2}{2m_{oo}}} \qquad (32)$$

where $m_2(\theta)$ is given by (97) of Sec. 12.4 and m_{oo} is the root mean square elevation of S above $\hat{S}$ (cf., (89) of Sec. 12.4). Except for the value n(0) the ζ-dependence of $n(\zeta)$ is intuitively obvious. A derivation of (32) may be found, e.g., in [166]. However, for our present purposes, (12) of Sec. 12.9 will do just as well. We may use (12) of Sec. 12.9 for our present problem as follows: the *relative* probability $q(\zeta)$ (i.e., without the property of normalization over all possibilities) that a horizontal path $\mathcal{P}$ a distance ζ above $\hat{S}$ meets S is:

$$q(\zeta) = n(\zeta)/n(0) = e^{-\frac{\zeta^2}{2m_{oo}}} . \qquad (33)$$

This relative probability may be used to obtain a rough but workable estimate of $Q(\hat{x},\xi)$ by computing the probability of intersection with S of an inclined path of direction ξ through a point $\hat{x}$ on $\hat{S}$, where ξ is in Ξ_-. In particular, it will yield the conditional probability of an intersection, given that ξ is in $\Xi_-(x,t)$. As shown in (a) of Fig. 12.61, the inclined path at each elevation ξ above $\hat{S}$ has a relative probability $q(\zeta)$ per unit length of meeting S, were it to continue horizontally. However, the path is continually changing its elevation so that the probability that it meets the surface S at least once is a sum of probabilities of the form:

$$\frac{1}{m_{oo}^{1/2}} \sum_{j=1}^{\infty} e^{-\frac{(j\Delta x \tan|\psi|)^2}{2m_{oo}}} \Delta x \tan|\psi| \qquad (34)$$

on the assumption that the elevations of S above $\hat{S}$ are independent from one discrete station to the next along the x-axis as shown in (b) of Fig. 12.61. The factor $m_{oo}^{1/2}$ is placed into (34) so as to make the relative probability dimensionless. Along the x-axis (which is to lie in $\hat{S}$), we have marked out unit distances Δx of arbitrarily small but fixed extent. The path through $\hat{x}$ is inclined at an angle ψ with the x-axis where ψ is positive above, and negative below $\hat{S}$. At j units out from $\hat{x}$, the relative probability that the surface is in the vertical slot beginning at $j\Delta x \tan|\psi|$ and of height $\Delta x \tan|\psi|$ is given by

$$\frac{1}{m_{oo}^{1/2}} e^{-\frac{(j\Delta x \tan|\psi|)^2}{2m_{oo}}} \Delta x \tan|\psi|$$

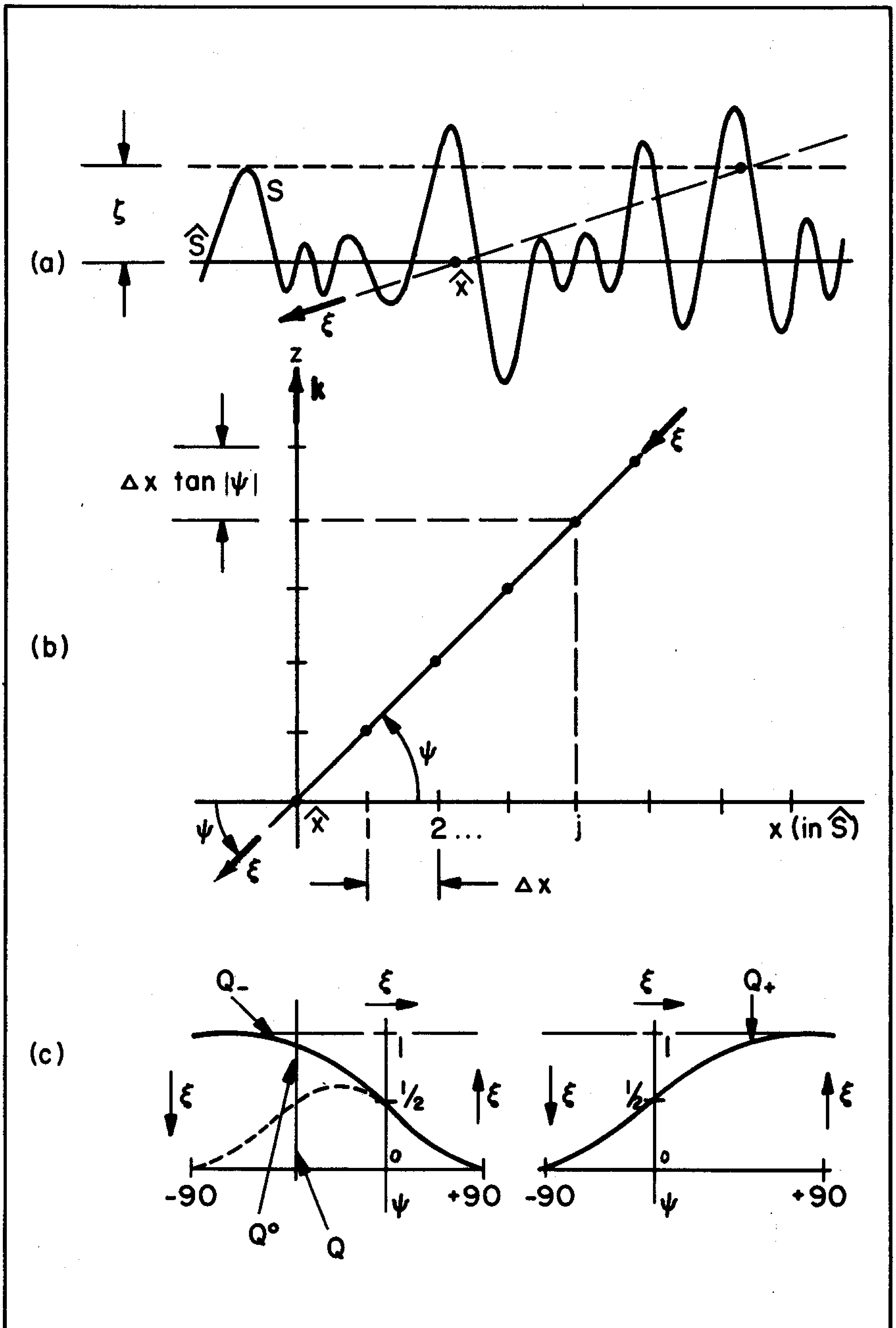

FIG. 12.61 Parts (a) and (b) depict the geometrical relations needed for the derivation of the form of the function q_{-}. Part (c) shows the general qualitative behavior of the Q-functions with respect to angle ψ.

The relative probability that the given path intersects S at least once is then the sum (34) given above. Actually, since the surface is statistically stationary, the horizontal distances $j\Delta x$ along the x-axis in (b) of Fig. 12.61 are only schematic. The whole diagram may be collapsed like an accordion onto the z-axis and the sum (34) is imagined to take place in a vertical strip of arbitrary unit width Δx and proceeding vertically in steps of magnitude $\Delta x \tan|\psi|$, beginning with $\Delta x \tan|\psi|$. Writing:

$$"\Delta\zeta" \quad \text{for} \quad \Delta x \tan|\psi| \quad ,$$

$$"\zeta_j" \quad \text{for} \quad j\Delta x \tan|\psi|$$

the sum (34) reduces to:

$$\frac{1}{m_{oo}^{1/2}} \sum_{j=1}^{\infty} e^{-\zeta_j^2/2m_{oo}} \Delta\zeta$$

and by requiring $\Delta x = \lambda m_{oo}^{1/2}$ where λ is at present an arbitrary fixed positive dimensionless parameter (to be discussed below), the sum can go over into its continuous version:

$$\int_{\lambda \tan|\psi|}^{\infty} e^{-y^2/2}\, dy$$

and which, when normalized, becomes the required conditional probability $q_-(\hat{x},\xi)$:

$$q_-(\hat{x},\xi) = \sqrt{\frac{2}{\pi}} \int_{\lambda \tan|\psi|}^{\infty} e^{-y^2/2}\, dy \quad . \tag{35}$$

Thus we have found $q_-(\hat{x},\xi)$, the conditional probability of at least one intersection of the path along ξ with the sea surface S, on the condition that ξ is in $\Xi_-(x,t)$. Our derivation has assumed that this conditional probability is independent of the position of ξ in $\Xi_-(x,t)$. Therefore:

$$\boxed{\begin{aligned} Q(\hat{x},\xi) &= q_-(\hat{x},\xi)Q_-(\hat{x},\xi) \\ &= q_-(\hat{x},\xi)(1 - Q_+(\hat{x},\xi)) \end{aligned}} \tag{35a}$$

which with (29b) determines $Q(\hat{x},\xi)$. From this and (13a) we find at once that:

$$\begin{aligned} Q_-(\hat{x},\xi) &= Q^o(\hat{x},\xi) + Q(\hat{x},\xi) \\ &= Q^o(\hat{x},\xi) + q_-(\hat{x},\xi)Q_-(\hat{x},\xi) \quad , \end{aligned}$$

so that for a gaussian sea surface model:

$$Q^{\circ}(\hat{x},\xi) = Q_{-}(\hat{x},\xi)[1 - q_{-}(\hat{x},\xi)]$$
$$= [1 - Q_{+}(\hat{x},\xi)][1 - q_{-}(\hat{x},\xi)] \quad (35b)$$

which with (29b) determines $Q^{\circ}(\hat{x},\xi)$. From the intuitive meanings of the Q-functions, we require the following limiting values, which will be helpful in checking specific statistical models (of which the gaussian is but a single instance):

Q-function	$\xi = \mathbf{k}$	ξ in $\hat{S}$	$\xi = -\mathbf{k}$
$Q_{+}(\hat{x},\xi) =$	1	1/2	0
$Q_{-}(\hat{x},\xi) =$	0	1/2	1
$Q^{\circ}(\hat{x},\xi) =$	0	0	1
$Q(\hat{x},\xi) =$	0	1/2	0
$q_{-}(\hat{x},\xi) =$	1	1	0

The general expected shapes of the Q-curves are schematically depicted in (c) of Fig. 12.61. Thus Q_{-} is expected to be a monotonic decreasing function of ψ, while Q is to increase in the range (-90°,0) and is to coincide with Q_{-} in the interval (0,90°). Q_{+} on the other hand is to be a monotonic increasing function of ψ and such that $Q_{+} + Q_{-} = 1$. These properties hold not only for the illustrative example of the gaussian sea presented above, but also for general sea surface statistics.

Some final comments are needed on the presence of the arbitrary dimensionless parameter λ in the lower limit of (35). The determination of λ can rest in the following observations. First observe that the choice of Δx in the preceding derivation is not fixed by any physical conditions associated with the simple model (32). A more detailed model would give, for example, the joint probability distribution for wave elevations above or below two neighboring points on $\hat{S}$. Second, observe that this distribution would then provide a horizontal distance between two points of $\hat{S}$ above and below each of which the air-water surface would oscillate independently. Suppose this distance were μ_{oo} Then we could write "λ" for $\mu_{oo}/m_{oo}^{1/2}$, and so λ would be intimately tied to the statistics of the sea surface. In particular, we choose Δx so that it is λ times $m_{oo}^{1/2}$ or simply μ_{oo}. For such a Δx the independence condition needed in the preceding derivation would be satisfied. The λ chosen should be the smallest of such ratios. It might be helpful in practice to keep in mind that μ_{oo} should *decrease* and $m_{oo}^{1/2}$ should *increase* with *increase* Errata of the generating wind speed (recall, e.g., (10) of Sec. 12.10). Observe finally that (35) is a special case of:

$$q_-(\hat{x},\xi) = \frac{1}{q_o}\int_{\lambda \tan|\psi|}^{\infty} q(y)\, dy \qquad (36)$$

where $\xi \cdot \mathbf{k} = \sin|\psi|$, and where we write:

$$\text{"}q_o\text{"} \quad \text{for} \quad \int_{o}^{\infty} q(y)\, dy \quad .$$

$q(\zeta)$ is the relative probability that S is at elevation ζ above $\hat{S}$; and (36) is derived under the same assumptions adopted for (35).

This completes the discussion of the structure of the weighting functions R_-, T_+, Q, Q^o, Q_+, and Q_-. For a statistically stationary air-water surface whose Fresnel reflectance is independent of location over the mean plane $\hat{S}$, these weighting functions and the time averaged radiance distribution $\hat{N}$ are independent of location $\hat{x}$ in $\hat{S}$; hence "$\hat{x}$" may be dropped from the notation when these conditions prevail. Equation (18) may now be solved in principle for air-water surfaces with known statistical properties.

The Instantaneous and Time-Averaged Equations for $N_-^+(S)$

Having developed the theory of the time-averaged radiance $N_+^+(S)$, starting with (3) of Sec. 12.10 and culminating in (18) above, we complete the general theory of time-averaged radiance fields for the dynamic air-water surface S by outlining the derivation of the time-averaged equations for $N_-^+(S)$, the radiance output of S into the body of the hydrosol (cf. Fig. 12.56).

Starting with (15) of Sec. 3.5 we have:

$$N_-^+(S) = A_-(S) + N_-^-(S)t_-(S) + N_+^-(S)r_+(S) \qquad (37)$$

where we have written

$$\text{"}A_-(S)\text{"} \quad \text{for} \quad N_-^o t_-^o(S) + N_+^o r_+^o(S) \quad . \qquad (38)$$

Here N_-^o is the *downward* incident radiance distribution on S from the sky and N_+^o is the upward incident radiance distribution on S from the hydrosol. By invoking the black-convexification hypothesis on the hydrosol side of S, (37) reduces to

$$N_-^+(S) = A_-(S) + N_-^-(S)t_-(S) \quad . \qquad (39)$$

Hence in view of (1), (6) of Sec. 12.10, $N_-^+(S)$ is completely determined once N_+^o and N_-^o are known. N_-^o is assumed given. N_+^o is determinable in averaged form in the manner to be shown in (10) of Sec. 12.13.

We can convert (39) into explicit numerical form in exactly the manner that (5) of Sec. 12.10 was converted into (12) of Sec. 12.10. The result is:

$$\begin{aligned} N(x,\xi,t) = \int_{\Xi} & N^o(x,\xi',t)[\chi(\xi',D^o(x,t))t_-(x;\xi';\xi) \\ & + \chi(\xi',\Xi_+(x,t))r_+(x;\xi';\xi)]d\Omega(\xi') \\ & + \int_{\Xi} N(x-r_m\xi',\xi',t)\chi(\xi',D(S,x,t))t_-(x;\xi';\xi)d\Omega(\xi') \\ \xi \in \Xi_-(x,t),\ & x \in S \end{aligned} \tag{40}$$

This is the requisite representation of the instantaneous radiance $N(x,\xi,t)$ with ξ in $\Xi_-(x,t)$, at an arbitrary point x of S at time t. The radiances $N(x-r_m\xi',\xi',t)$ are those governed by the integral equation (12) of Sec. 12.10. Equation (40) with (12) of Sec. 12.10 completes the description of the radiance distribution over $\Xi = \Xi_+(x,t) \cup \Xi_-(x,t)$ at an arbitrary point x of S at time t. The input radiance distribution $N^o(x,\cdot,t)$ is derived from knowledge of the sky radiance over $D^o(x,t)$ and of the hydrosol radiance over $\Xi_+(x,t)$. It will be shown in Sec. 12.12 how to find the hydrosol radiance given $N(x,\cdot,t)$ over $\Xi_-(x,t)$. Then in Sec. 12.12 a solution will be synthesized from the appropriate pieces of the analysis.

Continuing with the preparations for the present averaging process, let us write, analogously to (14) and (15):

$$\text{"}T_-(\hat{x};\xi';\xi)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_o^T t_-(x;\xi';\xi)dt \tag{41}$$

$$\text{"}R_+(\hat{x};\xi';\xi)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_o^T r_+(x;\xi';\xi)dt\ . \tag{42}$$

Finally, we write:

$$\text{"}\hat{N}_-^+(\hat{x},\xi)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_o^T N_-^+(x(t),\xi,t)dt \tag{43}$$

whenever ξ is in Ξ. We currently do not need two cases for $N^+(x,\xi)$, as in (1) and (2) for $\hat{N}^+_+(\hat{x},\xi)$, because of the black-convexification condition on the hydrosol side of S. We are now retaining superscripts and subscripts on "N" to keep $\hat{N}^+_-(\hat{x},\xi)$ and $\hat{N}^+_+(\hat{x},\xi)$ conceptually distinct. The latter, of course, is governed by (18). Finally, x(t) in (43) is obtained as in Figs. 12.57 and 12.58 by imagining the hydrosol and aerosol to be reversed in position relative to S(i.e., "reflected" in S). Hence (41)-(43) are well defined. Next, we apply the averaging operation (43) to each side of (40), and assuming the same stationary and independence conditions on the various quantities in (40), just as was assumed in obtaining (18), we arrive at:

$$\hat{N}^+_-(\hat{x},\xi) = \hat{A}_-(\hat{x},\xi) + \int_{\Xi} \hat{N}^+_+(\hat{x},\xi')Q(\hat{x},\xi')T_-(\hat{x},\xi';\xi)\; d\Omega(\xi')$$

where

$$\hat{A}_-(\hat{x},\xi) = \int_{\Xi} \hat{N}^o(\hat{x},\xi')[Q^o(\hat{x},\xi')T_-(\hat{x};\xi';\xi) + Q_+(\hat{x},\xi')R_+(\hat{x},\xi';\xi)] \cdot d\Omega(\xi')$$

$$\xi \in \Xi,\ \hat{x} \in \hat{S} \qquad (44)$$

This is the requisite equation governing the time-averaged surface radiance $\hat{N}^+_-(\hat{x},\xi)$ of the mean air-water surface $\hat{S}$ at point $\hat{x}$ with the time-averaged incident sky radiance $\hat{N}^o(\hat{x},\xi)$ as the basic input. The term $\hat{A}_-(\hat{x},\xi)$ is a convenient contraction of two terms, one of which is the integral of $\hat{N}^oQ^oT_-$ over Ξ, and the other the integral of $\hat{N}^o_+Q_+R_+$ over Ξ, as may be seen by comparing (44) with (40). In the first integral $\hat{N}^o$ is the time-averaged sky radiance over Ξ_-. In the second integral $\hat{N}^o$ is the time-averaged upward hydrosol radiance defined over Ξ. In sum (44) is the time-averaged version of (39), with the radiances $\hat{N}^+_+(\hat{x},\xi)$ as given in (18). Equation (44) is the companion formula to (18) and together they specify $N^+_+(S)$, $N^+_-(S)$ given N^o_+ and N^o_- as defined, e.g., in (38).

Under the same conditions leading to (26) we can deduce that:

$$T_-(\hat{x};\xi';\xi) = p(\xi',\xi)t(\xi',\xi) \qquad (45)$$

where $p(\xi',\xi)$ is evaluated, e.g., as in (24), but with the unit normal determined by requiring ξ',ξ to be a Fresnel transmittance pair of directions. In addition,

$$R_+(\hat{x};\xi';\xi) = p(\xi',\xi)r(\xi',\xi) \qquad (46)$$

which is the companion to (23). Observe that the ξ' in (46) is to be incident from the hydrosol side of S, whereas ξ' in (23) is incident from the aerosol side of S. A similar reversal of location of ξ' holds for (26) and (45). This information is to be kept in mind when the Fresnel reflectance $r(\xi',\xi)$ and the Fresnel transmittance $t(\xi',\xi)$ (as given in Sec. 12.1) are used for computations of $\hat{N}^{+}_{+}(\hat{x},\xi)$, $\hat{N}^{+}_{-}(\hat{x},\xi)$ using (18) and (45). Furthermore $p(\xi',\xi)$ in (45) is computed from knowledge of the orientation of the outward unit normal $\mathbf{n}$ to S given the transmittance pair ξ',ξ. The equation which determines $\mathbf{n}$ from ξ' and ξ in the Fresnel transmittance case is:

$$\mathbf{n} = \frac{n'\xi' - n\xi}{|n'\xi' - n\xi|} ,$$

which follows from (4) of Sec. 12.1. On the other hand $p(\xi',\xi)$ in (46) is evaluated when $\mathbf{n}$ is obtained from the reflectance pair ξ',ξ, by means of:

$$\mathbf{n} = \frac{\xi - \xi'}{|\xi - \xi'|}$$

which is (1) of Sec. 12.1. In the case of statistically stationary air-water surfaces "$\hat{x}$" may be dropped from the notation in (44).

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12.12 Instantaneous and Time-Averaged Radiance Fields Within a Natural Hydrosol

The integral equation and integral representation for the time-averaged radiance distribution over the dynamic air-water *surface*, as given in (18) and (44) of Sec. 12.11, will here be supplemented with a description of the time-averaged radiance field *below* the mean surface $\hat{S}$. In this way the theory of the time-averaged light field within natural hydrosols is made completely self-contained and can be reduced to a steady state plane-parallel medium problem. For, as a study of the derivation of (18) and (44) of Sec. 12.11 would show, the derivations began with the assumption that the initial sky radiance distribution N° was given over S, along with the white and black convexifications of the surface S so that the multiple interreflection process over S could be isolated and studied by itself. In reality, however, the incident radiances $N^{\circ}(x,\xi,t)$ on S from below (i.e., ξ in $\Xi_{+}(x,t)$) are intimately related to the fully self-interreflected radiances over S. Hence in a very definite sense, the input radiances $N^{\circ}(x,\xi,t)$ with ξ in $\Xi_{+}(x,t)$ leading to $\hat{N}(\hat{x},\xi)$ are dependent on the answer $\hat{N}(\hat{x},\xi)$ sought. It is precisely at this point that the power of the principles of invariance or, more generally, the interaction principle, becomes manifest. For by black convexifying the lower surface of S we could defer this complication of interactions between S and the body X of the natural hydrosol until the present stage of analysis. In the stage now to be considered, we imagine the dynamic air-water surface S peeled off the hydrosol leaving only the body X of the hydrosol. The instantaneous output of S (after full interreflections) will now be the instantaneous input to X at