is to be incident from the hydrosol side of S, whereas ξ' in (23) is incident from the aerosol side of S. A similar reversal of location of ξ' holds for (26) and (45). This information is to be kept in mind when the Fresnel reflectance $r(\xi',\xi)$ and the Fresnel transmittance $t(\xi',\xi)$ (as given in Sec. 12.1) are used for computations of $\hat{N}_+^+(\hat{x},\xi)$, $\hat{N}_-^+(\hat{x},\xi)$ using (18) and (45). Furthermore $p(\xi',\xi)$ in (45) is computed from knowledge of the orientation of the outward unit normal $\mathbf{n}$ to S given the transmittance pair ξ',ξ. The equation which determines $\mathbf{n}$ from ξ' and ξ in the Fresnel transmittance case is:

$$\mathbf{n} = \frac{n'\xi' - n\xi}{|n'\xi' - n\xi|} ,$$

which follows from (4) of Sec. 12.1. On the other hand $p(\xi',\xi)$ in (46) is evaluated when $\mathbf{n}$ is obtained from the reflectance pair ξ',ξ, by means of:

$$\mathbf{n} = \frac{\xi - \xi'}{|\xi - \xi'|}$$

which is (1) of Sec. 12.1. In the case of statistically stationary air-water surfaces "$\hat{x}$" may be dropped from the notation in (44).

12.12 Instantaneous and Time-Averaged Radiance Fields Within a Natural Hydrosol

The integral equation and integral representation for the time-averaged radiance distribution over the dynamic air-water *surface*, as given in (18) and (44) of Sec. 12.11, will here be supplemented with a description of the time-averaged radiance field *below* the mean surface $\hat{S}$. In this way the theory of the time-averaged light field within natural hydrosols is made completely self-contained and can be reduced to a steady state plane-parallel medium problem. For, as a study of the derivation of (18) and (44) of Sec. 12.11 would show, the derivations began with the assumption that the initial sky radiance distribution N° was given over S, along with the white and black convexifications of the surface S so that the multiple interreflection process over S could be isolated and studied by itself. In reality, however, the incident radiances $N^\circ(x,\xi,t)$ on S from below (i.e., ξ in $\Xi_+(x,t)$) are intimately related to the fully self-interreflected radiances over S. Hence in a very definite sense, the input radiances $N^\circ(x,\xi,t)$ with ξ in $\Xi_+(x,t)$ leading to $\hat{N}(\hat{x},\xi)$ are dependent on the answer $\hat{N}(\hat{x},\xi)$ sought. It is precisely at this point that the power of the principles of invariance or, more generally, the interaction principle, becomes manifest. For by black convexifying the lower surface of S we could defer this complication of interactions between S and the body X of the natural hydrosol until the present stage of analysis. In the stage now to be considered, we imagine the dynamic air-water surface S peeled off the hydrosol leaving only the body X of the hydrosol. The instantaneous output of S (after full interreflections) will now be the instantaneous input to X at every point x on S at time t. And, conversely, the instantaneous output of X will be the instantaneous input to S.

The solution of the time-averaged light field problem as outlined above will be developed in the following three stages: First we shall introduce two kinds of time-averaged radiance fields. The equation of transfer will be derived for each of these fields in the second stage. Each field performs a certain descriptive task and when these fields are linked together in the third stage, they will form a bridge between the theory and practice of measuring time-averaged light fields in the sea.

Two Types of Time-Averaged Radiance Fields

The measurement of time-varying light fields in lakes, oceans, and other natural hydrosols is most conveniently accomplished at an arbitrary *fixed* depth z below (or above) the mean surface $\hat{S}$. On the other hand, it was found convenient in Sec. 12.11 to describe the time varying light field over the dynamic air-water surface S at points fixed on and moving with S. Hence the time-averaged radiances $\hat{N}(\hat{x},\xi)$ in (18) and (44) of Sec. 12.11 are averages obtained by *moving with the surface* in a certain perscribed way (Figs. 12.57-12.59). The empirical and theoretical descriptions of time-averaged light fields therefore adopt two distinct types of averages. The empirical averages may be defined formally by writing:

$$"\overline{N}(z,\xi)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(z,\xi,t)dt \tag{1}$$

in which z and ξ are fixed ($-\infty < z < \infty$). Further for the theoretical averages, we write:

$$"\hat{N}(\zeta',\xi)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T N(\zeta'+\zeta(t),\xi,t)\,dt \tag{2}$$

in which ζ' and ξ are fixed ($\zeta' > 0$). For $\zeta' = 0$, $\hat{N}(\zeta',\xi)$ takes the form (2) of Sec. 12.11 for ξ in Ξ_-. The geometric setting for (1) and (2) is given in Fig. 12.62. $\zeta(t)$ is the elevation of S above $\hat{S}$. $\zeta(t)$ and all other distances are measured positive downward. For the purposes of the following discussion, we say that $\overline{N}$ is a *fixed depth* average, and $\hat{N}$ a *cosurface* average. Thus (18) of Sec. 12.11 is an integral equation for the cosurface average $\hat{N}(0,\xi)$, $\xi \in \Xi$, by virtue of the stationarity condition. For simplicity of notation the horizontal x and y coordinates have been suppressed in the various functions above, and these variables are also fixed in (1) and (2). The point $\hat{x}$ on $\hat{S}$ has the representation (x,y,0). The x and y coordinates are eventually averaged out in statistically stationary media, the media most commonly adopted for discussion in practice, and which we shall assume throughout this discussion. It is to be noted that the cosurface average for depths below $\hat{S}$ is simply a mathematical

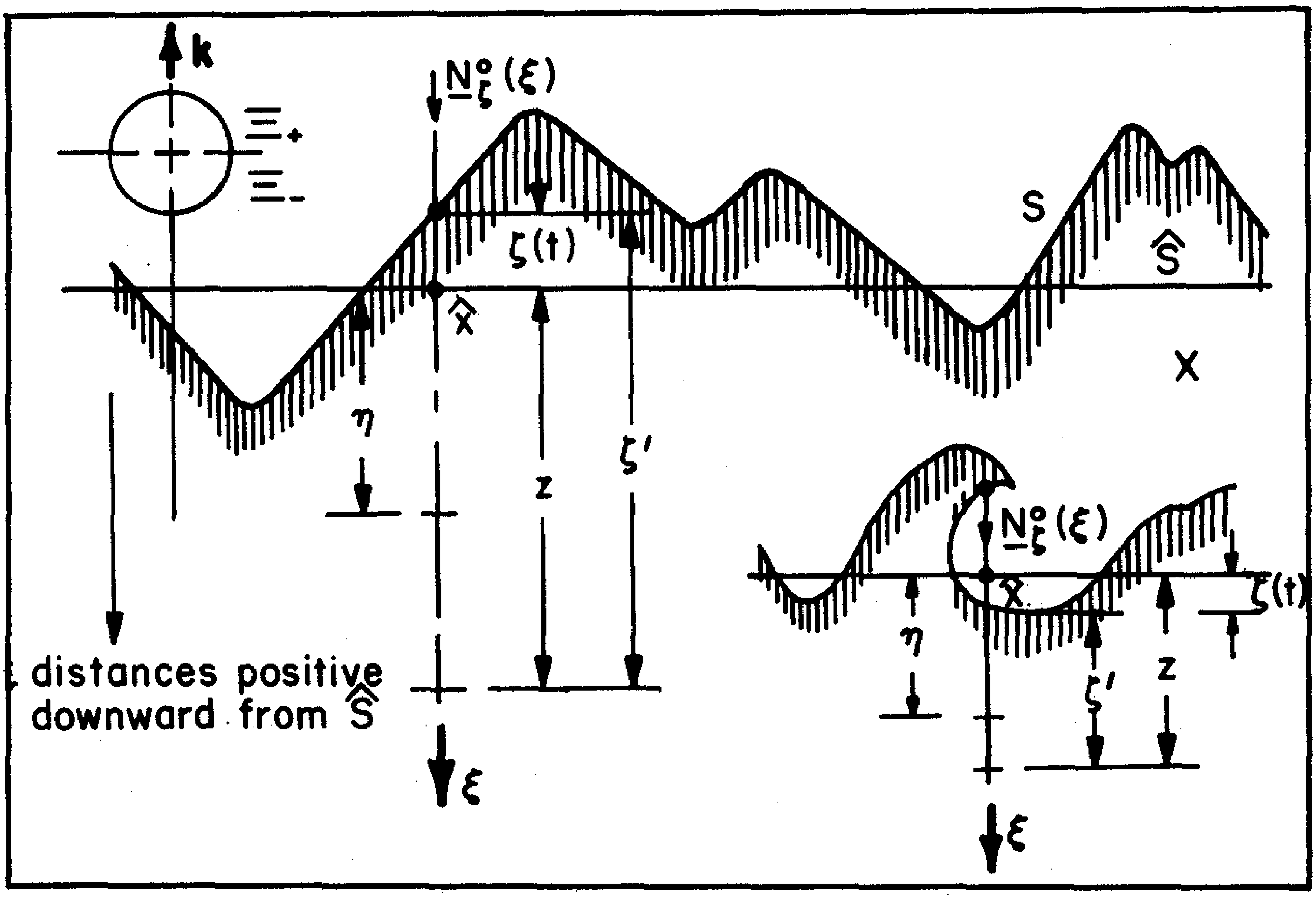

FIG. 12.62 Setting for the definition of fixed-depth and cosurface time averages (vertical path case).

device for facilitating the present averaging computations. Another possibility would be to solve the hydrodynamic equations (29) and (30) of Sec. 12.3 subject to suitable boundary conditions, and to make explicit use of the solutions.

Equations of Transfer for Time-Averaged Radiance Fields

We begin with the derivation of the equation of transfer for fixed depth averaged radiance fields, and to see the essential idea of the derivation, the vertical downward case (i.e., $\xi = -\mathbf{k}$) will be considered first. Thus for the moment, ξ, as in Fig. 12.62, will be directed vertically downward. At any time t, the radiance $N(z,\xi,t)$ consists of two parts (cf., (2) of Sec. 3.15) namely the residual radiance transmitted over the path through X from S to depth z, and the path radiance built up over the same path. However, because the depth z has been fixed, it is possible for a wave to pass which is so deep that a point at depth z would lie completely exposed to the atmosphere. In such an event the radiance reading is $N_\zeta^\circ(\xi)$, and is either the surface radiance of S or the zenith radiance of the sky as seen from above the surface. When $N_\zeta^\circ(\xi)$ is the surface radiance of S, it is given by (12) of Sec. 12.10. See inset of Fig. 12.62. To anticipate this case we must devise a means of describing uniformly the situations when the depth z is either exposed to the atmosphere or not. Thus let G be a function of a real variable with the property:

$$G(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} \tag{3}$$

Then from two applications of (2) of Sec. 3.15 for the cases $z - \zeta(t) \geq 0$ and $z - \zeta(t) < 0$, we have:

$$N(z,\xi,t) = N^{o}(\zeta(t),\xi,t)T_{z-\zeta(t)}(\zeta(t),\xi)G(z - \zeta(t))$$
$$+ \underline{N}^{o}_{\zeta}(\xi)[1 - G(z - \zeta(t))]$$
$$+ \int_{-\infty}^{z} N_{*}(\eta,\xi,t)T_{z-\eta}(\eta,\xi)G(\eta - \zeta(t))\, d\eta \tag{4}$$

The initial radiance $N^{o}(\zeta(t),\xi,t)$ is the instantaneous radiance of the lower part of S in the direction ξ at time t, and as given in (40) of Sec. 12.11.

The case of the inclined downward path (ξ in Ξ_{-}) now follows readily by measuring all distances parallel to the line along the direction ξ and going through a fixed point $\hat{x}$ on $\hat{S}$, as shown in Fig. 12.63. This figure is labeled analogously to Fig. 12.61. To emphasize that the distances for

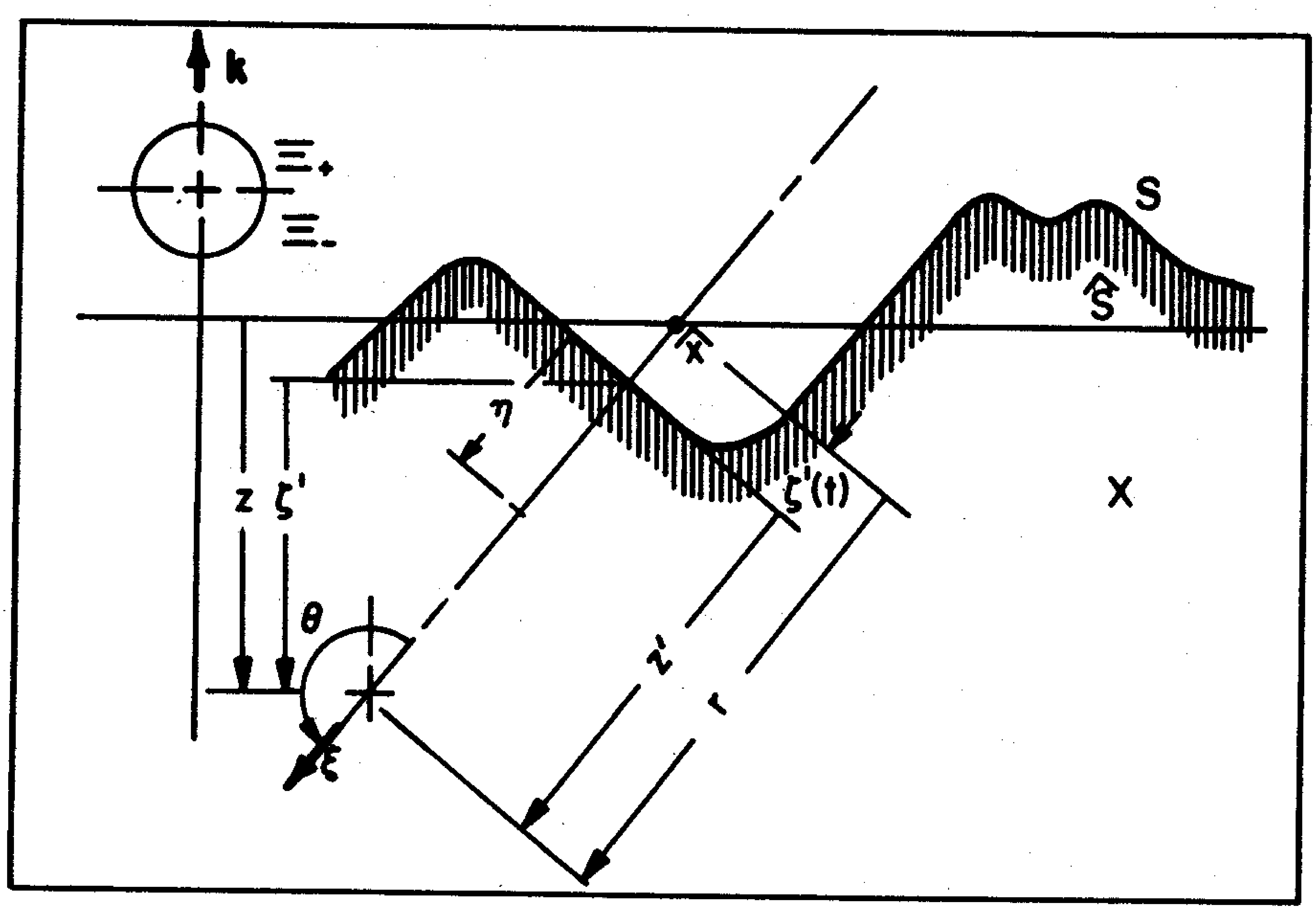

FIG. 12.63 Setting for the definition of fixed-depth and cosurface time averages (slant path case).

beam transmittance are measured parallel to ξ, "z" is replaced by "r" and "$\zeta(t)$" by "$\zeta'(t)$", as shown. Equation (4) may be converted to the slant path case by introducing these notational changes. In a similar manner the case for ξ in Ξ_+ can be constructed. However for our present purpose it is unnecessary to go into these cases in detail since they all are neatly expressible in the associated integrodifferential equation for $N(x,\xi,t)$:

$$\boxed{\begin{aligned}\frac{dN(z,\xi,t)}{dr} &= G(z-\zeta(t))[-\alpha(z)N(z,\xi,t) + N_*(z,\xi,t)] \\ &\quad + \delta(z-\zeta(t))[N^o(\zeta(t),\xi,t) - \underline{N}^o_\zeta(\xi)]\end{aligned}} \tag{5}$$

where z is any depth or altitude, $-\infty < z < \infty$, ξ is any direction in Ξ and $\zeta(t)$ is the elevation of S with respect to S (with "x", "y" omitted throughout this discussion for brevity). The last term is the net contribution to $N(z,\xi,t)$ by the radiance of the surface S; and this contribution is confined to the "paper thin" thickness of S, hence the presence Dirac-delta function $\delta(z-\zeta(t))$. It may be verified that,* for a given path $\mathcal{P}_r(x,\xi)$ with endpoints in or out of the medium X, the appropriate integral of (5) is (4), or its slant path counterpart as the case may be. It should be noted that (5) is a quasi-steady state form of the equation of transfer, and not a pure time dependent form as explained in the discussion of (7) of Sec. 12.10. The fixed-depth average form of (5) can now be taken, on application of definitions (1), and (2), and the invocation of the independence condition (10) of Sec. 12.11 for the four functions N^o, N, G, and δ; the result is:

$$\boxed{\frac{d\bar{N}(z,\xi)}{dr} = \bar{G}(z)[-\alpha(z)\bar{N}(z,\xi) + \bar{N}_*(z,\xi)] + \bar{\delta}(z)[\hat{N}^o(0,\xi) - \underline{\hat{N}}^o_\zeta(\xi)]} \tag{6}$$

where we have written:

$$\text{"}\bar{G}(z)\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T G(z-\zeta(t))\, dt \quad , \tag{7}$$

and

*Mainly what is required here is knowledge of the identity $d(f(z)G(z))/dz = (df(z)/dz)G(z) + f(0)\delta(z)$, where G is given in (3), δ is the Dirac-delta function, and f is any depth dependent function. Recall also the derivation of (3) of Sec. 3.15, and note that $[G(z)]^2 = G(z)$; $[1 - G(z)]G(z) = 0$; and in general $G(y)G(z) = G(y)$ for $y \le z$. (The latter property yields the preceding two.)

$$"\overline{\delta}(z)" \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(z - \zeta(t))\, dt \quad , \tag{8}$$

$\overline{N}_*(z,\xi)$ is the fixed-depth time-average of $N_*(z,\xi,t)$ and is easily seen to be related to $\overline{N}(z,\xi)$ in the expected way:

$$\overline{N}_*(z,\xi) = \int_\Xi \overline{N}(z,\xi')\sigma(z;\xi';\xi)d\Omega(\xi') \quad . \tag{9}$$

The averaged radiance $\hat{N}^o(0,\xi)$ is, by (2) (in which $\zeta' = 0$), the cosurface time-averaged radiance and therefore by the stationarity assumption is governed by (44) of Sec. 12.11 (in which $\hat{x} = (x,y,0)$). In the case of an air-water surface S whose elevations $\zeta(t)$ are governed by a gaussian distribution (cf. (12) of Sec. 12.9) we have: $\overline{\delta}(z) = \phi(z)$ and:

$$\overline{G}(z) = \frac{1}{\sqrt{2\pi m_{oo}}} \int_{-\infty}^{z} e^{-\frac{\zeta^2}{2m_{oo}}} d\zeta \quad . \tag{10}$$

Equation (6) is the requisite equation of transfer for the fixed-depth time-averaged radiance function $\overline{N}$. Observe that (6) is a steady state equation whose radiance function is defined for every z in the depth range $[-\infty,\infty]$ and that (6) has the gestalt of an equation of transfer applied to an infinite plane-parallel medium with continuous source $\hat{N}_\eta(z) = \overline{\delta}(z)$ $[\hat{N}(0,\xi) - \hat{N}_\zeta(\xi)]$, $-\infty < z < \infty$, and in which the volume attenuation function is $\overline{G}\alpha$ and the volume scattering function is $\overline{G}\sigma$. *It follows that the theory of Chapter 7, and in particular that of Sec. 7.13, is directly applicable to the present setting and can in principle determine $\overline{N}(z,\xi)$ for all z in $[-\infty,\infty]$ and all ξ in Ξ, given the "internal source" $\hat{N}_\eta(z)$ (with $\xi \in \Xi$) and the sky radiance $N^o(\xi)$ (with $\xi \in \Xi_-$) as a boundary condition.*

The cosurface time-averaged radiance field has an equation of transfer obtained by applying (2) directly to the equation of transfer for $N(\zeta'+\zeta(t),\xi,t)$:

$$\boxed{\frac{d\hat{N}(\zeta',\xi)}{dr} = -\alpha(\zeta')\hat{N}(\zeta',\xi) + \hat{N}_*(\zeta',\xi)} \tag{11}$$

where the connection:

$$\hat{N}_*(\zeta',\xi) = \int_\Xi \hat{N}(\zeta',\xi)\sigma(\zeta';\xi';\xi)d\Omega(\xi') \tag{12}$$

is readily established using the independence condition (10) of Sec. 12.11. Equation (11) is vastly simpler to work with than (6) and indeed the entire theory of radiative transfer on stratified source-free plane-parallel media is available for the solution of the problem of determining $\hat{N}(\zeta',\xi)$ for ζ' in $[0,\infty]$ and all ξ in Ξ. The boundary radiance of (11) is $\hat{N}(0,\xi)$ as governed by (44) of Sec. 12.11. *Therefore by virtue of (18) and (44) of Sec. 12.11, and (11) above, the problem of the time-averaged radiance field over a dynamic air-water surface has been reduced to the case of the static air-water surface considered in Sec. 12.2, and all results of that section may now be transferred,* mutatis mutandis, *to the present context for the cosurface radiance field $\hat{N}$ in a medium whose upper boundary is $\hat{S}$.*

Connection Between Fixed Depth and Cosurface Time-Averaged Radiances

By observing on the one hand the relative ease with which the mathematical problem of the time-average radiance field is solved by adopting the cosurface average $\hat{N}$ and its governing equation (11), and observing on the other hand the fact that the experimentally determined average light field $\bar{N}$ is more conveniently expressed in terms of fixed depth averages, we are led to seek a connection between the two averages $\hat{N}$ and $\bar{N}$.

Consider a vertical path as in (a) of Fig. 12.64 and the differential of path radiance $N^*(z,\xi,t)$ at fixed depth z at time t as generated by the path function value $N_*(\zeta' + \zeta(t),\xi,t)$ over the differential path length $d\zeta'$ with fixed cosurface depth ζ':

$$dN^*(z,\xi,t) = N_*(\zeta'+\zeta(t),\xi,t)G(y(t))T_{y(t)}(\zeta'+\zeta(t),\xi)\ d\zeta' \ . \quad (13)$$

The presence of $G(y(t))$, (cf. (3)) takes into account the possibility that depth z may lie above the water surface at time t. Applying the time averaging operator to each side of (13), and using (10) of Sec. 12.11, we have, by (1) and (2):

$$d\bar{N}^*(z,\xi) = \hat{N}_*(\zeta',\xi)\Phi(z-\zeta')d\zeta' \quad (14)$$

where we have written:

$$\text{"}\Phi(z-\zeta')\text{"} \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T G(y(t))T_{y(t)}(\zeta'+\zeta(t))dt \quad (15)$$

The difference $z-\zeta'$ arises because of the defined form of $y(t)$ and the fact that under random conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T y(t)dt = z-\zeta'$$

so that the time average (15), under random conditions on S, would depend only on the difference $z-\zeta'$. From (14) we have on integrating over all cosurface depths ζ':

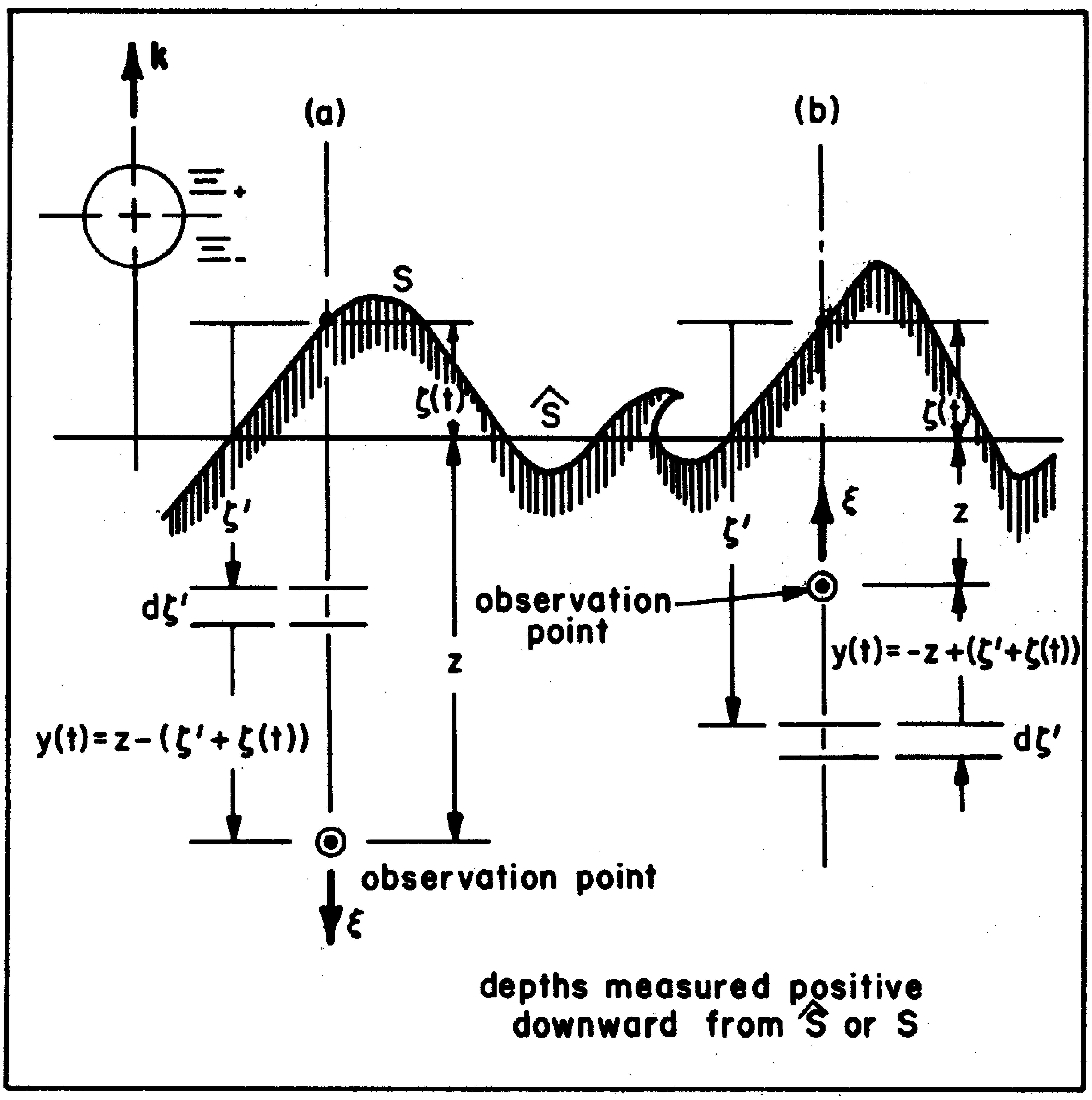

FIG. 12.64 Deriving the connection between fixed-depth and cosurface time averages.

$$\overline{N}^*(z,\xi) = \int_0^\infty \hat{N}_*(\zeta',\xi)\Phi(z-\zeta')d\zeta' \qquad (16)$$

which is the desired connection between the two time-‑ ›raged radiances $\overline{N}^*$ and $\hat{N}_*$.

The connection between the time-averaged residuε radiances is obtained by reconsidering the general form oi the argument leading to (4), but now within the setting (ε of Fig. 12.64: The instantaneous connection for downward radiε ce is:

$$N^o(z,\xi,t) = N^o(\zeta(t),\xi,t)G(z-\zeta(t))T_{z-\zeta(t)}(\zeta(t),\xi)$$

$$+ \underline{N}^o_\zeta(\xi)[1 - G(z-\zeta(t))] \qquad . \qquad (17)$$

Averaging this, results in:

$$\overline{N}^{o}(z,\xi) = \hat{N}^{o}(0,\xi)\Phi(z) + \hat{\underline{N}}^{o}_{\zeta}(\xi)[1 - \overline{G}(z)] \quad . \qquad (18)$$

In this way we come to the requisite connection (for $\xi = -\mathbf{k}$):

$$\begin{aligned}\overline{N}(z,\xi) &= \overline{N}^{o}(z,\xi) + \overline{N}^{*}(z,\xi) \\ &= \hat{N}^{o}(0,\xi)\Phi(z) + \int_{0}^{\infty} \hat{N}_{*}(\zeta',\xi)\Phi(z-\zeta')d\zeta' + \hat{\underline{N}}^{o}_{\zeta}(\xi)[1-\overline{G}(z)] \quad .\end{aligned}$$

(19)

In a similar manner the upward radiance connections (i.e., $\xi = \mathbf{k}$) can be made (see (b) of Fig. 12.64). The requisite form is:

$$\overline{N}(z,\xi) = \hat{N}^{o}(0,\xi)\overline{G}(-z) + \int_{0}^{\infty} \hat{N}_{*}(\zeta',\xi)\Phi(\zeta'-z)d\zeta' \quad . \qquad (20)$$

The asymmetry between (19) and (20) arises from two sources: first there is the usual asymmetry indigenous to deep plane-parallel media (such as those now being considered) where there is incident external downward radiance, but no such upward radiance. This accounts for the term $\hat{\underline{N}}^{o}(\xi)[1-\overline{G}(z)]$ in (19) and its absence in (20). Secondly, we agreed at the outset of the present study (in Sec. 12.10) to white convexify the upper surface of S. This is tantamount to assuming that the atmosphere is relatively transparent compared to the hydrosol, so that $T_r = 1$ in air. This assumption is quite realistic for distances r on the order of ten moderate gravity wave lengths. Under such conditions $\Phi(z) = \overline{G}(z)$, as may be verified from (15). This accounts for the $\overline{G}(-z)$ rather than $\Phi(-z)$ in the residual radiance term in (20). The minus sign and the use of $\zeta'-z$ rather than $z-\zeta'$ in the integral term of (20) reflects the change of direction in going from (19) to (20).

We note in passing that a completely symmetric formulation of the present problem is possible if both the upper and lower surfaces of S are black convexified at the outset and the aerosol and the hydrosol are both treated exactly alike--namely as two contiguous general scattering-absorbing media with an interface. (See Example 7 of Sec. 3.9.) We shall leave such a formulation to interested students of the subject. The derivations above are readily extended to cover such a symmetric formulation. Such a formulation is useful when a dense layer of fog rests on a dynamic air-water surface and the radiative transfer interaction is required in an instantaneous or averaged form. We have chosen the present route to (19) and (20) because of expository reasons and because they represent the more frequently occurring situation.

In the event that the air-water surface is to be viewed from a great height (aircraft and satellite heights) then one may use $\hat{N}(0,\xi)$, with ξ in Ξ_+, as determined in the manner shown in Sec. 12.13, as the time-averaged inherent radiance of the surface and then go on to compute its apparent radiance in the usual manner for a given path of sight.

Equations (19) and (20) are particularly designed to give the exact time-averaged radiance field in the immediate vicinity of the air-water surface, in particular over the depth region with $2m_{oo}$ above and below $\hat{S}$, and generally at depths in the hydrosol where bright moving beams of refracted sunlight are still observable.

One may generalize (19) and (20) from the vertical up-down case to the case where ξ is arbitrary nonhorizontal paths by merely replacing $y(t)$ in $T_{y(t)}$, and $G(y(t))$ by $-(\sec\theta)\cdot y(t)$ in (19) and (20) to account for the slant path attenuation (see Fig. 12.63). All other terms are unaffected by virtue of the assumed statistically stationary character of the motions of the hydrosol X and its boundary S. Hence (19) applies to all directions ξ in Ξ_- and (20) to ξ in Ξ_+. Furthermore, by invoking the ergodic hypothesis we may represent $\Phi(z-\zeta')$ for a path directed along ξ, by:

$$\Phi(\mathrm{sgn}(r)(z-\zeta')) = \int_0^\infty \overline{G}(r)T_{|r|}(\zeta',\xi)d\zeta' \qquad (21)$$

where we have written:

"r" for $-\sec\theta(z-\zeta')$

and where $\xi\cdot\mathbf{k} = \cos\theta$. "sgn (r)" means the algebraic sign of r, namely + or -. The beam transmittance $T_{|r|}(\zeta',\xi)$ is readily evaluated in homogeneous media. In fact:

$$T_{|r|}(\zeta',\xi) = e^{-\alpha|r|} \qquad (22)$$

and the time-average $\overline{G}(r)$ may be evaluated by means of (14) for seas with gaussian elevation distributions. Hence in such practical settings $\Phi(z-\zeta')$ is known. This completes the establishment of the connections between fixed-depth and cosurface averages of the radiance distributions beneath a dynamic air-water surface.

As a check on the connections (19) and (20) we can let the wind speed U_a go to zero, so that m_{oo}, σ_c, and σ_u go to zero. Consequently $\Phi(z-\zeta)$ goes to $T_{z-\zeta'}$, $\overline{G}(z)$ for $z > 0$ goes to 1, and $\overline{G}(-z)$ goes to zero for every positive z; and so (19) and (20) reduce to the integral forms of the equation of transfer for the static case (re: Sec. 12.2).

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12.13 Synthesis of Time-Averaged Radiance Fields

The complete description of the time-averaged light field in a natural hydrosol with a wind-blown air-water surface will now be attained by gathering together the various pieces of the description fashioned in the preceding three sections.