$$\Delta = \frac{N}{2}\begin{vmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \frac{N}{2}\begin{vmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

$$= \frac{N}{2}\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix} = N\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

$$= N\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 2N(-2+1) = -2N \neq 0 \qquad (25)$$

Hence the four given vectors above are linearly independent.

13.7 Direct Measurement of the Volume Total Scattering Function

Our purpose in this section is to outline a method of direct measurement of the volume total scattering function s. We will be concerned principally with the derivation of the formula behind the method and a few words about the kinds of equipment that may be needed for the realization of the method. The general method outlined below will be applicable to any scattering medium, but it appears to be most useful in the experimental study of natural hydrosols. A few additional preliminary observations will help orient the reader and place the present discussion in the appropriate perspective.

The two fundamental inherent optical properties of scattering-absorbing media are the volume scattering function σ, and the volume attenuation function α. The pair (α,σ) is *fundamental* in the sense of Def. 3 of Sec. 9.1, i.e., that from these two, all other attenuation functions, either inherent or apparent, may effectively be found in accordance with certain well-defined rules of computation. However, the pair (α,σ), while being fundamental in the sense just explained, is by no means unique. Thus the pair (a,σ) is also fundamental, where a is the volume absorption function. The connecting link between these two fundamental pairs of optical functions is provided by the notion of the volume total scattering function s, defined by writing:

$$\text{"s"} \quad \text{for} \quad \int_{\Xi} \sigma \, d\Omega$$

(cf., (3) of Sec. 4.2). For then, by definition (i.e., (4) of Sec. 4.2) we have $\alpha = a + s$, so that the pair (α,σ) is known once (a,σ) is, and conversely.

By means of the work in Chapter 3, and also that of Secs. 13.4-13.6, the theory and practice of the direct measurement of α and σ, is now well established. As was shown, the measurement of α is accomplished by various beam transmittance procedures all of which, at their core, spring from a single analytical relation, the interaction principle of Chapter 3. The direct measurement of σ is accomplished by specially designed instruments known as σ-meters which mechanically mimic the general definition of σ. Even the volume absorption function a, once an elusive quantity which could be determined only indirectly, now has a direct, simply realizable method of determination either in the field or in the laboratory as we shall see in Sec. 13.8. We now round out this list by discussing a direct method of determining s, and thereby supplementing the usual indirect methods which obtain s from an integration of σ over Ξ, or from a and α by means of the formula: $s = \alpha - a$.

The General Method

The general direct method of determining s makes use of the close connection between the volume scattering function σ and the path function N_* at some point x in an arbitrary optical medium X (cf., (8) of Sec. 3.14):

$$N_*(x,\xi) = \int_\Xi \sigma(x;\xi';\xi)N(x,\xi')d\Omega(\xi') \quad . \tag{1}$$

Recall that $N_*(x,\xi)$ is a *radiance per unit length* at x in the direction ξ generated by scattered radiances $N(x,\xi')$ at x arriving along the directions ξ' (see (7) of Sec. 13.6). Here Ξ is as usual the collection of all units vectors (the unit sphere) in E_3. The practice of measuring $N_*(x,\xi)$ is relatively highly developed both in the atmosphere and the hydrosphere, and we shall make use of this fact in developing the general method of measuring s.

Recall that the value s(x) of the volume total scattering function at x is given by:

$$s(x) = \int_\Xi \sigma(x;\xi';\xi)d\Omega(\xi') \quad , \tag{2}$$

in any isotropic medium. Then returning to (1) and integrating each side over Ξ we have:

$$\int_\Xi N_*(x,\xi)d\Omega(\xi) = \int_\Xi\left[\int_\Xi \sigma(x;\xi';\xi)N(x,\xi')d\Omega(\xi')\right]d\Omega(\xi) \quad . \tag{3}$$

The order of integration may generally be reversed on the right hand side of (3), the result being:

$$\int_\Xi N(x,\xi')\left[\int_\Xi \sigma(x;\xi';\xi)d\Omega(\xi)\right]d\Omega(\xi') \quad . \tag{4}$$

The inner integral is the value s(x) of s at x. Since s(x) is independent of ξ', it may be placed before the outer integral sign of (4), thus:

$$s(x)\cdot\int_{\Xi} N(x,\xi')d\Omega(\xi') \quad . \tag{5}$$

The integral term in (5) is the value h(x) of the scalar irradiance function h at x. As a result, the right side of (3) becomes:

$$s(x)h(x) \quad .$$

In analogy to h(x), we define (as in (23) of Sec. 9.3) the left side of (3) by writing

$$\text{"}h_*(x)\text{"} \quad \text{for} \quad \int_{\Xi} N_*(x,\xi)d\Omega(\xi) \quad . \tag{6}$$

Combining these results, we have the desired basic formula for the general direct method of determining s(x):

$$\boxed{s(x) = \frac{h_*(x)}{h(x)}} \quad . \tag{7}$$

Observations

Observe first of all that equation (7) is quite general. No assumption has been made about the angular distribution of the radiance about point x. Furthermore, σ is quite arbitrary in the angular structure at x. The only assumption made, and a quite reasonable one at that, concerns the isotropy of X at point x, i.e., that $\sigma(x;\cdot;\cdot)$ is invariant under a *rotation* of the local coordinate frame about x; in other words, that $\sigma(x;\xi';\xi) = \sigma(x;\xi_1';\xi_1)$ whenever $\xi'\cdot\xi = \xi_1'\cdot\xi_1$ (re: (8) of Sec. 7.12).

Observe next the structural similarity between (7) and the classical formula:

$$\boxed{\alpha = \frac{N_*}{N}} \tag{8}$$

for the determination of the volume attenuation function α (see (9) of Sec. 13.4). In order to obtain the simple form (8), quite a few assumptions about the medium and light field must be made. This is not, however, the case for s.

Further, it is interesting to observe that (8) is associated with summations of N over all *points along a line of fixed direction*, while (7) is associated with summations of N over all *directions of lines through a fixed point*. In this sense, the concepts α and s may be classed as *dual concepts* with respect to the phase space $X \times \Xi$ (cf., Fig. 3.34).

Finally, we note that the measurement of $s(x)$ can be accomplished by rotating a short-path radiance meter about x, which thereby determines $N_*(x,\xi)$ for each ξ. The result is then integrated either automatically or manually over Ξ to obtain $h_*(x)$. The theory and practice of measurement procedures for $h(x)$ are well known (cf., (17) of Sec. 13.1).

Two Special Methods

If one has control over the lighting conditions and the homogeneity of the medium, as may be possible in the laboratory, the basic relation (1) and the general formula (7) yield a number of particularly simple methods for the determination of x. We now briefly consider two such methods.

Cylindrical Medium

Suppose a narrow circular cylindrical tube of length r is filled uniformly with the scattering material under study. The inner walls of the tube are lighted so that at all points along the axis of the tube, the radiance distribution is angularly uniform of magnitude N. Thus, under the assumptions of an angularly uniform N and a homogeneous medium, (1) reduces to:

$$N_* = N\,s \quad . \tag{9}$$

If an observation point is at one end of the tube so that a line of sight may be directed along the axis of symmetry to the other end, which has zero inherent radiance, one would expect to observe an apparent radiance N_r of magnitude:

$$N_r = \frac{N_*}{\alpha}\left[1 - e^{-\alpha r}\right] = \frac{Ns}{\alpha}\left[1 - e^{-\alpha r}\right] \quad . \tag{10}$$

Suppose the tube is constructed so that r may be varied. Then if αr is small, (10) yields:

$$\boxed{N_r = N\,s\,r} \quad . \tag{11}$$

Hence by plotting N_r vs. r, one may look for the region of linearity of N_r. The slope of the plot in this region is simply Ns. If N is known, then s is determinable.

By lengthening r, (10) indicates that eventually the readings N_r must level out, the plateau being of magnitude:

$$\boxed{N_r = \frac{N_*}{\alpha} = \frac{Ns}{\alpha}} \quad . \tag{12}$$

Knowing N_r and N, we may then estimate the ratio $\rho = s/\alpha$, which is the well-known *scattering-attenuation ratio* or *albedo for single scattering*, a quantity which plays an important role in both the theory and application of radiative transfer.

It is clear that if experiments leading to both (11) and (12) have been made, then α and s are both determinable by this simple scheme, whence follows the volume absorption coefficient also: $a = \alpha - s$.

Spherical Medium

Suppose an integrating sphere of internal radius r is filled uniformly with the scattering material under study. Suppose further that the inherent radiance distribution of the inner surface is uniform and of equal magnitude at each point. The sphere is fitted with a small viewing port which allows a radiance tube an unrestricted view along a diametral line from the port, through the center, to a small circular region of inherent radiance zero on the far portion of the inner surface. Let the observed radiance of the circular region be N, and suppose a scalar irradiance probe records the amount h at the center of the spherical cavity. Then, because of the symmetry of the light field at the center of the cavity, except perhaps for the tiny dark patch and observation port, we would expect N_* to be essentially independent of direction and (for not too dense a medium so that the black patch is clearly visible) very nearly of magnitude $N/2r$ (cf. (3) of Sec. 13.3). It follows that $h_* = 2\pi N/r$ and that, by means of (7):

$$\boxed{s = \frac{2\pi N}{hr}} \quad . \tag{13}$$

If, on the other hand, the medium is made optically dense, so that the black patch is not visible, then the observed N would be very nearly N_*/α, where N_* is the value of the path function at the center of the sphere. Therefore in this case:

$$\boxed{s = \frac{4\pi N\alpha}{h}} \quad , \tag{14}$$

so that under the present circumstances, unless α is already known, one may determine only:

$$\boxed{s/\alpha = \frac{4\pi N}{h}} \quad . \tag{15}$$

Several variants of the above methods are immediately realizable. For example, a set of three spheres of diameters one, two, and three units, say, are constructed and, using optically rare media of *equal* density, plot the quantity $2\pi N/h$.

Then, according to (13), if this varies linearly with r, the conditions of (13) are satisfied and the slope of the line is none other than the required value of s.

<< Table of Contents << Title Page >> Next Page >>

13.8 Operational Definition of Volume Absorption Function

The volume absorption function a(x), by its very nature, must initially enter the domain of radiative transfer theory in an indirect manner, that is, by means of the defintion which characterizes a(x) as the difference between $\alpha(x)$ and s(x), as was done in (4) of Sec. 4.2. In this section we shall deduce from this definition alternate operational procedures which lead to determinations of a(x) by means of direct measurements of radiometric quantities.

The formula which serves as the basis for the present operational definitions is (15) of Sec. 8.8. Thus, solving for a(x), we have:

$$\boxed{a(x) = -\frac{1}{h(x)} \nabla \cdot \mathbf{H}(x)} \quad . \qquad (1)$$

This operational definition of a(x) inverts the usual way of looking at the divergence relation (15) of Sec. 8.8. In (1) we imagine the relation to have its roots in the operations of the real radiometric world. By performing the indicated operations on the right side in (1), a(x) is thereby determined virtually independently of $\alpha(x)$ and s(x).

Fig. 13.13 depicts an operational realization of equation (1). Suppose a janus plate (i.e., an instrument which measures net irradiance $\overline{H}(x,\xi)$, as depicted in Fig. 2.21) is oriented so that it measures $\overline{H}(x,\mathbf{i})$, $\overline{H}(x,\mathbf{j})$, $\overline{H}(x,\mathbf{k})$ at a point x in an optical medium. Suppose further that the janus plate is moved back and forth along the i-axis a distance r about each side of x, as shown in Fig. 13.13. Then:

$$\frac{\overline{H}(x + r\mathbf{i}) - \overline{H}(x - r\mathbf{i})}{2r} ,$$

approximates to:

$$\frac{\partial \overline{H}(x,\mathbf{i})}{\partial r} ,$$

for suitably chosen r. The remaining two derivatives occurring in the representation of the divergence $\nabla \cdot \mathbf{H}$ can be approximated similarly. If the scalar irradiance h(x) at x is determined, then (1) leads to a(x). Equation (1) also suggests several alternate modes of measurement of a(x) in which spherically or cylindrically symmetric light fields, artificially induced, can lead to appropriate forms of $\nabla \cdot \mathbf{H}$ using spherical and cylindrical coordinate systems. These special instances of (1) are best left for adaptation, by interested researchers, to individual cases (cf., e.g., (43) of Sec. 9.2).