problem, for with each change in the lighting conditions four more new operators appear and so there will always be two more operators than radiometric equations available for solution.

General Observations on Inverse Problems in Hydrologic Optics

Observe that the inverse problem of determining the local optical properties $\rho(y)$ and $\tau(y)$ is readily solved in hydrologic optics by using the results of Examples 2 or 3 and the theory of Sec. 7.3. Hence the inverse problem in hydrologic optics is completely solvable by appropriately using the general concepts assembled in Chapter 7.

One important reason why the inverse problem presents no novel difficulties in hydrodologic optics (in principle) rests on the fact that the optical medium whose optical properties are sought is directly accessible to experimental probing. This fact was used throughout the preceding examples.

In branches of radiative transfer other than geophysical optics, such as the current fields of astrophysical optics or planetary optics, the problem of determining, say, α and σ throughout a stellar or planetary atmosphere is much more difficult when the atmosphere cannot be directly probed internally. Indeed, under such a condition, unless some specific laws governing at least the internal depth behavior of σ and α are available, or some equivalent information is available or even good guesses possible, then the general (inverse) problems of the second class, are insoluble.

13.11 On the Consistency of the Operational Formulations

We conclude this chapter with a check on the consistency of the operational definitions of the main optical properties introduced throughout the chapter. The method we shall employ is that which attempts to assemble all the various operationally defined pieces into a structure which, hopefully, will be recognizable as one of the forms of transport equations--either for radiance, irradiance, or some other appropriate radiometric quantity.

To see how the method proceeds, we select for illustration those concepts which should fall together into the form of the equation of transfer for unpolarized radiance fields. Thus consider a regular neighborhood C(A,B) of paths, as shown in Fig. 13.16. The common beam transmittance $T_r(x,\xi)$ for the members of C(A,B) is given by (7) of Sec. 13.2 as:

$$T_r(x,\xi) = \frac{N(y_1,\xi) - N(y_2,\xi)}{N(x_1,\xi) - N(x_2,\xi)} \tag{1}$$

The common path radiance $N_r^*(y,\xi)$ for the members of C(A,B) is given in (2) of Sec. 13.3 in the form:

$$N_r^*(y,\xi) = N(y,\xi) - N(x,\xi)T_r(x,\xi) \tag{2}$$

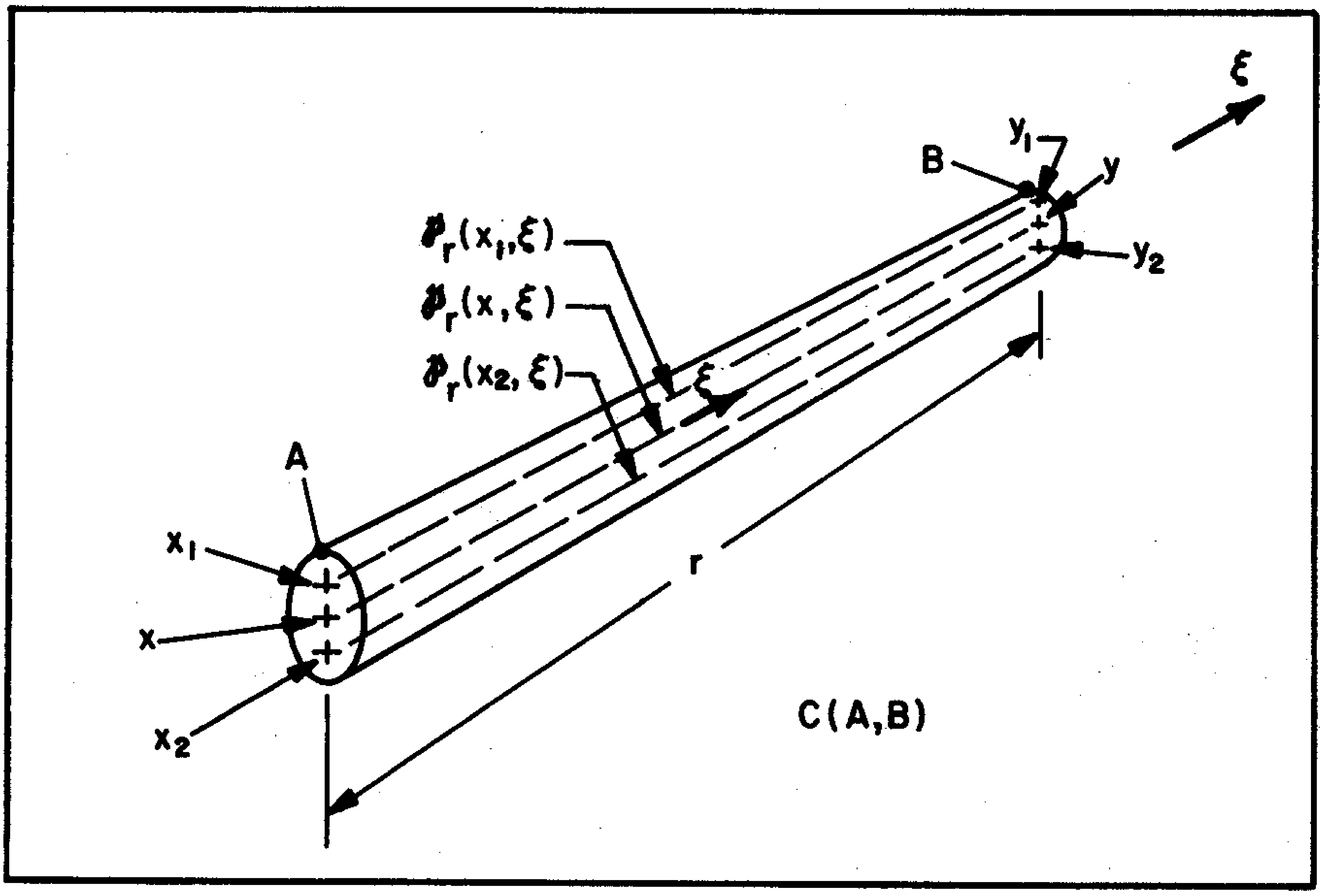

FIG. 13.16 A regular neighborhood C(A,B) of paths in a general optical medium used in the consistency check of the operational formulations of the concepts of unpolarized radiative transfer theory.

In particular, for paths $\mathcal{P}_r(x_1,\xi)$, $\mathcal{P}_r(x_2,\xi)$ in C(A,B), (2) implies:

$$N_r^*(y,\xi) = N(y_1,\xi) - N(x_1,\xi)T_r(x,\xi)$$

$$N_r^*(y,\xi) = N(y_2,\xi) - N(x_2,\xi)T_r(x,\xi)$$

Hence (1) and (2) are mutually consistent. Therefore, we can proceed with (2) in which $T_r(x,\xi)$ is defined as in (1), and write (2) as:

$$N(y,\xi) = N(x,\xi)T_r(x,\xi) + N_r^*(y,\xi) \quad . \tag{3}$$

We next introduce into (3) the operational definition of α by means of (1) of Sec. 13.4:

$$T_r(x,\xi) = 1 - r\alpha(x,\xi) + ro(r) \quad .$$

The result is:

$$N(y,\xi) = [1 - r\alpha(x,\xi)]N(x,\xi) + N_r^*(y,\xi) + o_1(r) \tag{4}$$

where $o_1(r)$ is a quantity such that $o_1(r)/r$ goes to zero with r. Equation (4) can be rearranged so as to read:

$$\frac{N(y,\xi) - N(x,\xi)}{r} = - \alpha(x,\xi)N(x,\xi) + \frac{N_r^*(y,\xi)}{r} + \frac{o_1(r)}{r} \quad . \quad (5)$$

Using (4) of Sec. 13.3, (5) can be written as:

$$\frac{N(y,\xi) - N(x,\xi)}{r} = - \alpha(x,\xi)N(x,\xi) + N_*(x,\xi) + \left(\frac{o_1(r)}{r} + \frac{o_2(r)}{r}\right) \quad (6)$$

where $o_2(r)/r$ goes to zero with r. Now letting $r \to 0$, (6) leads to

$$\frac{dN(x,\xi)}{dr} = - \alpha(x,\xi)N(x,\xi) + N_*(x,\xi) \quad (7)$$

or simply

$$\frac{dN}{dr} = - \alpha N + N_*$$

which is the desired equation of transfer for N with volume attenuation function α and path function N_*. By introducing σ into (7) via (1) of Sec. 13.6 or any of the operational formulations in Sec. 13.6 built up from that equation, we see that the operational definitions of $T_r(x,\xi)$, $N_r^*(x,\xi)$, $N_*(x,\xi)$, $\alpha(x,\xi)$ and $\sigma(x;\xi';\xi)$ are indeed mutually consistent.

On the Relative Consistency of the Unpolarized and Polarized Theories of Radiative Transfer

There remains the question of whether the equation of transfer (7) is consistent with respect to the finer-grained theory of the polarized light field, and whether it forms a faithful picture of the radiometric features of an optical medium. The problem, essentially, is this: Suppose one measures radiance distributions in the sea or air using a radiance meter without a polarizer attached (cf. Fig. 2.25 and Sec. 2.10). Suppose the measurements determine the α, and σ of the medium under study, as defined operationally in the present section. Suppose further that we then place this α and σ into (7), compute the radiance field throughout the medium and then compare the computed field with the measured field. The question now is: *Can these two radiance fields be equal, in principle?* Notice that we qualify the question as one of a matter of principle. Surely, measurement and calculation techniques, even in the relatively advanced technology of today, cannot culminate in an exact corroboration of the two radiance fields. Therefore, what we are primarily after here is the resolution of a subtler matter concerned with the physical foundation, and the internal mathematical consistency of the polarized theory. The question therefore splits into two parts. The first question is: *Does the polarized theory contain the unpolarized theory as a special case, and if so, does the special case agree with that which* *we have been using all along in this work?* (The fact that we dare ask this question at this late hour, and the fact that this work exists publicly, means that the answer to the preceding question is happily in the affirmative. However, there is a surprise ending to the story which, like the denouement of any interesting mystery, should not be glimpsed prematurely.)

To answer the first question, as stated above, we need only return to (7) of Sec. 3.15, the equation of transfer for polarized radiance, and write it out in component form, using the notation of Sec. 2.10:

$$\begin{cases} \dfrac{d({}_jN)}{dr} = -\alpha({}_jN) + \displaystyle\int_{\Xi} \left[\sum_{i=1}^{4} ({}_iN)p_{ij}\right] d\Omega \\ j = 1, 2, 3, 4 \end{cases} \tag{8}$$

where p_{ij} is the entry in the ith row and jth column of the standard observable volume scattering matrix **p**, and ${}_jN$ is the jth component of the radiance vector **N**, j = 1, 2, 3, 4. **p** is related to σ in (24) of Sec. 13.6, in the manner shown in Sec. 112 of [251]. According to (8) of Sec. 2.10, the observed radiance N without a polarizer attached to the radiance tube is given in terms of the polarized radiance vector's components as:

$$N = {}_1N + {}_2N \quad . \tag{9}$$

This suggests that we consider (8) for the cases j = 1 and j = 2, and add the associated equations together, thus:

$$\frac{d({}_1N + {}_2N)}{dr} = -\alpha({}_1N + {}_2N) + \int_{\Xi} \sum_{j=1}^{2} \left(\sum_{i=1}^{4} ({}_iN)p_{ij}\right) d\Omega$$

$$= -\alpha({}_1N + {}_2N) + \int_{\Xi} \sum_{i=1}^{4} ({}_iN)(p_{i1} + p_{i2})\, d\Omega \quad . \tag{10}$$

We now specifically adopt the assumption that: *the radiance field in a given optical medium is unpolarized.* It follows from the list of observable radiance vectors in Sec. 2.10 that the unpolarized radiance vector **N** has the form:

$$\mathbf{N} = \frac{1}{2}\,(N,N,N,N) \quad , \tag{11}$$

i.e., all ${}_iN$ are equal, to a common function (1/2)N, where N would be measured by the radiance tube without a polarizer. Using this form of **N** in (10) we deduce that

$$\frac{dN}{dr} = -\alpha N + \frac{1}{2}\int_{\Xi} N\left(\sum_{j=1}^{2}\sum_{i=1}^{4} p_{ij}\right) d\Omega \quad . \tag{12}$$

Writing

$$"\sigma" \quad \text{for} \quad \frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{4} p_{ij} \quad \left(= \frac{1}{2}\sum_{i=1}^{4}\left(p_{i1} + p_{i2}\right)\right), \tag{13}$$

we see that the *mathematical structure* of (12) is identical* to that of (7). *Hence the theory of the unpolarized light field is a logical consequence of the theory of the polarized light field, and so they are relatively consistent in a mathematical sense.* An added dividend to the present inquiry is the exact form of the volume scattering function σ in terms of the components p_{ij} of **p** under the present assumption. (See (24) of Sec. 13.6 and the remarks following (7) of Sec. 3.15.)

Note carefully that all we have shown is that the *form* of (7) is correct in unpolarized light fields. Its *content* (i.e., the magnitudes of α and σ as determined by the operations of this chapter), while mutually consistent, as shown above, need not agree, e.g., with the σ of (13). We shall look into this matter in a moment.

It is one thing to postulate that the world behaves in a certain way and another to experimentally determine whether or not it does indeed behave that way. It is an experimental fact (look up into the sunlit sky with polaroid sunglasses) that the process of scattering of light, on either the phenomenological level (such as that inhabited by a radiative transfer theory) or the microscopic electromagnetic level, alters the state of polarization of the light. Thus unpolarized light becomes polarized after being scattered (i.e., ingoing (1/2)(N,N,N,N) emerges as polarized flux after encountering a scattering volume) and polarized light subtly alters its state to another state after another scattering. The net effect of a multitude of such scatterings in an extensive optical medium on the radiance distribution within it is to reach an asymptotic state indigenous to the phase function of the medium (see the discussion of (19) of Sec. 4.6, and the consequences of the asymptotic radiance theorem in Sec. 10.7). The second of the two main questions in the present discussion hinges on these physical facts, and may now be phrased as follows: *Is the basic equation of transfer for the radiance* $N = {}_1N + {}_2N$, *strictly of the form (7) in a polarized light field?*

To answer this second question, we turn again to (10) and observe immediately that it cannot generally be placed into the form (7). Hence the answer to the second question is negative. Just where and by how much do the two equations differ? They differ in the path function term. Thus by definition (9) we can write (7) as:

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*What could have gone wrong here, for example, would have been the appearance of an unexpected source term on the right side of (12) arising from contributions to N by ${}_3N$ and ${}_4N$; or the inability to factor the integrand into an N-type and a σ-type factor.

$$\frac{d({}_1N + {}_2N)}{dr} = -\alpha({}_1N + {}_2N) + \int_{\Xi} ({}_1N + {}_2N)\sigma d\Omega \qquad (14)$$

where σ is determined as in (4) of Sec. 13.6. Subtracting the right-hand side of (14) from that of (10) term by term, we are led to write, *ad hoc:*

$$"\overline{N}_*" \quad \text{for} \quad \int_{\Xi}\Big[{}_1N(p_{11} + p_{12} - \sigma) + {}_2N(p_{22} + p_{21} - \sigma)$$

$$+ {}_3N(p_{31} + p_{32}) + {}_4N(p_{41} + p_{42})\Big]d\Omega \quad . \qquad (15)$$

The function $\overline{N}_*$ has the dimensions of the path function, namely radiance per unit length. Clearly the function values $\overline{N}_*(x,\xi)$ are zero for each x and ξ over a given optical medium if and only if (7) and (14) are equivalent equations of transfer over that medium under all lighting conditions. From (15) we see at once that such an equivalence holds if and only if

$$\left.\begin{aligned} p_{11} + p_{12} &= p_{22} + p_{21} = \sigma \\ p_{31} + p_{32} &= 0 \\ p_{41} + p_{42} &= 0 \end{aligned}\right\} . \qquad (16)$$

In order words, if (16) holds in a certain optical medium, then the equation of transfer (7) is an exact model of the light field (i.e., $_1N + {}_2N$*) in that medium even though the light field in that medium is polarized. On the other hand, if (16) does not hold, then (7) cannot in principle describe the polarized radiance field in such a medium under all lighting conditions.* Equation (16), therefore, provides a practical test for the theoretical applicability of (7) to natural optical media with polarized light fields. The set (16) can also be written in terms of the components σ_{ij} of σ, if desired. A convenient measure of the values $\overline{N}_*, N_*$ would be via the corresponding equilibrium radiances $\overline{N}_q = \overline{N}_*/\alpha$, $N_q = N_*/\alpha$.

We can write the exact equation of transfer (10) for $N(= {}_1N + {}_2N)$ in source-free steady media, in a form which is directly tied to the preceding equivalence criterion:

$$\boxed{\begin{aligned} \frac{dN}{dr} &= -\alpha N + N_* + \overline{N}_* \\ N &= {}_1N + {}_2N \end{aligned}} \quad . \qquad (17)$$

Hence if the conditions (16) do not hold, then this fact is manifested in (17) by the appearance of a nonzero increment $\overline{N}_*$ of the path function N_* (a spurious source term) which arises from the scattering contribution of the four components of the polarized light field. We have observed above that this in reality will always be the case. In such a case, then, it is the *magnitude* of $\overline{N}_*$ which critically gauges the departure of the classical equation of transfer (7) from its exact counterpart (17) within polarized light fields. It follows that the classical equation of transfer in real light fields exhibiting polarization, is only an approximate equation. To the author's knowledge, no systematic test of the conditions (16) nor any estimate of the magnitude of $\bar{N}_*$ in (17) has been made at the time of publication [1976]. An important question for the classical unpolarized theory of radiative transfer now rests in the practical applicability of (7) to the study of light fields in natural hydrosols: *while (7) generates a mathematically self-consistent theory of unpolarized light fields, how well (in a quantitative sense) does it describe $N(= {}_1N + {}_2N)$ within actual (polarized) light fields found in nature?*

As anticipated above, the denouement of this problem still stands at this late date in the history of the theory, and awaits a definitive answer from those who are the only ones who can definitively answer it: the experimenters. Theoretical reasoning, such as that above, can be carried only so far. There eventually comes a time in the construction of any physical theory when all the theorizing must momentarily stop, and the court of last appeal be faced: Nature herself.

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13.12 Bibliographic Notes for Chapter 13

The development of the operational formulation for beam transmittance in Sec. 13.2 is based on the work in Ref. [238]. The operational formulations of path function and path radiance, as given in Sec. 13.3, are patterned in part on the theory of Sec. 3.12. The K-method of determining the path function was developed in Ref. [219]. Path function calculations using the integral method (6) of Sec. 13.3 were explored in Ref. [214]. The operational definition of volume attenuation function in Sec. 13.4 is based in part on Sec. 3.11 and the work of Ref. [238]. Interesting and useful parallels of the methods of Sec. 13.4 as developed in meteorologic optics may be found in Ref. [231]. Further parallels with α measurements in meteorologic optics may be found in Ref. [177] and Ref. [80]. The theory of Sec. 13.5 is based directly on Ref. [217]. The developments of the volume scattering function in (11)-(25) of Sec. 13.6 are for the most part new, with basic points of view developed in Sec. 18 of Ref. [251]. Some rudimentary forms of σ-definitions may be found in Ref. [177] and which are recognizable as early forms of (4) of Sec. 13.6. The discussion of the general empirical properties of σ, as given from (5) to (10) of Sec. 13.6 is based in part on observations in Ref. [229]. The theory of the volume total scattering function in Sec. 13.7 is drawn from Ref. [241], while that of Sec. 13.8 is based on Ref. [241] and Ref. [220]. An experimental device based on this theory is described in Ref. [299]. Alternate means of measuring s are discussed in Sec. 9.4.4 of Ref. [177]. The remarks in Sec. 13.9 are a consolidation of those on apparent optical properties found in Ref. [305].

The discussions of Sec. 13.10 are based in part on Ref. [243] and only begin to explore the inverse problem in radiative transfer theory. The inverse problems (problems of the second class) of radiative transfer are classified in Ref.