1.3 Three Simple Models for Light Fields

How do we seek order in all that we have encountered above? How do we incorporate those few evidences of order, already glimpsed, into some greater scheme, satisfying for its accuracy, comprehensiveness, and relevance to the main stream of modern physical theory? The number of effects to be described is great, and their intricacy has a tendency to initially intimidate those who attempt a precise description: nature's ways are orderly but infinitely complex, the theorists are few and finite; therefore, each stage of theoretical knowledge inevitably rests on chosen compromises. Three such theoretical compromises are selected for study here; each is designed to describe one facet of the radiometric complex encountered in the seas and lakes of the earth: the first two describe the light fields generated by sunlight and skylight and give simple models for the radiance distributions and two-flow irradiance fields; the third describes artificial light fields set off in the water by man-made point sources and extended artificial sources of radiant flux,

The Two-Flow Model

The two -flow model of the light field pictures the radiant flux in a natural hydrosol X, free of internal sources, as divided into two streams at each depth z below the boundary: a downward stream of radiance H_- and an upward stream of irradiance H_+ (see Fig. 1.38). The primary purpose of the model is to predict H_+ and H_- at each depth z, given H_+ and H_- at the upper boundary, or more generally, given H_+ at some depth and H_- at another (possibly the same) depth. The hydrosol, therefore, is viewed by this model as a *plane-parallel* medium, i.e., an infinite region of space caught between two horizontal parallel planes, which are the *boundaries* of the medium. The physical properties of the hydrosol are described in the present model by means of two optical properties a b; and the geometrical flow of the radiant energy is described by means of a *distribution factor D.* These three concepts are defined in detail as follows. We write:

for the amount of irradiance *absorbed* from a narrow vertical beam of radiant flux of unit irradiance as it crosses a horizontal layer of unit thickness in X.

"$_b$"

for the amount of irradiance *back scattered* without change in wavelength from a given arbitrary stream of radiant flux of unit irradiance as it crosses a horizontal layer of unit thickness in X.

Finally, if h_+, h_- are the scalar irradiances associated with

water
1 (measured positive
downward)

H_+ ~--thin layer
AZ

FIG. 1.38 Setting for the two-flow model for irradiance.

the two given streams of radiant flux in X, we write:
"D t #1
for h_t/H_+
D_+ give the mean distances traversed by each stream through a horizontal layer of unit thickness. They are also convenient measures of the diffuseness or collimatedness of the flows. This latter interpretation can be made plausible by a few examples. If the downward stream, say, is collimated, i.e., in the form of a narrow beam which makes an angle 9 with the vertical, then from (9), (10) of 1.1 it is easy to see that
O_W = sec e. Further, if the downward radiance distribution is uniform, then by (11) (15) of Sec. 1.1, we have D_ = 2. In the model currently under study, it is assumed that:

and we shall write "D" for this common value. (On the basis of this assumption, we occasionally call the resultant two-flow model the *one-D (two-flow irradiance)* model.) It is easy to see that the amount of irradiance lost by absorption from a flow of unit irradiance and of distribution factor D, as it traverses a unit thickness layer in X, is other hand the amount of loss by backscattering with the quantity D not appearing explicitly.

absorption is treated differently than scattering in the above sense, rests in the fact that these processes manifest themselves differently geometrically: when flux is absorbed it disappears from the scene; when it is scattered, it must

still be contended with in the radiometric scene. This is discussed further throughout Chapter 8, along with precise definitions of D and b.
We are now ready to derive the basic differential equations of the two-flow model. Consider the downward stream of radiant flux as it passes through a horizontal layer of thickness Az, where z is measured positive in the downward direction. (Fig. 1.38) As the stream progresses through the layer, it is partially absorbed and partially scattered backwards to join the upward stream of flux: The total amount of irradiance lost from H_ by these two processes is, according to the definitions of a and b
aDH A z + bH A z
4n the other hand, H_ will be increased by that amount of flux, namely bH+az, scattered backwards from the upward stream. The net change OH. of the downward irradiance, after traversing the layer of thickness Az, is therefore:
AH_w _ - (aD + b(H,,ez + bH_+Az (2) In the same way we find that for the upward stream of radiant flux, which moves through the same layer (so that its asso ciated Az is negative) the net change All_+ of H_+ is:
AH_+ _ - (aD + b) H_t(·ez) + bH. (-,&z) . (3) Dividing each side of (2) by dz, and each side of (3) by -Az, and letting Azi0, we have:
These equations constitute the two-flow model for light fields in homogeneous stratified natural hydrosols. This model (the one-D model), in undecomposed form, in essence goes back to Schuster in 1905 who first formulated similar equations in the astrophysical context. In Chapter .8 we review the high points of the model's

history and place it on a sound physical and mathematical basis. For the present, however, we indulge in a relatively uncluttered derivation and solution of the model, in order to point up its central ideas and its simple beauty.

The solution of the system (4), (5) is*
where m+, m.. are arbitrary constants to be fixed by specifying either one of H_+ and H_ at each of two chosen depths (distinct or not), and where we have written:
This completes the construction of the two-flow model. We shall put it to work in Sec. 1.4.

The Radiance Model

The radiance model connects the radiances at the beginning and end of an arbitrary path, such as AB, in a natural hydrosol X (Fig. 1.39). Thus, given the radiance at A in the direction , the model yields the radiance at B in the same direction , This model is quite general, for we can choose point A to be on the upper or lower boundary of X and so the radiance at the end B will give the apparent radiance of the boundary; and this is just the radiance one sees or measures at B with a radiance meter.

In order to construct such a model we need to know what happens to the radiance as it travels along a straight path in the water. If we imagine the radiance to be generated by
a swarm of photons traveling along the path, then on the one hand we would expect this swarm to lose some members via scattering and absorption at each point along the path. Accordingly, let us write:

"a" for the amount of radiance *absorbed from a* narrow beam of radiant flux of unit ra*diance traveling* a unit distance along
and
a path.

"s" for the amount of radiance *scattered without* change in wavelength from a narrow beam of radiant flux of unit radiance travelling a unit distance along a path.

*H(z,+) is the value of the function H_+ at depth z. Similarly, H(z,-) is the value of H_ at z. The functional notations "H±" and "H(•,±)" are to be considered synonymous and may be used interchangeably.

FIG, 1.39 Setting for the radiance model.

We note in passing that the *volume absorption function a for* X just defined is identical with that defined for the two flow model. The function s is the volume total scattering *function for X.*

Now, on the other hand, we would expect the swarm of photons to gain new members from the surrounding environment simply as a result of some of the nearby photons being scattered into the swarm as it passes along a small segment of its path, Thus, let us write:

"N_*" for the amount of radiance scattered without change in wavelength into a narrow beam of radiant flux travelling a unit distance along a given path past a given point: 1

„ *If N_o is the inherent radiance,* of the path, i.e., the beginning radiance at point A in Fig. 1,39, and N_r is the apparent *radiance of point* A as seen at point B a distance r along the path, then according to the above remarks the change AN_r of
N_r in the next increment of distance Dr along the path is expected to be:
where we have *written*
«a« for a + s (II) Equation (10) is the *equation of transfer* for radiance. It is the central equation of radiative transfer theory. We
call a the volume *attenuation function and N*-the path function. The* equation is used to connect the value Nr(z,8) of
N_r at depth z, in the direction e with the value $N_o(z_©,e)$ *of* N_o at depth z_o in the direction e. (See Fig. 1.39*)
As it stands, (10) looks like a simple differential equation, and, indeed, it is readily integrated if we know a and N_* along the path. We shall assume a to be constant along the path, and N_* to be given along the path, and that N_* varies only with depth. Then it is easily verified that the general solution of (14) is (see, e.g., (l)-(3) of Sec. 3.15):

The simple model we are interested in at present rests on the assumption that $N_*(z,e)$ in optically very deep media depends *only on depth z in X,* in the manner:
N (z 8)= N*(0,9)e (
-Kz 13
*~) where K is the empirical depth rate of decay of the general light field in X. For example it may be taken as the empirical K in (7) of 1.2, or the theoretical k in (9) above encountered in the two-flow model (cf. .(6 I) of Sec. 1.4) . At any rate, using (13) in .(12). performing the integration and simplifying, we have:
This is the requisite simple model for radiance. We shall study it later to see if it helps us understand some of the observed properties of the underwater light field surveyed in Sec. 1.2. It is a simple matter to generalize (14) to the case where $N_*(z,8)$ depends also on the azimuth angle #. (See Chapter 4:) For the present we can think of (14) holding in an arbitrary given azimuth plane.
The Diffusion Model
The diffusion model is designed to describe the spatial variation of scalar irradiance in a natural hydrosol. This model together with the two-flow model for irradiance, and
the model for radiance, forms a reasonably exhaustive battery of elementary descriptions of most of the natural and artificial light fields encountered in everyday practice.
A simple and instructive route to the diffusion model can be made via the two-flow model (4), (5), as follows. Let us add together, term by corresponding term, the two equations $(4)_{so}$ (S). We find:
_.. _ -aD(H+ + H-) (is) Now, according to (8) of Sec. 1.1 and the definition *of net irradiance f($_{z,}$+),* which is defined by writing:
v
for H(z,+) - H(z,-) f
or more briefly:

«H+~t for H+ - H
irH rr for H . H , -
AID +
we can cast (15)-into into.the form
~, dH. dH+ _ ah dz dz

(16)
using the definition of the distribution factor D, and (1). This states that the depth rate of change of the net upward irradiance at a point is jointly proportional to the volume absorption coefficient and the scalar irradiance at that point:
Readers familiar with the rudiments of vector analysis will see that either derivative term on the left side of (16) is simply the negative of the divergence of the vector irradiance H (cf. (4) of Sec, I.I). The other two (the x,y) derivatives of the components of H *are missing from (16) be*cause the two-flow model applies to *stratified* media, i.e., media whose properties are constant over horizontal planes in the hydrosol. However, this recognition of the nature of the left side of (16) permits us to write:
V•H _ -ah (17)
in place of (16).
Equation (17), despite the route we have just taken, is a quite general law which holds in source-free media of arbitrary shape and inhomogeneities and whose light fields are of arbitrary spatial and directional structure. We have in this way made a leap from the special to the general by making a simple observation on the mathematical form of the divergence of a vector field. (For further details, see (5) of Sec. 2.8 and (15) of Sec. 8.8.) An even more general form can be obtained if we allow the presence of sources in the medium;

(is)
where h_n is the radiant flux generated per unit volume by internal sources.
Now, the diffusion model we are interested in springs from (18) once we have made a .special assumption about the behavior of the light field and the nature of the term h_n.
The requisite assumption is concerned with the scattered light field in the medium of interest, so that we shall look only at the components of H and h which consist of radiant flux having been scattered at least once, In order to point this up in the notation, it can be shown that we may write (17) in a form quite analogous to (18)
(i9)

This star notation is standard notation for scattered radiant flux. To indicate how ~we may arrive at (19), we first observe that the full vector and scalar irradiance fields are represented as:
$_g$o
(20)
h h + h (21) where 8°, h$^{\circ}$ consist of residual radiant flux directly transmitted from the sources and boundaries. When written in this form we say that the light field H

has been *decomposed into* its residual and scattered parts; This mode of decomposition is not new to our discussions in this chapter. For we have in effect represented the apparent radiance Nr in (12) in precisely this way. Indeed, if in the context of (12) we write

"$_N$ot, r

for

soar

(22)

and

"N *11 r

for J N_*e~^{a}($^{r-r}$)dr' (23) p

then the equation (12) for apparent radiance N_r becomes (in *functional form):*

where N^o *is the residual radiance and Nr the path radiance.* This form is completely analogous to 24 (21). In fact all we have to do to get (2.0),.(21) is integrate (24) over all directions and apply (3), (4) of Sec. 1.1 (cf. Secs. 6.5 and 6.6). Hence if we integrate each side of (10) over all directions in this manner, we can obtain (19) quite rigorously. The complete details of this derivation may be found in the derivation of (63) of Sec. 6.6.

We return to (19), and make the assumption about H^*. which invokes the desired diffusion model. The assumption is simply this.

$_H$*

shall be proportional t

-ah*

(25)

Here oh* is the gradient of h^*. For example, in a stratified plane-parallel medium, this amounts to saying that:

- a~ (-k) x (constant)

i.e., that the scattered irradiance vector--which in the sea clearly points downward in the direction of greatest net irradiance--is simply the derivative of the scattered scalar irradiance times the unit downward vector (-k}, i.e., the vector pointing along the direction of increasing z. It is interesting to note that this is a .sort of backwards version of

(16), obtained from the latter essentially by moving the derivative operation from its left to its right side. Notice that $* is required by (25) to point in the direction of decrease of h. In natural waters dh/dz is negative (with in¨ft creasing z measured downward as usual), we shall use the conventional symbol "D" for the diffusion constant of proportionality. Notice that its dimension is that of a length. (We use the letter I'D" here without fear of confusion with our distribution coefficients.) Hence assumption (25) can be written as n equality:

H* _ Dvh* . (26) and when this assumption is used in (19) we have;

V"• -DVh*

)
-ah* + h i
or, since D is a constant we have, finally,

(diffusion equation for decomposed light field)
(27)
which is the ? resent desired form of the diffusion model*
tt is symbol 'IV s the laplacian operator used i_n vector analysis. In this model we assume that the source term h* describes the origin of the scattered scalar irradiance h* and thereby is of the form: .

where s is the volume total scattering coefficient defined in the preceding radiance model discussion and h^o is the scalar irradiance associated with the residual flux from the source and boundaries, The diffusion model takes its name from the *assumption (26), which is Fick's law of diffusion,* now applied to the diffusion of photons.

Equation (27) as it stands constitutes s reasonably good model of the scattered (or diffuse) scalar irradiance in both natural and artificial light fields. By way of contrast,
we observe that it is more accurate than the diffusion model that comes from applying (26) (without the stars) to (18), instead of (26) to (19). For in the former case, i.e., when applying (26) (without the stars) to (18) we find

$-DV^2h + ah$ _ h n
(diffusion
equation for (29) decomposed
light field)

and even though the mathematical forms of (27) and (29) are the same, an essential difference between them arises by virtue of the nature of the source term hn. In the case of (29), h_n for artificial point sources is a Dirac delta function, whereas in (27), as we see by (28), h* is a relatively

smoothly varying function throughout the medium. Since diffusion models become more accurate the smoother the spatial variation of the source terms, the superiority of (27) over (29) is quite clear.

However, it takes correspondingly more effort to solve (27) than it does (29). The formal solution of (29) for a point source is straightforward, and takes the form:

h (r)
a J_{oe} -rcr Dr
(undecomposed h, and point source)
(so)
where we have written

for

4W
(31)
and P_o is the radiant flux output of the point source, assumed to be uniform in all directions: Furthermore r is distance from the observation point to the point source, and we have written:
of $_K$it
for

(32)
where a is the volume absorption coefficient for the medium, and D is the diffusion constant (C-f. (27)of Sec. 6.5),
The general solution of (27) is now forthcoming by means of (30) and a straightforward Integration, To see this, *we imagine that at each point* x' of the Medium X (which is an extensive region without perturbing boundaries) the residual scalar irradiance ho(x') is scattered, there to give rise to an entirely new point source problem whose solution at an observation point x is described by (30), now written in the form:

-Kr
(33)
where

and
h 0 . (XI)s h (35)
Hence if,the original point source is at the origin (i.e., at x'0), and of a relatively mild directional output, then the scalar irradiance field h(x) at x is given very nearly by:
h x) h^o (X)
h. (X)
(36)

66
where
and
and where

h^* (x) == j h_s (x') dV (x') (37)

X

ho(x,) a Non(x,) $_e$-ar $_s$ (38)

$_r$' _ 1x'1 (39)

and n(x') is the solid angular subtense of the point source as measured at x'. The source is actually a small finite sphere of surface radiance No in the direction C' = x/jx'j. V is the volume measure in X. We shall not go into further details here. See (66) of Sec. 6.6 in particular, and Sec. 6.6 in general for complete details.