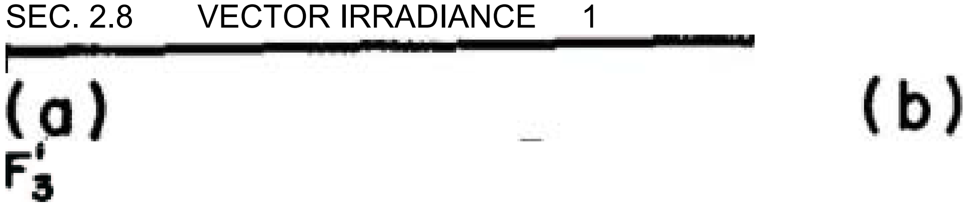
2.8 Vector Irradiance

The radiometric concept of vector irradiance, which wil-1 now be considered, constitutes an interesting and useful complementary concept to that of scalar irradiance. Whereas scalar irradiance in essence measures the volume density of radiant energy at a point and does so without emphasis on the directions of incidence of the component flows but only their magnitudes, vector irradiance in contrast gives a measure of the direction of the preponderant flow of radiant energy at the point without emphasis on the magnitude of the various component flows. Besides serving to complement the geometric properties of scalar irradiance in this way, vector irradiance forms a rigorous tool in deriving the transfer equations for scalar irradiance, and also a powerful means of measuring precisely and directly the absorption properties of real optical media. The basis for the latter means (the divergence relation for H) is considered in Chapter 8 and some of its applications are discussed in Chapters 6 and 13. In this section emphasis will be on introducing and explicating the geometric and physical meanings of vector irradiance.

A Mechanical Analogy

The notion of vector irradiance can be introduced by means of an analogy with the vectorial treatment of forces in static mechanics. Figure 2.19 (a) depicts a force diagram familiar to beginning students in static mechanics. A particle at point P is subject simultaneously to two steady forces of magnitude F1 and F2 along directions E1 and C2. In order to establish equilibrium of the particle--i.e., to balance out F1 and F2 so that the particle is stationary, another force of magnitude Fs' must be applied along direction E3'. The magnitude Fs and direction Es of the equivalent force that may replace Fi and F2,is found by means of the familiar parallelogram of forces shown in Fig. 2.19 (b). `The required balancing magnitude is then -F3'and its direction is -~s', which follows directly from Newton's Third Law. The central observation to be made here is that, for the purpose of static equilibrium, two forces P1(= F_1&$_1$) and F_2(= F_2 ~$_2$) can be replaced by a single force Fs (= F3~s) which serves as a mechanical equivalent of the set of forces consisting of F1 and F2 together. Thus, Ps is, for the purposes of an-equilibrium computation, equivalent to F1+F2•

Consider now a point P irradiated by two beams of radiant flux which are flowing along directions t1 and &2 with



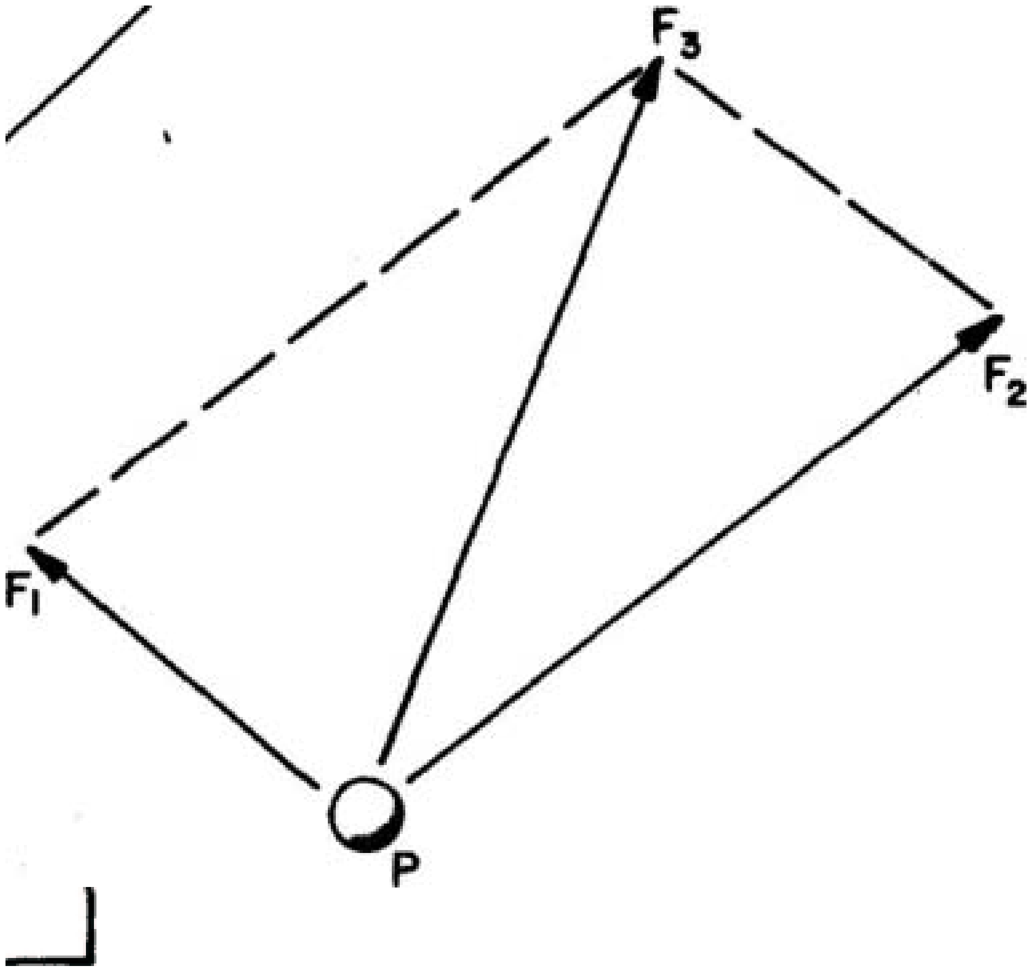


FIG. 2.19 The parallelogram law in mechanics

radiance N, and NZ, respectively, as in Fig. 2.24 tai. Each radiance has a fixed small solid angle a. Now whereas the mechanical context of Fig. 2.19 (a) is meaningful in terms of sets of directed forces and equivalent single forces, the context of Fig, 2.20 (a) is meaningful in terms of sets of directed radiances and equivalent single radiances. In the mechanical setting, a single force could, for the purpose of an equilibrium computation, replace the two given forces by a single force F3. We now ask: can we replace the two directed radiances Ni and Nz by a single equivalent radiance N3?

Some thought will show that, before the preceding question can even be entertained, the sense of "equivalent" must be defined. Clearly, the replacing radiance can be "equivalent" in any one of several desirable ways. For example, if it is required that the replacing radiance produce the same scalar irradiance at P, then there are many possible candidates for N3. If on the other hand it is required that the replacing radiance produce the same net irradiance on an arbitrary collecting surface at P, then there is generally one and only one radiance N3 that can replace N, and Nz in this sense. Observe that the replacing radiance N_3 must be equivalent to N 1 and N Z in this sense not just for one fixed position of a collecting surface at P; if that were the case, then Ns could be chosen from any of an infinite number of radiances. Rather, N3 is to produce the same effect for alt possible orientations of a collecting surface at P.

The analogy here with the mechanical context is essentially exact: in the mechanical context F 3 establishes the same net force on

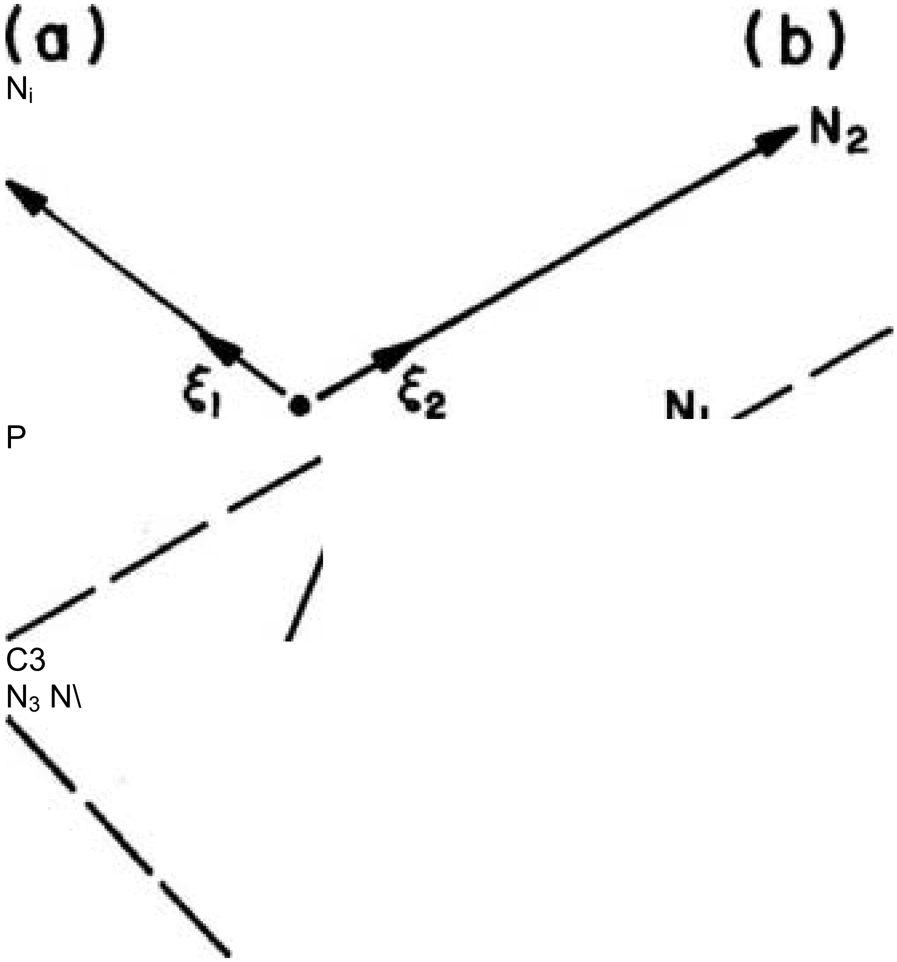


FIG. 2.20 The parallelogram law in radiometry

any particle at P as does F, and F2; in the radiometric context Ns is to establish the same net irradiance on any surface at P as does N1 and N2. It is a simple matter now to prove that the parallelogram law may also be used for the radiometric context to solve the analogous problem in that setting. Thus, the requisite replacing radiance Ns and its associated direction ~3 of flow follow from a parallelogram construction as in Fig. 2.20 (b). In particular, if we write:

and

"N~" for ItN2t# for &2N2

then Nj+N_2 is the requisite vector radiance provided it has a solid angle 0. For if E is the inward unit normal to a collecting surface S at P, then we have by (6) of Sec. 2.a:

~'~INI + C'~2N2

as the expression for the total net irradiance on 5 produced by the two beams. This sum may be written:

~•NIO + ~•N2S2

or, as:

E`(NI + N2)Q

This representation suggests that if we direct a radiance beam of solid angle a at P and along the direction of Ni + N2

and with the magnitude of N1 + N2, then this single beam will produce the same net irradiance across S at P as the two given beams. The vector N1 + N2, which we have denoted by "N3" in Fig. 2.20 (b), is found exactly as in any vectorial addition operation.
The observations just made can be generalized to the case of any finite set of beams irradiating a point x in a radiometric environment. Toward this end, suppose that the various beams have radiances N1,..., Nk along directions
and that, for generality, they have generally distinct solid angles, 91, …, 2k, respectively. Then by repeated use of (6) of Sec. 2.5, the net irradiance produced on a surface S with unit inward normal ~ at x is:
~'~,NiDi + E'EZN2D2 + ... + E•ekNk11k
Suppose we write:
k
"H $(_x)$ +~ for L E_j N_j n_j S
Clearly H(x) is a vector and its magnitude
sions of irradiance. Furthermore H(x) has
r
JH(x)J has dimenthe property that:
$\xi \cdot H(x)$
is the net irradiance on the surface S at set N1,..., Nk of radiances at x. In the considered above H(x) was EINYD + UN20. the vector irradiance associated with the distribution Nl,...f Nk.
x produced by the introductory example We shall call H (x) discrete radiance
General Definition of Vector Irradiance
We now can go one step further in the development of the idea of vector irradiance. Instead of a discrete finite set of radiances N1,..._, Nk at x, we consider a general radi ance distribution N (x, •) . Instead of the finite summation over the sets of directions of the radiances in (1), we use the continuous counterpart to the sum, namely an integral over all the directions E at x. Thus let us write:
11H$(_x)$ if
for &N (x, dslC }- (2)

and where, in turn, the integral uses field radiance and is to be understood as an ordered triple of integrals, as is customary in vector analysis. That is, we have written:
ff tN (x, t) dQ (&) " for w

f N (x., ~) cos $t1_l$ dil (e) , f N (x, E) cos $1Y_2$ dil (&) , fN(x,t) cos 0, dH (e)
(3) and where 1Ji, 2,fz , andV 3 are as defined in Sec. 2.5 (cf., e.g,,,, (18) of Sec. 2. 5) .We call H (x) the vector irradiance at x. The alternate form (3) of the integral in (2) is the form in which H(x) is computed in actual practice. The integral in (2) is a compact symbol for the ordered triple of integrals in (3). A researcher

requiring the direction and magnitude of H(x) at x knowing N(x,•) at that point, computes the three components of H(x) in accordance with (3). Thus, if we write:

"H_i (x) " for

f N (x ,) cos I did ( ) (4)

V

for i = 1,2,3, then:

H(x) = (H_1 (x) , H2 (x) ,Hs (x)) 1 (5)

The magnitude of I H (x) I of H(x) is

(HI' (x)+ H A x) + $H3^2$ (x)) II2 (5) and the direction of H(x) is the unit vector:

(HI(x),H2(x),,H3(x))/(H,2(x)+H22(x)+HS2(X))1/2 . (7)

The General Cosine Law for Irradiance

The cosine law for irradiance was introduced in Sec. 2.4 in a rather special context. Our purpose here is to show how the law can be given the status of a general theorem in radiometry. Thus we will free the cosine law in (16) of Sec. 2.4 from the restrictions placed on it in that section. The means by which the generalization can be accomplished is the notion of vector irradiance. The law may be stated as follows: Let N(x,•) be a radiance distribution at point x in an optical medium. Let ~ be the unit inward normal to a surface S at x. Then the vector irradiance H(x) at x, as defined in (2) has the property that:

&•H(x) _ IH(x)I cos -V

where " I H (x) I " denotes the magnitude o f H (x) , as given by (6), ,and '[1]/0" denotes the angle between &, and the direction o f

H(x) as given by (7). Furthermore,

111(x) I $max_{\&}$ H(x,) (9

_i.e., 1H(x) I is the maximum of the set of a1Z net irradiances H(x, C) at x, where the net irradiance H(x, C) at x across S in the direction o f C is as defined in (14) o f See. 2.4. The proof of (8) is immediate, since (8), stripped of all phys i - cal connotations, simply constitutes an elementary theorem in vector analysis. Equation (9) is the more deep of the two and follows from the observation that each Hi(x), i = 1,2,3, can be written out in full form as:

H I (x) N (x , 0 cos V'I do (E)

$\Xi(\xi)$

f N(x,~) cos-V_I dQ(~)

H(x,i) - H(x, -i) IT(x,i) (1D)

where the first equality results from writing w as the union of two disjoint hemispheres ~M and W(-E) and where the second equality follows from two applications of (8) of Sec. 2.5. In a similar way we show that:

H 2 (x) H(x,J) - H(x, ¯J) = H(x,j) (11)
H(x,k) ff(x,-k) = H(x,k) .
In this way we uncover the physical significance of the three components li I (x) , H 2 (x) , and Hs(x) of H (x) . For example, H I (x) is the net irradiance across a plane at x whose inward normal is the coordinate unit vector i along the x axis. Continuing on our way to establish (9), we now examine g•H(x) directly:
%
~.H(x) = ~• f C'N(x, E') dQ(&')
M
V
= f &•g'N(x9E') dQ(&')
~•C'N(x,t') dil(C) + ~'E'N(x,C') dn(C')
V r•

$$= \int_{\Xi(\xi)} \xi\cdot\xi' N(x,\xi')\ d\Omega(\xi') - \int_{\Xi(-\xi)} (-\xi)\cdot\xi' N(x,\xi')\ d\Omega$$

=
H (x, C) H (x , - C) a fi (x, c)13

This sequence of five equations is justified analogously to the sequences culminating in (10), (11), and (12). Now, however, we have included more detailed steps. Clearly, (13) subsumes (10)-(12). In view of (13), we may write (8) as

H (x, ~) = I H (x) I cos h
(14)
and_ now (9) follows immediately from this. The maximum value of H(x, ~) occurs when -2~^{0}= a . From this, we have (9). Thus IH(x)l is simply the net irradiance across that surface S at x whose unit inward normal is the same as the direction of H(x).
The results (8)-(14) are of importance in both theoretical and experimental radiative transfer. An intuitive feeling for HW and for equations (8) and (14) may be obtained by imagining an experimental device of the kind schematically depicted in Fig. 2.21. The device has two collecting surfaces S_+ and S_ placed so that S_+ and S_ together receive radiant flux from every direction in '. Further, the unit inward normal ~ to S_+, may be represented by a wire with a pointer welded to one end, and the whole arrow fastened to the material collecting surfaces as shown schematically in Fig. 2.21. The meter for the device is wired to read H(x,t)-H(x,-~), i.e., the recorded irradiance on S_+ minus the recorded irradiance on S_. A device so constructed is called a subtracting Janus plate, (where "Janus" has the same etymology as "January") and may be used to empirically determine H(x) in natural optical media. To operate the device, one

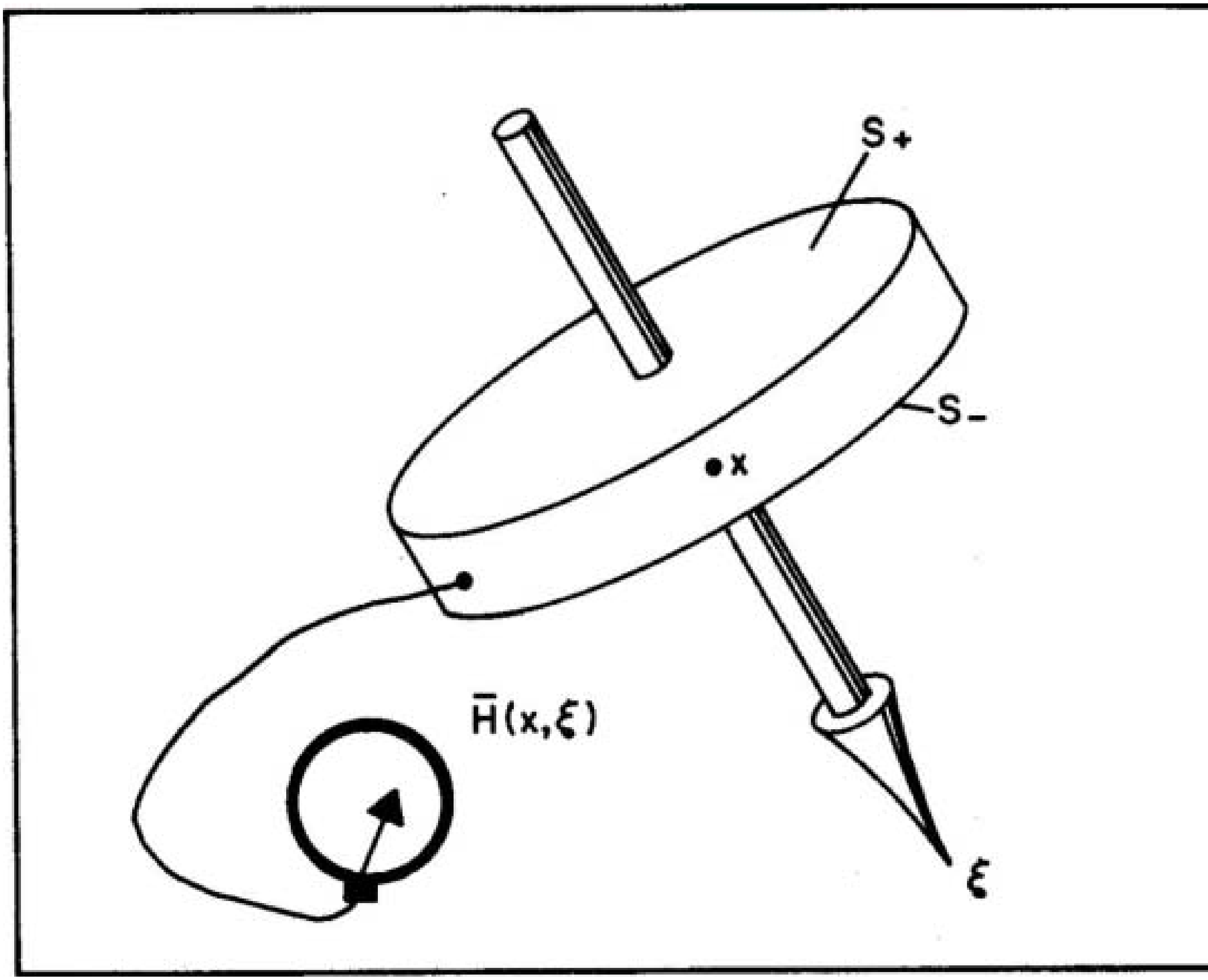


FIG. 2.21 Schematic of a subtracting Janus plate$_s$ used in measuring the vector irradiance field.

orients it at a point x so that the reading !T(x,~) attains a maximum. Then by (9) the magnitude of H (x) is this maximum reading, and the direction of H(x) is the direction of the arrow fastened to the device. The full geometric and physical significance of (8) and (14) can now spring forth: regardless of the complexity of the radiance distribution N(x,•) --and as the eye is a witness, such complexity can be subtle and of infinite variety in natural optical media--the meter reading of the Janus plate varies precisely in a sinusoidal fashion as the meter's direction is varied and as a increases from 9 to n. By means of H(x) we can develop a theory of the light field which is similar in many respects to certain classical fluid flows in hydrodynamics. This analogy has been explored by Gershun [98], and by Moon and Spencer [187].

Equations (8) and (14), aside from their intrinsic mathematical interest appear to have in store potential practical applications. For example, (14) may be of use in facilitating the practical task of computing and tabulating irradiances H(x,&) over all directions t at given points x in optical media. Such a task is encountered, for example-, when contrast calculations in submarine environments are desired. Specifically, by (14), it is clear that at a point x a full tabulation of H(x,~) over all C in need not require a correspondingly full computation. Indeed, appropriate use of Equation (14) cuts the requisite work just about in half. Thus, one can compute H(x,Q for each direction E over some pre-selected hemisphere, say and also compute H (x) , whence IH(x)I is obtained. Then from (14) we have

H(x,-~) = ff(x,~) - I H(x) I cos -d (15)

where C is in ~(C_o), and so -C is in- $^-$($^-$Co), the complement of H(Ca) with respect to .. Thus for every -C in ($^-$Co) one computes H (x, - C) using the already tabulated value H (x, C) , the angle W , and I H (x) I .

We conclude the present discussion of the general cosine law for irradiance by casting its basic form (8) into one which comes as close as possible to its special counterpart (16) of Sec. 2.4. Thus let "m" denote the unit vector associated with H(x), i.e., m is the direction of H(x) as computed by (7). Then, by (14), we have H (x ,m) JH(x)J which is the maximum net irradiance at x. Further, in (14), cos z fm &•m; hence (14) becomes:

$$\bar{H}(x,\xi) = \bar{H}(x,m)m\cdot\xi$$

(16)

Clearly (16) of Sec. 2.4 is a special case of (16) above when radiant flux is incident on x in accordance with the restrictions on the earlier equation.