

### 3.1 A Preliminary Example

We shall develop an example of the interaction principle in this section with the purpose in mind of fixing, on a relatively simple intuitive level, the salient features of the principle preparatory to stating the principle in its full form.

Empirical Reflectances and Transmittances for Surfaces A prerequisite for the development of the example is the definition of the empirical reflectance of a small plane surface  $S$ . Figure 3.1 depicts such a surface  $S$  with unit outward normal  $k$ , which is irradiated at each point by radiant flux  $\star$  through a narrow solid angle  $D'$ , the flux passing through a hypothetical collecting surface  $S'$  on its way to  $S$ . The observed (empirical) field radiance of the incident flux is  $N(S',D')$  and the observed (empirical) surface radiance arising from reflection of  $N(S',D')$  by  $S$  in a narrow solid angle  $D$  is  $N(S',D';S,D)$ . We write:

$$N(S',D') \quad Q(D')$$

and call  $r(S',D';S,D)$  the (empirical) reflectance of surface  $S$  for the incident and reflected directions  $D'$  and  $D$ , respectively. Here  $S'$  is the projection of  $S$  on a plane perpendicular to a direction  $V$ , the central direction of  $D'$ . The function which assigns to  $(S',D')$  and  $(S,D)$  the number  $r(S',D';S,D)$  is called the (empirical) reflectance function for  $S$ . For the purpose of the present example, we assume  $r(S',D';S,D)$  is known for all pairs  $(D',D)$  of incident and response (reflected)

For simplicity in exposition, throughout this work all radiant flux quantities will be assumed unpolarized, unless specifically stated otherwise. For an outline of the task of extending all results below to the polarized context, see Chapter XII of [251]. The interaction principle, however, holds implicitly for the polarized case. For the relative mathematical consistency of the assumption of the unpolarized light field with respect to the complete theory of the polarized field, see Sec. 13.11.

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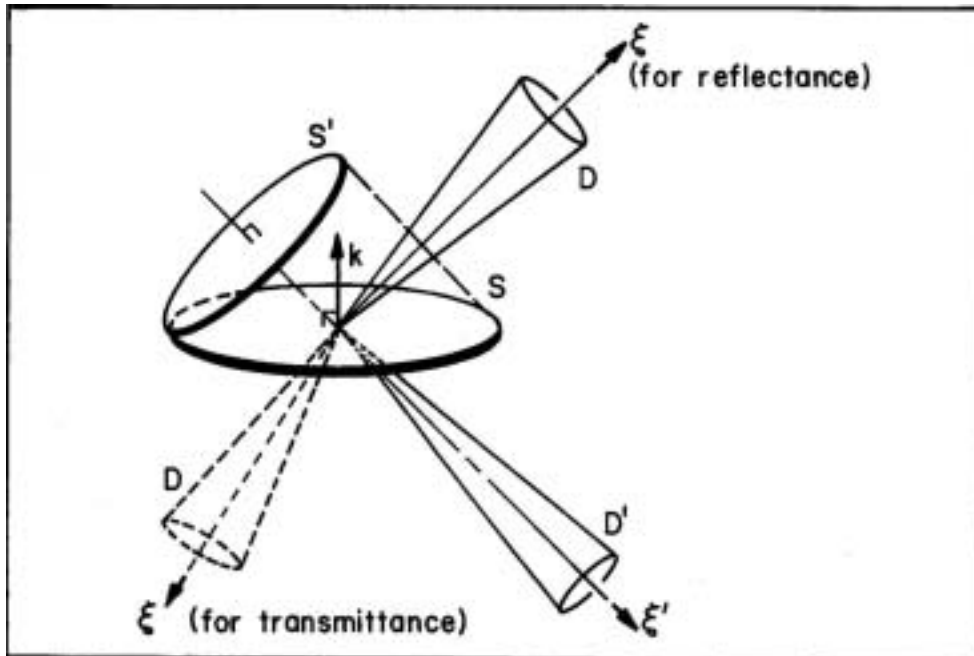


FIG. 3.1 Setting for empirical reflectances and transmittances of surfaces.

direction sets such that  $\int_V \mathbf{k} \cdot \mathbf{k} < 4$  and  $\langle \mathbf{k} \rangle \cdot \mathbf{a}$ , respectively. If  $\mathbf{e}_1$  and  $\mathbf{C}$  are arbitrary central directions of the sets  $D'$  and  $D$ , respectively. These two conditions merely require  $D'$  and  $D$  to lie on opposite sides of  $S$ , as in Fig. 3.1.

Now the essential property of the response radiance  $N(S', D'; S, D)$  is that it is additive with respect to  $D'$ . More precisely, experimental evidence indicates that we may assert the following property of  $N(S', D'; S, D)$ . In each case let the sets  $D, D'$  of directions be circular, conical sets with central directions  $\mathbf{e}, \mathbf{C}'$ , respectively. Then:

(i) ( $D'$ -Additivity) If  $S$  is a surface in an optical medium  $X$  and  $S$  is irradiated in turn by radiances  $N(S', D'; S, D)$  and  $N(S_2', D_2'; S, D)$ , with  $N(S_1', D_1'; S, D)$  and  $N(S_2', D_2'; S, D)$  as the respective observed response radiances, then  $N(S_1', D_1'; S, D) + N(S_2', D_2'; S, D)$  is the observed radiance of the  $S$  under simultaneous irradiation.

Furthermore:

(ii) ( $D'$ -Continuity) Let the geometric setting be defined as in (i). If  $D'$  is a set of directions  $s$  and  $D$ , then  $N(S', D'; S, D)$  is a D.

By letting  $D$  lie on the same side of  $S$  as  $D'$ , these two empirically-based properties of reflected radiant flux are readily turned into the corresponding laws for

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transmitted flux (see dotted direction cone in Fig. 3.1). Observe that, by virtue of (i),  $r(S', D'; S, D)$  is independent of the magnitude of  $N(S', D')$ . It should be particularly noted that (i) and (ii) are new laws which are independent of the  $D$ -additivity and  $D$ -continuity properties of  $0$  in Sec. 2.3. The present laws are intended to characterize a particular type of interaction of radiant flux with matter, whereas the earlier laws were intended to characterize certain intrinsic radiometric (principally geometric) properties of radiant flux regardless of its interaction with matter. These two properties permit a limiting process to culminate in rigorously defined reflectance and transmittance functions for surfaces. The details of such definitions will be considered in (b)-(9) of Sec. 3.3. For the present we use (i) and (ii) as they stand to help solve the following radiometric interaction problem.

The Problem

Two plane surfaces,  $S_1$  and  $S_2$ , in a vacuum are mutual point sources. In addition, they are mutually visible and are irradiated by sources of radiance  $N_{10}$  and  $N_{20}$  over solid angles,  $D_{01}$  and  $D_{02}$ , respectively, as shown in Fig. 3.2. These incident radiances initiate an interreflection process between  $S_1$  and  $S_2$  with resultant surface radiances  $N(S_1, D_{12})$  and  $N(S_2, D_{21})$ . Here  $D_{12}$  is the set of directions from a point in  $S_1$  to every point in  $S_2$ . Since  $S_1$  and  $S_2$  are mutual point sources (i.e., each is a point source as seen from the points of the other),  $D_{12}$  does not vary appreciably as location is varied over  $S_1$ , and so may be assumed constant over  $S_1$ .

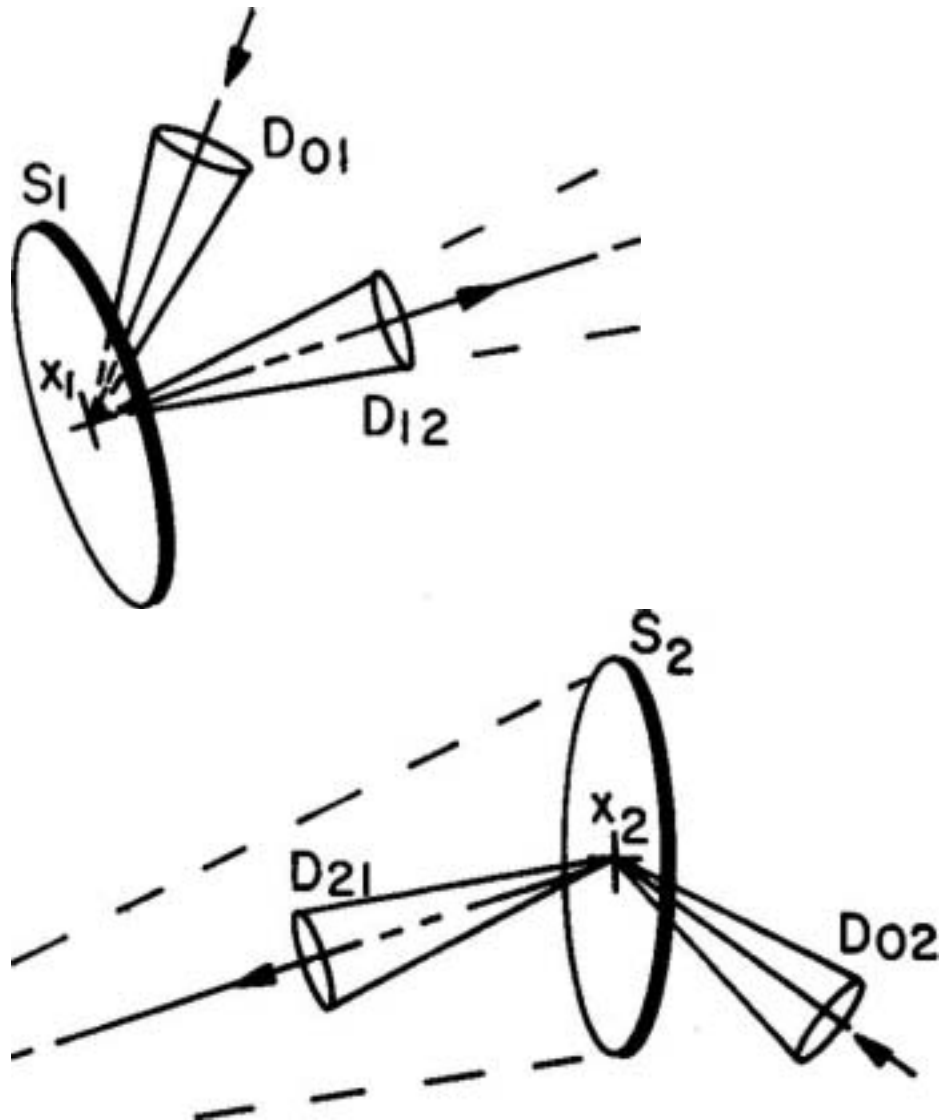


FIG. 3.2 Setting up an interaction calculation for surfaces  $S_1$  and  $S_2$ .

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Similarly  $D_{21}$  is the set of directions from a fixed point of  $S_2$  to every point of  $S_1$ ; and  $D_{21}$  has the same general property as  $D_{12}$ . With these preliminaries out of the way, we can now state the present problem: Given  $S_1$  and  $S_2$ , as above, with known empirical reflectance functions,  $r_1$  and  $r_2$ , and initial irradiances,  $N_1^0, N_2^0$ .

Required. the steady state empirical radiances  $N(S_1, D_{12})$  and  $N(S_2, D_{20})$ .

$N_2^0$  for

$N_1^0$  for

it  
 $r_{12}$  of for

to  
 r<sub>121</sub> « for  
 "rit  
 o<sub>12</sub> for  
 'fro<sub>2</sub>  
 jf' for

The Present Instance of the Interaction Principle To facilitate the present discussion let us write:

$$N(S_1, D_{12}) = N_{02}, D_{21} J$$

$$r_1(S_{12}, q-D_{12}; S_1, sD_{12}) = r_2(S_{21}', o'D_{21}, l'S_2, OD_{21}) = r_1(S_{01}', sD_{01}, S_1, sD_{12}) = r_2(S_{02}, D_{02}; S_2, pD_{21})$$

where  $S_{12}'$ , e.g., is the projection of  $S_1$  on the plane normal to the direction from  $x_1$  to  $x_2$ . Similarly, for  $S_{21}'$ ,  $S_{01}'$ , and  $S_{02}'$ . In the case of  $S_{01}'$ , e.g., imagine an external source point  $x_0$ . The set  $-D_{12}$  consists of all negatives of directions in  $D_{12}$ . Thus if  $E$  is in  $D_{12}$ , then  $-D_{12}$  contains  $-E$ . Now by virtue of the definition of empirical reflectance, the  $D'$ -additivity property (i) above, and the fact that the intervening space between  $S_1$  and  $S_2$  is a vacuum (so that the radiance invariance law can be used) we have:

$$N_{12} = N_{1r012001} + N_{21r212a12}$$

$$N_{21} = N_{2r021no2} + N_{12r12IQ2r}$$

where we have written:

$ot_D$  of

for  $D(D_{0i}, i = 1, 2)$

and

$l'_{ij}$  for  $Q(D_{ij})V_{ij}, i, j = 1, 2$

For later purposes it is convenient to make one final set of definitions, We write:

for  $r_{0ij}, D_{0i}, i = 1, 2$

$J$

" $\{ijj\}$ "

for  $r_{ijj} \Omega_{ji}, i = 1, 2$

,  $j = 1, 2$

,  $j = 1, 2$

=  $1, 2$

=  $1, 2$

198 INTERACTION PRINCIPLE VOL. II Then (2) and (3) become:

$N_{1z}$

$0 r_1^{01} + N_{z1Ez12}$

$N_{21} = N_{2E21} + N_{12E1z1}$

Equations (4) and (5) together constitute the algebraic core of the statement of the present form of the interaction principle. In the present case we have two relatively small

plane surfaces which are interacting radiometrically. Each surface  $S_i$  ( $i=1$  or  $2$ ) is irradiated by two incident parcels of radiant flux in the form of the empirical radiances,  $N_{i0}$  and  $N_{ji}$ , and  $S_i$  itself has a resultant surface radiance  $N_{ij}$ .

To the sets of such incident radiances,  $N_{i0}$  and  $N_{0i}$  and response radiances  $N_j$  associated with  $S_i$  ( $i=1$  or  $2$ ), there correspond four interaction operators (numbers in this case), namely  $L_{iA}$  and  $E_{Ai}$ , such that (4) and (5) hold. The main role of the interaction principle in the present case would be to assert the existence of these operators and to yield the interaction equations (4), (5).

**Solution of the Problem**

The interaction principle formulation (4), (5) of the present problem leads to the solution of the problem by means of the theory of simultaneous algebraic equations. Thus, multiplying each side of (4) by  $E_{111}$ :

$$N_{12}E_{121} = N_{i0}E_{1z1} + N_{21}E_{a12}E_{1x1}$$

and using this representation of  $N_{1z}E_{1a}$  in (5):

$$N_{21} = N_{00}E_{z1} + (N_{00}E_{02}E_{121} + N_{21}E_{z12}E_{1z1})$$

whence:

$$N_{00}E_{01} + N_{00}E_{00}$$

$$N_{00}E_{01} =$$

$$1 - E_{212}E_{121}$$

The radiance  $N_{12}$  can be found by permuting the symbols "1" and "2" in this equation. The complete solution of the system (4), (5) is then:

$N_{12} = \frac{N_{00}E_{01}}{1 - E_{212}E_{121}}$

$$N_{12} = \frac{N_{00}E_{01}}{1 - E_{212}E_{121}}$$

$$N_{21} = \frac{N_{00}E_{02}}{1 - E_{121}E_{212}}$$

$$+ \frac{N_{00}E_{02}E_{121}E_{212}}{1 - E_{121}E_{212}}$$

$$- \frac{N_{00}E_{02}E_{121}E_{212}}{1 - E_{121}E_{212}}$$

$$N_{00}E_{01} + N_{00}E_{00} \frac{E_{121}E_{212}}{1 - E_{121}E_{212}}$$

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**Discussion of Solution**

A sufficient condition that  $N_{12}$  and  $N_{21}$  are determinable via (6) and (7) is that the product  $E_{121}E_{212}$  is less than 1. We shall now show that a sufficient condition that this latter property holds is that at least one of  $E_{121}$  and  $E_{212}$ , is strictly less than 1. An examination of these definitions of  $E_{121}$  and  $E_{212}$  shows that this condition may be based on a particular form of the principle of conservation of energy. To see this in the case of  $E_{212}$ , we need only systematically unfold the definitions leading to it. Thus,  $E_{212}$  is:

$$r_{21}(\text{S}_1, \text{D}_1; \text{S}_2, \text{D}_2)$$

The quantity  $T_{212}$  is:

$$r_1(\text{S}_1, \text{D}_1; \text{S}_1, \text{D}_1)$$

This in turn is the value of the empirical reflectance function for  $S_1$ . By (1), and the fact that this value of  $r_1$  is independent of the magnitude of the irradiating flux, we can select any incident radiance, say  $N_{i0}$  over  $-D_1$ , and let  $N_{12}$  be the response (reflected) radiance over  $D_1$ . Then:



N

$$r_1(S_1, D_{12}; S_1, D_{12})Q_{12} = \frac{1}{2} N_{12}$$

If  $A(S_{12})$  is the projected area of  $S_1$  on a plane normal to the direction from  $x_1$  to  $x_2$  [see Fig: 3.2), and  $P(S_1, D_{12})$  and  $P(S_2, D_{12})$  are the radiant fluxes associated with

$N_{12}$ , then the incident radiant flux is given by:

$$P(S_1, D_{12}) = N_{12} A(S_{12})$$

and the surface (response) radiant flux is given by:

$$P(S_1, D_{12}) = N_{12} A(S_{12}) r_1(S_1, D_{12}; S_1, D_{12})$$

Here we have used the fact that  $Q(-D_{12}) = a(D_{12}) = Q_{12}$ ,

also the operational definition of surface radiance (Sec. 2.6). Hence:

and

$$r_1(S_1, D_{12}; S_1, D_{12}) = \frac{P(S_1, D_{12})}{N_{12} A(S_{12})}$$

At this point we choose the energy conservation principle in the form which states that:

if  $P^-$  is the total radiant flux incident on a given surface  $S$  and  $P^+$  is the total radiant flux leaving the surface  $S$  and  $P^+$  and  $P^-$  are independent of time, then  $P^+ = P^-$ . We

shall assume this statement is true. From this we deduce in particular that:

$$r_1(S_1, D_{12}; S_1, D_{12}) \leq 1$$

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so that:

$$\sum_{i,j} r_{ij} \leq 1$$

A similar inequality now follows for  $E_{ij}$ . These inequalities are the most we can say, without further qualifications, about any reflectance (or transmittance) operator occurring in the theory of radiative transfer. Thus in a particular geometrical situation we must explicitly postulate or demonstrate

that at least one of  $E_{12}$  and  $E_{21}$  in (B) and (7) is strictly less than 1; and as our analysis has now made clear, this is a sufficient condition that (6) and (7) uniquely determine  $N_{12}$  and  $N_{21}$ ,

Related Problems and their Solutions

The solutions (6) and (7) of the problem considered above can be used to solve related problems centering on the radiometric interaction of  $S_1$  and  $S_2$ . Suppose, for

example, we require the surface radiance of  $S_1$  in some set  $D_i$

of directions other than  $D_{12}$ . Here  $x_0$  is associated with point  $x_0$  which may be any point in the surrounding medium either in or not in  $S_1$  or  $S_2$ . Toward this end we write:

$$N_{10} \text{ for } r_1(S_1, D_{10}; S_1, D_{10})$$

$$r_{210} \text{ for } r_1(S_1, D_{10}; S_2, D_{10})$$

$$r_{10} \text{ for } r_2(S_2, D_{10}; S_1, D_{10})$$

Then by the D---additive property (i) above and the radiance invariance law we have:

NIB a Narojsnaj + N11rz10012

In an exactly similar manner we arrive at the surface radiance of  $S_2$ :

$$N_{i0} = N_{z0} r_{20} + N_{j0} r_{21}$$

Once again we can contract these solutions into a fixed form which clearly reveals the underlying unity of the interaction concept. Thus by writing:  
and

$$t_{E_{i0}} = \text{for } r_{ij}$$

$$t_{E_{j0}} = \text{for } r_{ij}$$

$$i, j = 1, 2$$

the preceding equations become:

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f

$$N_{i0} = N_{z0} r_{20} + N_{j0} r_{21}$$

$$+ N_{z1} E_{z1g}$$

$$N_{i0} = N_{z0} r_{20} + N_{j0} r_{21}$$

$$N_{i0} = N_{z0} r_{20} + N_{j0} r_{21}$$

where  $N_{1z}$  and  $N_2$ , are as given in (6) and (7), For the purposes of later comparison with the general statement of the interaction principle we observe that: to the incident radiances  $N_{i0}$  and  $N_{j0}$  on  $S_i$  ( $i = 1$  or  $2$ ) and response radiance  $N_i$  there correspond four interaction operators (numbers in this case), name  $t_{E_{i0}}$  and  $E_{j0}$  such that

(B) or (9) hold. The index 0 in (8)  $H^0$  (9) and in the equations below) may be replaced by distinct indices, if desired. In other words, surfaces  $S_1$  and  $S_2$  may give off radiances in distinct directions, which may be computed by (8), (9) by replacing index  $a$  in (9), say, by a new index  $y$ .

An Alternate Form of the Principle

We now abruptly change our conceptual orientation in Fig. 3.2 from that of two radiometrically interacting surfaces  $S_1$  and  $S_2$  to that of a single subset  $S$  of the optical medium

irradiated from without by radiant flux. This change in orientation can be encouraged by imagining  $S_1$  and  $S_2$  in Fig. 3.2 to be encircled by a closed dashed curve and to think of the curve as holding a single subset  $S$  of space (that is,  $S$  is a disconnected subset which happens to consist of two separate surfaces,  $S_1$  and  $S_2$ ). This subset  $S$  is irradiated at two plates by incident radiances  $H_{i0}$  and  $N_{j0}$ , and the response of  $S$  is imagined in the form of two streams of flux characterized by  $N_{i0}$  and  $N_{j0}$ . This conceptual compression of  $S_1$  and  $S_2$  into a single radiometrically responsive entity can

be expressed symbolically as follows. We first write the system (8) and (9) in matrix form (replacing "\$" in (9) by "y", for generality)

(NjOpNzy)

$$\begin{pmatrix} N_0 & N_0 & E_1 \sim 0 & E_2 & 1 \\ (1 & 9 & z) & (N_{21} & , N_{12}) \end{pmatrix}$$

( U Ezy (

$$\begin{pmatrix} 0 & \Sigma_{2\gamma} / & \backslash 0 & \Sigma_{12\gamma} \end{pmatrix}$$

Further, from (6) and (7) we can write:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 & a z \\ (N_{21} & jhx2) & (N_{12} & NI) & 0 & 0 \\ ('V_2 & 1 & \bar{f} & 2 & 2) \end{pmatrix}$$

where, in turn, we have written:

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f1 12 is for 12

(1-EI. -r. ,)

$$\Psi_{21}^0 \text{ for } \frac{\Sigma_{21}^0}{(1 - \Sigma_{121} \Sigma_{212})}$$

0 f1 to for

(1 Eez ~ E2ex)

1 IT o2 1 f

for

Ez i Z212 (1-EIzixi)

Then going one step further and writing:

and:

0 0 Ti: 0 0 ~ 2x 22

E° ~ 0

fIT 0 f1 for y

E2Y

$\Psi_{01}$

for

and

EZI

Jilt a it ~

for ( ,

E12y

we arrive at the following alternate representation of the system (8) and (9):

$$(N_1 \quad N_2)_y = (N_1 \quad N_2)_{y0}$$

Let us write:

$$1 \quad 0 \quad 1 \quad f \quad \sim \quad 0_y$$

for  $T_A + T_0 T_Y = 0_Y$

and thereby arrive at the desired form of the system (8), (9)

$$(N_1 \quad N_2)_y = (N_1 \quad N_2)_{y0} \quad (10)$$

The significance of (10) may be discerned as follows: for the given subset S we have shown that to any arbitrary pair of incident radiances ( $N_1, N_2$ ) and response radiances ( $N_1, N_2$ )

there corresponds a unique interaction operator  $f$  a  $2 \times 2$  matrix

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of Peat numbers in this case), namely be  $f$ , such that (10) holds.  $\sim Y$

Equation (10) constitutes an alternate form of the interaction principle to that displayed in (4), (5) [or in](#) (8), (9). This alternate form is designed to give an indication of the potential internal complexity of an object S to which the interaction principle may assign an interaction operator. It is not too great an extension of ideas from the setting of (10) to the setting of an arbitrary finite number of interacting surfaces. However, the systematic study of such systems of interacting surfaces or solids is the domain of discrete space radiative transfer theory and lies far beyond our present concerns, For those interested in pursuing this matter further, we observe that the complete theory of such systems is developed in Ref. [251].

The Natural Mode of Solution

we conclude this preliminary example of the interaction principle by displaying an alternate mode of solution of the problem of the radiometric interaction of the two surfaces  $S_1$  and  $S_2$  considered above. Our purpose is to show that this alternate mode of solution and the interaction principle mode of solution are equivalent. As our developments proceed into the next chapter, we shall also see that each mode of solution possesses a valuable conceptual kernel which is capable of extension to quite wide domains of application in radiative transfer theory in general, and hydrologic optics in particular. This alternate mode of solution we call the natural mode of solution, for it appears to be conceptually the simplest and most natural approach to interreflection problems.

The natural mode of solution may be described quite briefly as follows, we imagine a hyper-fast camera filming the radiometric interaction of two surfaces,  $S_1$  and  $S_2$ . The filmed episode begins the instant the incident radiances  $N_{10}$  and  $N_{20}$  simultaneously impinge on  $S_1$  and  $S_2$ , respectively.

In a playback of the filmed episode in slow motion, we see part of  $N_{10}$  reflected from  $S_1$ , and start to travel toward  $S_2$ . This reflected flux eventually reaches  $S_2$  and part of it is redirected back toward  $S_1$ . In the meantime  $N_{20}$  has been reflected at  $S_2$  and part of the reflected flux moves on to  $S_1$ , there to be reflected, and to have some flux begin to return to  $S_2$ . As the film continues, the sources  $N_{10}$  and  $N_{20}$  continue to steadily pour

flux on S1 and S2. After a while S1 is being irradiated by photons, some of which come directly from N1o, some of which are making their first arrival from S2, and some their second arrival from S2, etc. By and by the fluxing and interfluxing reaches a measurable steady state (while, in principle, however, there will always be some interreflection number which has not yet been attained). The following argument develops the symbolic representation of this steady state interreflection process.

Retaining the notation of the preceding discussions, let us go on to write:

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"N12" for N°E12

and

'IN ts for

$$N_2^0 \Sigma_{21}^0$$

Further, for every j a Z,3,...., we write:

"Ni tt

for

$$N_{21}^{j-1} \Sigma_{212}$$

"N<sub>21</sub><sup>j</sup>"

for

j-i

N12 E121

By recalling the moving-picture allusion it is easy to see that NJ is interpretable as the surface radiance of S, in

the directions of Sz consisting of radiant flux having undergone precisely j reflections.

Again, by means of the analogy, we are led to write

"N12"

for  $\sum_{j=1}^j N_{12}^j$

"N21"

for  $\sum_{j=1}^j N_{21}^j$

{11}  
{12}

The numbers N12 and Nxl obtained in this way are called the natural solution of the present problem of the radiometrically interacting surfaces S, and S2. That N12 and Nz1 are indeed solutions of the steady-state interreflection problem associated with S, and Sz will now be shown. By starting with the definitional identity arising from-(11):

$$N_{12} = \sum_{j=1}^{\infty} N_{12}^j$$

we deduce the following chain of equalities:

... The last equality follows from (12) and the preceding definition of  $N_{12}$ . By comparing (13) and (4) we see that the natural mode of solution implies the interaction mode of solution. Evidently the steps in (13) are reversible, so that the interaction mode of solution implies the natural mode of solution. Thus the two modes of solution are equivalent in this case. Since the interaction mode of solution clearly represents the solution of the interreflection problem of  $S_i$  and  $S_i$ , the natural mode of solution therefore is also, by virtue of the preceding equivalence, a solution of the interreflection problem. This equivalence actually holds in very general settings and has been established in detail for these settings, in Ref, [251]. We shall have occasion to study and use once again this equivalence of the two techniques later in the present work. Finally, we observe that the sums in (11) and (12), being reducible to a simple geometric series with ratio  $Z_{11}E_{21}$  and initial term of the form  $(N_{4E_{ij}} + N_{EOE})$  ( $i=1, j=2$  for (11);  $i=Z, j=1$  for (12)),  $j_i j_{ij}$  are readily evaluated; these sums are given by (6) and (7).