3.7 Applications to Plane-Parallel Media

The application of the interaction principle to plane-parallel media, which is the main theme of this section, will perhaps be most interesting from the following two points of view. First, the interaction equations for interacting plane-parallel media will be seen to be identical in form to those for interacting planes illustrated in Sec. 3.4, This point of similarity of the two types of interaction equations for ostensibly dissimilar radiative transfer contexts should encourage a closer examination of the classical modes of solution of the problems associated with these media with the purpose in mind of obtaining a unified means of solution for both types of settings. The natural mode of solution, as we shall see, is one candidate for such a unification.

The second interesting observation that can be made about the illustrations below concerns the ontogenetical foundations of radiative transfer theory, that is, the basic concepts on which the theory rests. Advanced students of radiative transfer theory know that the classical framework of the subject can be made to rest on the equation of transfer for radiance and on the principles of invariance. For some time there was a question as to the primacy of one or the other of these concepts; which was more fundamental:' the equation of transfer or the principles of invariance? This question is naturally of interest to those who are concerned with the logical connections between these two tap roots of the subject, The principles of invariance have been developed and made a powerful tool of radiative transfer theory by Chandrasekhar

and his followers. The systematic use of the principles of invariance by Chandrasekhar's school has led to brilliant solutions of long standing intractable problems in radiative transfer theory, problems which were insuperable using the standard approach to them by means of the equation of transfer alone. In view of this fact, one is led naturally to speculate on whether the principles of invariance incorporate a deeper and logically independent insight into radiative transfer phenomena than does their predecessor, the classical equation of transfer. This speculation was examined in an earlier work (Ref. [251]) with the purpose of resolving the question of the logical status of the principles of invariance within the theory of radiative transfer. It was found that the principles of invariance were logically deducible from the equation of transfer. Moreover, it was found that, by suitably enlarging the domain of valid applicability of the principles of invariance--that is, by capturing their abstract essence in a sufficiently general physical setting--the equation of transfer was in turn logically forthcoming from these more comprehensive principles. The net result was the establishment of the logical equivalence of the principles of invariance and the equation of transfer. By continuing the abstraction process of the principles of invariance still further, the interaction principle eventually was attained.

With this brief historical sketch in mind, the following illustrations may take on a deeper significance than would a mere enumeration of examples of the application of the interaction principle. Thus, the illustrations are intended to summarize a unification and an extension, by means of the interaction principle, of the classical framework of radiative transfer theory on plane-parallel media. In subsequent sections, the illustrations will be extended to cover ever wider applications of the interaction principle.

Example 1. Irradiances on Plane-Parallel Media

We consider a general plane-parallel medium x bounded by two distinct transparent parallel planes, a and b, as in Fig. 3.19. The medium consists of scattering-absorbing material with no reflecting surfaces on the boundary or on parallel planes within x, and is irradiated by external sources only at its upper and lower boundaries. These sources are radiance distributions $N_-(a)$ on a and $N_+(b)$ on b of fixed directional structure and are independent of location on a and b. That is, the directional structure (but not the size) of $N_-(a)$ is fixed and is independent of position in a. Similarly for b. Furthermore, we assume the medium to be stratified, i.e., its optical properties and light field are independent of position on each intermediate plane at depth y within X,
a -s y: b . Let H (y , +) and 11(y,-) be the resultant irradiances at depth y (for notation, see Sec. 2.4). For brevity and uniformity of exposition throughout this chapter, we write
"$H_+(y)$ " for the irradiated H (y, +) , and "$H_-(y)$ " for the irradiance H(y, -) . Similarly "$w_+(y)$", "$w_-(y)$" will denote the ra
diant emittance of plane y in the upward (+) and downward (-) directions. The incident radiance distribution $N_-(a)$ induces

FIG. 3.19 The basic interaction setting for irradiances on plane-parallel media.

an irradiance which we denote by "H-(a)". Similarly, $N_+(b)$ induces an irradiance H+(b). Our main purpose in this example is to show how the interaction principle can be used to systematically formulate the radiative transfer problem on X for irradiance: given $H_-(a)$, $H_+(b)$; required, the irradiances $H_\pm(y)$ at every depth y, as y sb.
The application of the interaction principle is facilitated by having a convenient designation of the present optical medium X and its various plane-parallel subsets. Let us write "X(x,z)" for a plane-parallel medium bounded by planes x and z, a s x :sz ! b .
We use, for convenience, "x" , etc., now to denote both the plane and its depth in the medium. The present medium X is of the form X(a,b). As a first application of the interaction principle, we consider the subset
X(a,y) of X(a,b), as y fib. The enumeration of the sets of incident radiometric functions on X(a,y) in the present case is:

X(a,y) is

Ai: all irradiances like $H_-(a)$

A2: all irradiances like $H_+(y)$

The enumeration of the sets of response functions of

B_l: all radiant emittances like $W_+(a)$ B_2: all radiant emittances like $W_-(y)$

Thus, in the case of medium X(a,y), m = 2, n = 2, and the interaction principle supplies four interaction operators sib (in the present case these are numbers) of the form:

stj -- R(ay) s12 ~- T(a,y) -- T(y,a) s22 - - R(y, a)

The four numbers R(a,y), T (a,y) , T (y, a) , and R(y,a) can be represented, if need be, in terms of the S-function of Sec. 3.6 (see paragraph on Variations of the Basic Theme),

and the discussion of that section shows how they can come by an alternate route from the interaction principle. For practical numerical work, one may integrate the Riccati equations obeyed by R and T as shown in Ref. [251] and in Chapters T, 8 below. Hence these four numbers depend in a known manner on
the depth y in X(a,b), we have then four functions R[a,•], T (a, •) , T(' , a) , R(• , a) associated with X (a,b) which take on specific values for each choice of subset X(a,y) of X(a,b). For the purposes of the present example, we assume specific knowledge of these four functions. In later discussions, throughout Chapter 8, we will show how the functions can be obtained.
According to the interaction principle, W+(a) and W_ (y) are given by:

$W_+(a) = H_-(a)R(a,y) + H_+(y)T(y,a)$ (1)

$W_W W = H_-(a)T(a,y) + H_+(y) R(y,a)$ (2)

Hence if all six quantities on the right side are known, $W_+(a)$ and $W_-(y)$ are determinable. Of these six, five are known as given properties of X(a,y) or as given radiometric data. The remaining quantity, namely H+(y) is not generally known. This indicates that we should apply the interaction principle once again, now to the subset X(y,b) of X(a,b) ; a:sy s b,
Isolating X(y,b), we then enumerate the sets of all incident radiometric functions on X(y,b):
A,: all irradiances like H-(y) A2: all irradiances like $H_+(b)$. The enumeration of the response functions of X(y,b) is: B_Y: all radiant emittances like $W_+(Y)$
B2: all radiant emittances like W_(b). The associated four interaction operators s_{ib} are:

Sii -- R(y,b)

s~2 __ T(y,0b)

S21 -- T(b,y)

S22 - w R(b,y)

The interaction principle then states that:

$W_+ (Y) =$
$H_+ (b) T (b, y) + H_- (y) R(y j b)$

$W_- (b) =$
$H_+ (b) R (b , y) + H_- (y) T (Y, b)$

The auxiliary equations for the set (1)-(4) are:

$W_+(y) =$
$H_+(y)$

$W_- (y) =$
$H_- (y)$

which follow from the equality of field and surface radiance at a given point and in a given direction and the fact that no parallel planes within x(a,b) are reflecting planes (see statement (b) following (13) of Sec. 3.4, and see (32) of Sec. 2.5). With these two auxiliary equations, Eqs. (2) and (3) become autonomous:

$H_- (y)= H_- (a)T(a,Y) \quad + H_+(y) R(y,a)$
$H_+ (y) = H_+ (b) T (b ,Y) \quad + H_- (y) R(y,b)$
(7) (8)
These are the principles of invariance for irradiance in plane-parallel media. They will play an important role the .studies of Chapter 8. In essence, (7), (8) are two options for $H_+(y)$ with solutions:
in equation
$H_+ (Y)$

H_ (y)
H_+ (b) T (b,y) + H_ (a) T(a,y) R(y,b)

R(y, a) R(y,b)
H_ (a)T(a,y) + H+(b)T(b,y)R(y,a)
1 ^ R(y, a) R(y,b)
(lo)
From H_t (y) we
can determine W_+(a) and W
We conclude this example with several general
tions. First. of all, the complete solution of H+(y)
W_+(a), W_(b) is contingent on knowledge of the eight
tances and transmittances associated with the subsets

and x(y,b) of x(a,b). Methods of finding these numbers will be discussed in Chapter 8. Even without referring ahead to these methods, the following properties of these numbers can be brought to light. A reflectance such as R(a,y) is dependent not only on the material comprising X(a,y) but also the

directional structure* of N_(a) irradiating X(a,y). This fact may be checked by studying (13) of Sec. 3.6. It turns out that R(a,y) is not an inherent optical property of X(a,y) but rather only an apparent optical property (cf., Sec. 9.1). It is for this reason that we hypothesized incident radiance distributions on X(a,b) of fixed directional structure. The matter of R and T factors for plane-parallel media shall be discussed in detail in Chapter 8.
Secondly, we can see that the magnitudes of R(a,y) and R(y,b) must not exceed 1 if there is to be a determinate solution of H±(y). By appealing to the energy conservation law of general physics, suitably tailored to the radiative transfer context (Sec. 3.1), we can show that:
05 T(a,b) o 1 (11)
0:5 R(a,b) !i 1 (12)
for every plane-parallel optical medium X(a,b).
We can go further than (11), (12) and state that:
T(a,a) = 1 (13)
R(a,a) = 0 (14)
for every degenerate optical medium X(a,a). This state of affairs is expected since the transmittance of a transparent plane should be 1 and its reflectance should be 0. These limiting values follow directly from (1), (2) and the auxiliary equations (5), (6), after setting a = b and noting that (1) and (2) hold for every 1-1-(a) and H+(b).
Example 2. Radiances in Plane-Parallel Media
As a second illustration of the interaction principle we consider once again the plane-parallel medium X(a,b) of Example 1 but now with attention directed toward incident

and response radiance distributions. The incident external radiance distributions on X(a,b) and N_(a) and N_+(b) are arbitrary. The stratification assumption is now dropped. We require the determination of radiance distributions N±(y) over any intermediate plane y in X(a,b), as shown in Fig. 3.20.

We begin by partitioning X(a,b) into two parts: X(a,y) and X(y,b), a :s y -a -b. Isolating X(a,y), and enumerating the sets of incident radiance distributions, we have:

This is the case for the irradiance context and generally the radiometric quantities derived from radiance by various integral operations. The analogous operators R(a,y), etc_., considered below for the radiance context, are independent of the light field and are therefore inherent optical properties of X(a,y)

r

FIG. 3.20 The basic interaction setting for radiance distributions on plane-parallel media.

Al: all field radiance distributions like N _(a) Az: all field radiance distributions like N+(y) Enumerating the sets of response radiance distributions, we have:

B1: all surface radiance distributions like N+(a) B2: all surface radiance distributions like N±(y)

The four interaction operators sij are: s1l -- R (a s y) S22 T(ay)
sz1 '` T(yla) S22 -- R(y,a)

._.._..........b

The four operators above are integral operators as defined in (8)-(11) of Sec. 3.6. Hence these operators ultimately come from the interaction principle in the sense that the existence of the kernel function S(X;.,•-.,.) is guaranteed by the interaction principle for every optical medium (see Sec. 3.16).

The interaction principle then states that:

N_+ (a) = N. (a)R(a,Y) + N_+ (Y)T(Y.a) (l5)

N+ (Y)= N_ (a) T (a, Y) + N_+ (Y) R(Y, a) (16)

By repeating this process of application of the interaction principle to medium X(y,b), we arrive at the analogous pair of statements:

N+ (Y) =
N_+ (b) T (b,Y) + N_ (Y) R(Y pb)

N+ (b) =
N_+ (b) R(b,Y) + N_ (Y) T (Y,b)

When we append the following two auxiliary equations:

N+ (Y) =
N_+ (Y)

N± (Y) - N
_ (Y)

the set of equations (15)-(20) becomes autonomous. These auxiliary equations are specific instances of (32) of Sec, 2.5, and are a succinct way of ruling out the presence of any internal reflecting interfaces in the medium being analyzed. Thus, we see that, besides making the system (15)-(20) autonomous, the auxiliary equations allow us to write the system (15)-(18) in terms of surface radiance only, without possibility of ambiguity. It follows that N+ (Y) in (15) and (16)
i+ equal to N_+ (y) . N_(y) in (17) and in (18) is equal to N^+-(y). The incident radiance distributions N- _(a) and N+(b) are from external sources and are immediately convertible
to surface radiances using (32) of Sec. 2.5. Hence the surface radiance signature "+" may be dropped from the superscript position on "N". The set (15)-(18) thus becomes:

N_+(a) = N_ (a)R(a.,Y) + N+(Y)T(Y,a) (21)

N_ (Y) =
N„ (a)T(a,Y) + N+(Y)R(Yfa) (22)

N+(Y) =
N_+(b)T(bfy) + N_ (Y)R(Yfb) (23)

N_ (b) =
N_+(b) R(b.,Y) + N_ (Y)T(b,Y) (24)

The middle two equations are autonomous. Their solutions are;
N_+(Y)$^=$ [N+(b)T(b,y) + N_ (a)T(a,Y) R(Y,b)] [I - R(Y,a)R(Y,b)] -I
(25)
N_(Y)= [N_ (a)T(a,,y)+ N+(b)T(b,,y)R(Y,a)] [I - R(Yfb)R(Y,,a)] -, (26)
From (21) and (24) we can determine. N_+ (a) and N., (b) . The term [I - R(y,a)R(y,b))-; is understood to be the inverse of the integral operator [I - R(y, a) R(y,b)) , and the term
R(y,a) R(y,b) is the iteration of R(y,a) with R(y,b). That is, with the help of the definitions (8) and (10) of Sec. 3.6.
we have written:

"R(y,a)R(y0b)" for
f_w f $f_=$ f [ ] S (X (y, a) ; x" , $_V$' ; x' X) dA (x$^=$') dsI
(E") .. - ..+ Xy
S (X(y,b) ;x' ,, E' -x,, E) dA(x I) do(EI) (27)

in exact analogy to the iteration r_+ (a) r- (b) for surface integral operators in (67) of Sec. 3.4. A similar definition is made for R(y,b)R(y,a). Therefore, in order for the inverse of [I - R(y, a) R(y,b)] to exist, at least one of R(y, a) or R(y,b) must be norm contracting (re. (60) of Sec. 3.4). This condition is invariably found to hold in every natural (real) optical medium encountered in atmospheric and hydrologic optics. The theoretical details of iteration of (27) are covered in Sec. 3.4. Therefore if the norm contraction condition holds,

CI ` R(y,, a) R(y,b)] -1 = F_- [(y, a) R(y .,b)] 3 (28)

A similar equality is obtained by interchanging "a" and "b". Thus N+(y) is uniquely determinable via (25), (26) using the natural mode of solution. The practical truncation process discussed in Example 7 of Sec. 3,4 (cf. in particular (88) of Sec. 3.4) holds for the present setting also.

It may be of interest to observe that only the inverse operation (28) need be computed in order to find both N+(y) and N_ (y) by means of (25) and (26) when,say, N_ (a) = 0.

This observation is based on the identity:

$$B [I - AB]^{-1} = [I - BA]^{-1} B$$

which holds for every pair of operators A and B such that $[I - AB]^{-1}$ exists. By means of this, the operator combination

R(y,a) [I - R(y~b)R(y,,a)] -k

in (26) can be written:

~I - R(y, a) R(y,b)] $^{-1}$ R(y, a)

If, on the other hand, N_+(b) = 0, then only [I-R(y,b) R(y,a)] -' need be evaluated, for similar reasons.

We conclude by observing that the solutions N±(y) are predicated on knowledge of the operators associated with X(a,y) and X(y,b), in particular knowledge of the function S(X;•,.;.,_) when X is X(a,y) and X(y,b). We shall consider some means of arriving at this °knowledge in Chapter 7.

Finally, from the same considerations leading to (13) and (14), or formally from (25) and (26), or from Ia, Ib of Sec. 23, of Ref. [251], it is easy to deduce that the integral-operators T (a, a) , R(a, a) obtained by setting y = a satisfy the conditions:

T(a,a) = I (29)

R (a, a) = 0 (30) where "I" and "0" now denote the identity and zero operators, respectively, of the operator algebra of Sec. 3.4.

Example 3. The Classical Principles of Invariance

We pause in our illustrations of the interaction principle to show how the four classical principles of invariance emerge by applying the interaction principle to a suitably dissected plane-parallel medium X(a,b) without internal sources of radiant flux. Figure 3.21 exhibits the requisite partitioning of,X(a,b). We consider an arbitrary subset X(x, z) , which in turn is partitioned into X(x,y)and(y, z) , a ._ x ..~ y s z s b . Thus the geometric setting for the principles of invariance requires consideration of a partitioned internal slab X(x, z) arbitrarily located within X(a,b). This partitioning is of sufficient generality to subsequently allow functional relations to be written down for the four operators R(a,b), R(b, a) , T (a,b) , T (b , a) associated with a general plane parallel medium X(a,b),(see Sec. 7.1).

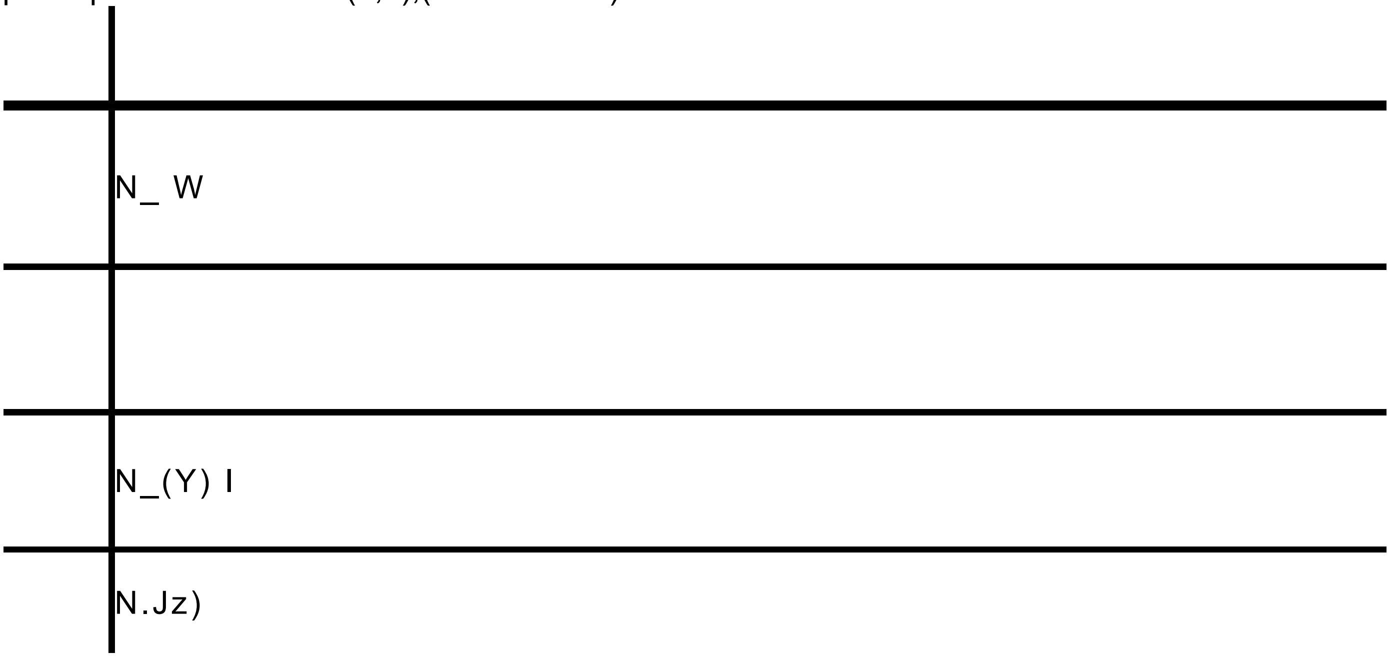

FIG. 3.21 The setting for the classical principles of invariance on plane-parallel media.

By repeating, line for line, the derivations of Example 2, now for X(x,z) (so that "a" is replaced by "x", and "b" by "z" and "y" is unchanged), we have the present counterparts to (23) and (22)

I• $N_+(y) = N+(z)T(z,y) + N-(y)R(y.z)$

II.. $N\_(y) = N\_(x)T(x,y) + N_+(y) R(y,x)$

These are the two main principles of invariance I and II. Principles III and IV are obtained from them as follows. For III, use I twice: first let y = a, z = b; then let y = a, with z arbitrary:

III. $N_+(a) = N+(b)T(b,a) + N_-(a) R(a,b)$

$= N+(z)T(z,a) + N-(a)R(a,z)$

For IV, use II twice: first let y = b, x = a; then let y = b with x arbitrary:

IV. $N\_(b) = N-(a)T(a,b) + N+(b)R(b,a)$

$= N\_(x) T(x, b) + N_+(b) R(b,x)$

Statements I-IV are the prineipU8 of invariance for a plane-parallel medium X(a,b). They are rules by which one can formulate the laws of radiative transfer on X(a,b), including the equation of transfer (cf. Sec. 25 and Sec. 126 of Ref. [251])e

The numbering of these principles is designed to facilitate their comparison with those in Sec. 50 of Ref. [43].

It should be noted that the present forms of the principles are written for generally inhomogeneous plane-parallel media so that four operators (rather than two as in [43]) are required for a complete determination of the light field in X(a,b). It should also be noted that the apparent simplicity and symmetry of I-IV above relative to their counterparts in [43] results from judicious use of operator concepts and also from leaving the light field in undecomposed form, i.e., the radiance distributions are not decomposed into reduced and diffuse flux (cf. Sec. 5.2). In this way the basic algebraic properties of the principles emerge and encourage formal manipulations such as those leading to (25), (26). (For the details of decomposition of light fields and their operators, see Sec. 7.1.) Of course, in the last analysis one must grapple with the realities of S(X;•,•;•,•). However, one of the, virtues of the interaction method resides precisely in its ability to defer such activity until the most propitious moment in a given analysis. In particular, one acquainted only

with the interaction principle and elementary algebra can analyze the most complex interaction problem to a stage typified by (25) and (26). From that stage onward, the natural mode of solution can be invoked for manual service, or service in automatic computer programs.

One further observation can be made which will facili tate the comparison of I-IV above with their classical coun terparts in [43]. Suppose, as in [43], that X (x , z) is homo geneous. Or. more generally, suppose that X(x,z) and the light field within X (x, z) are stratified, as x s z s b . We fur ther agree for the case where ' is in _ and & is in +, to write:

"R(x,z!C';0" f or lk' I f S(XIy' 0~' ;yft) dA(y') (31)

$_{X}X$

and for the case where ' is in $4 and E is in M „, to write:

"T(xizi.gl !.~)" for jk*&j f S(X;y' ,~' ;y,,&) dA(y') (32)

$_{X}X$

where the domain of integration is over the upper plane boundary $X\sim_c$ of X(=X(x, z)) at depth x. The point y in (31) is in X_x; the point y in (32) is in X_z. Two more definitions can be made in a similar manner for upward reflectance and transmittance functions. However, if X(x,z) is homogeneous, then it is easy to see that, under the present conditions, these functions are all non negative valued and depend spatially only on the difference z -x of the depth parameters for X (x , z) . (The reason for this will be established in Sec. 7.1.) Hence to homogeneous X(x,z) are associated two functions, the R and T functions defined above. The R-function in (31) is the present counterpart to the S-function in [43], and the T-function in (32) corresponds to the T-function in [43]. With the definitions (31) and (32) in mind we may represent the operators R(a,b) and T(a,b) (for downward incident flux) in (8), (9) of Sec. 3.6 as:

R(a,b) = [I R(a,b;t' ~t) dQ(t') jkstj

T(a,b) . f [ ]T(a,b;~',E) da(t') Ik*&I ~_

V

A similar pair of operators is associated with upward incident flux on X(a,b) 0

Example 4: The Invariant Imbedding Relation

The solutions of the radiative transfer problem in plane-parallel media X(a,b) studied in Example 2 above were accomplished by using the interaction principle to set up the interaction equations for each of two subsets X(a,y) and X(Y,b) of X(a,b). The resultant equations were then solved for the radiance distributions Nt (y) , as ys b. In the present example we will apply the interaction principle directly to a subset X(x, z) of,X(a,b) of Example 3 and ask it to give at once the interaction operators which yield Nj(y) at some level y within X(x,z). The resultant operator equation is called the invariant imbedding retation, and employs the important concepts of the complete reflectance and transmittance operators.

We begin with the setting of Fig. 3.21. The subset X (x , z) is isolated, as x z z s b . The sets of incident radiometric functions on X(x,z) are enumerated as:

Ai: all incident radiance distributions like $N_+(z)$

Ai: all incident radiance distributions like N_(x) .

The sets of response functions on X(x,z) are those at level y,

$x \leq y \leq z$:

B;: all response radiance distributions like N+(y)

Bt: all response radiance distributions like N_(y)

The interaction principle then asserts the existence of four interaction operators sib:

- 9"'(z:y.x) _ 4^7(Z 0 Y 9 x) Sz1¯ A(xoysz) S32¯ X(xoypz)

These four operators are not the simple integral operators of the kind in (8)-(11) of Sec. 3.5. Their structure will be considered shortly, For the present we go on to the assertion of the interaction equations in the present case. The interaction principle states that for N +(y):

$N_+(Y)$ = N+(Z)X(Z,Y=x) + N_(x)a(x,Yfz) (33)

N_ (Y)N+(z) a(z oYpx) * N_ (x) (x,Yfz)(34) This pair of equations can be written in matrix form. Let us first write:

I'M (x.Y, z) '1 for

X(=py.x) ~;?(y9Y9X)

(35)

.&'R(x,Y, z) T(x,Y * z)

Then (33), (34) become:

(N+ (y) N_ (Y)) = (N+ (z),,N_ (x)) ?tj(xmy,z)
(36)

Equation (36) is the invariant imbedding relation. Equation (36) is reminiscent of (10) in the preliminary example of Sec, 3.1 which resulted from considering the two surfaces S_1 and S2 of that example as a radiometrically self-interacting entity. Such an interpolation may also be made in the case of (36) ; that is, we imagine X(x,z) isolated with $N_+(z)$ and N-(x) as incident radiometric functions on x(x,z). Then the internal radiance distribution N(y), (=(N+(y),N-(y)) arises in response to this input by imagining the parts X(x,y) and x(y,z) to interact radiometrically. We denote the collection of all operators ?&(x,y, z) associated with x(a,b), and with
parameters x,y, z in the interval[a,b] , by 'r, (a,b)".

The four operators occurring in the invariant imbedding relation will now be related to the standard reflectance and transmittance -operators (8)-(11) of Sec. 3.6. No loss in generality is engendered in setting x = a and z = b in (36), since x(a,b) was initially arbitrary. The resultant equation is:

(^{N}+ (Y) N_ (y)) (^{N}+ (b) N_ (a)) ~(a py,b) • (37)

In particular, we have:

N_+ (Y); N_+ (b) '"(b,Y, a) + N_ (a) .6~(a mY,b) (^{38})

N_ (y)+ N+(b).(oT(b1,Y,a) + N_ (a) T(a.Y,b) . (39)

Now according to the interaction principle these Gland ,r operators are unique, and yield the functions N±(y) corresponding to every member of-the set of incident radiance distributions in Aj and A2. Thus, in particular by setting $N_+(b)$ = 0 (the zero radiance distribution) in (38) and doing likewise
in (25), we find that, since N_ (a) is arbitrary:

4(a my,b) = T(a my) R(y mb) [I $^{-1}$
- R(y f a) R(Y vb)]

In a similar manner we find:

(~;?(b,Y,, a) -
'~ T(b vY) R(y }
-
0'(a my,b) " T(amy) [I $^{-1}$
- R(Y fb) R(Ym a)]

9$^-$(b,Y,a) = T(bsy) [I $^{-1}$
- R(Y,4a)R(Y,b)]

The operator R(a,y,b) is the complete reflectance operator and T (a,y,b) the complete transmittance operator. Equations (40)-(43) show how these new operators may be constructed from the standard operators of Sec. 3.6. The methods of Chapters 7 and 8 will show how Zok and 7' may be obtained directly by integration of the equation

of transfer in two-flow form. We can now rewrite, if necessary, (40)-(43) for a general sub
set x(x, z) of x(a,b) . From (29), (30) and (40)-(43) we deduce for every x, z , as x :sz: b ; that

Z' (x ,,z .,z) = T (x, z) (44)

.~e(x,x, z) = R(x, z) (45)

(x ,x, z3 I (46)

J~0P,(xlz,z) 0 (47)

The invariant imbedding relation (36) contains the four principles of invariance I-IV of Example 3 as special cases. Thus, let x = y, then (33), (44), (45) yield:

• $N_+(y)$ '~ N+(z)T(z,y) + N_ (y)R(y, z) Further, let z = y, then (34). $(44)_9$ (45) yield: I.I. N_(y) " N_ (x)T (x,y) + N_+ (y)R(y f x)

From these first two principles of invariance follow principles .III and IV after the manner explained in Example 3.

Examples of the use of the invariant imbedding relation in extended computations in discrete space settings may be found in Sec. 70 of Ref. [251]: Further examples are given in Chapters 7 and 8 below.

We conclude this example with a few observations of historical interest. The invariant imbedding relation (36) was first given in Ref. [233] in an attempt to put into precise analytical form the verbal statement of the invariant imbedding principle of Bellman and Kalaba [13]. The latter principle was, in turn, an extension of the ideas of Ambarzumian [1], [2] and Chandrasekhar '[42] centering around the classical forms of the principles of invariance. The work of Bellman and Kalaba was an important impetus to the eventual formulation of the invariant imbedding relation. This relation, in turn, motivated the algebraic formulation of the classical radiative transfer principles. This algebraic formulation evolved and eventually culminated in the interaction principle of Ref. [251], which is the foundation of the present work.

Example 5 Semi-group Properties of Transmitted and Reflected Radiant Flux

One of the most primitive of intuitions we have of light in natural optical media such as the atmosphere or the sea is that of its transmission from one point to another.

In this example we show how the invariant imbedding relation yields a general analytical embodiment of this intuitive idea of transmitted light. In particular we shall derive the exact analytical expression of the following property of transmitted radiant flux: the amount of light transmitted over a path from point a to point b is equal to the amount first transmitted from a to an intermediate point y and thence transmitted from y to b. This type of property will eventually

yield the volume attenuation function, one of the two main inherent optical properties used, in the equation of transfer.
To derive the transmission property of radiant flux we return to the invariant imbedding relation (36) applied to an arbitrary level y in X(a,b) and with N+(b) 0 and N_(a) arbitrary:
(^{N}+ (Y) ,N_ (Y)) ' (^{N}+ (b) ,N_ (a)) M(a,,y,,b)
'~ (N_ (a) a (a.Yvb) 0N_ (a) mr(a y0b)) . (48) Using (36) once again now for level b in X(Y,b) with N+(b) =0: (N $_+$(b) 0N_ (b)~ ' (N_+ (b) 9N_ (Y)) 21? (Yjb0b) '~ (00N.(y))~(yjbjb)

(0jN.(y)) 0"(b,bfY) A(b,b ;Y)
&?(Y.b,b) 7(y,b,b)'

(0,N_ (Y) tT(Y9bsb))

From this, we have:

N_ (b) ' N_ (Y) Lr($y_o b_o$b)

From (48) we have

N_(Y) = N_(a) ,~7(a,Y,b)

Hence

N_ (b) ' N_ (a) .7(a,y,b) ^{7} (y,b,b)

From (48) once more, now applied to level b in X(a,b) N. (b) = N„ (a) Zr(a,b,b)

From this and (54), since N_(a) is arbitrary
(a,b ,b) " '(a,Y,b) (Y ,b sb) (51)
which is the desired semi-group property of the complete transmittance operator. Observe that the last "b'' in each transmittance operator is fixed and plays the role of a passive background parameter indicating the size of the medium in which the transfer takes place. When reading the equations, the attention of the reader should be directed to the first two parameters in each transmittance operator; then it will become clear that (51) indeed expresses in a very general form our basic intuition of transmitted radiant flux. Once the idea of the derivation is clear, the reader may derive a slight generalization of (51) wherein the subset x(a,z) replaces x(a,b) and y is an arbitrary level between a and z, The result is:
$5^{1,0}$(a, z,b) _ X(a.Y,b) ~7(Y t z,b)
(52)
Another semi-group relation similar to (51) but now for transmission from b to a is derivable from (36). This is left as still another exercise for the reader. A further relation is forthcoming from (36) which exhibits an interesting quasi-semi-group property for the complete reflectance operators:

.6?(a, z b) = 51"'(a,Y,b) ./,4R(y, z,b)
(53)
for every level y in an arbitrary subspace x (a, z) of x (a,b) .
The setting in which the semi-group properties for the complete operators .? and r is best viewed is that of the generalized invariant imbedding relation which is considered in 'Examples 6 and 7 below. Furthermore, the full semi-group relations for members of the partial group r_3(a,b) are developed (in the irradiated context) in Example 4 of Sec. 8,7., See in particular, (52) - (55) of Sec, 8.7.
Example 6: The Generalized Invariant Imbedding Relation
The generalized invariant imbedding relation, which we now consider, is the result of an attempt to increase the structural symmetry and comprehensiveness of (36) and such semi-group relations as (52) and (53), The setting for the present example is again that of Example 4: a general plane-parallel optical medium x(a,b), with no internal sources of radiant flux, and irradiated only at its upper and lower boundaries x_a and X~ by N_(a) and N+(b), respectively. We shall work with radiance distributions N±(y) on arbitrary levels y in X(a,b). Our goal in this example is the derivation of a generalized version of (36) which has greater analytic power and symmetry than (36), This will be bought,

however, at the expense of some intuitive value of the result. However, the sacrifice is soon lost sight of in the glare of the analytic and algebraic light shed by the operatorial degrees of freedom opened up by the extension.
We could begin the derivation of the generalized invariant imbedding relation by simply invoking the interaction principle with the appropriate choice of incident and response

functions. However, the requisite choice of incident and response functions is not immediately intuitively clear, and a few motivational comments in this direction will now be made.

An examination of the invariant imbedding relation (36) shows that the matricial operator 2t1(x,y,z) may be viewed as a single interaction operator mediating between the single incident radiometric function (N+(z),N_(x]] and the single response radiometric function CN+ (y), N,. (y)] . This follows from the fact that the purview of the interaction principle need not be restricted to the run-of-the-mill kind of single radiometric functions. Indeed, its statement allows, for example, ordered sets of any finite (or infinite) number of radiance functions to play the role of a single incident radiometric quantity. Continuing to examine the invariant imbedding relation (36), we see that the planes x,y,z are customarily con
strained to lie in the relations x s y s z to each other. Furthermore, the response function [N+(y),N_(y)) is limited to a single plane y in X(a,b) .

Suppose we now relax the latter of the two conditions just cited. Then we would be supplied by the interaction principle with an operator s such that

$(N_+ (W), N_- (x))$ = (N+'(u),N.(v))s

where w and x are in the interval defined by [v,u], v n u. We could go on to explore the properties of the four components

s ~ - of s in much the way we did those of ~ (x,y, z) in Examples 4 and S. But we wish to go one step further and relax the remaining condition on the incident and response functions. We now do not even require w and x to lie in [v,u], nor do we even require that v : u. The resultant operator arising from these relaxed conditions is the desired operator ?7(v,x;u,w) of the generalized invariant imbedding relation.

We now isolate X(a,b) and for an arbitrary pair of depths u,v in X(a,b) we enumerate the following sets of incident functions:

A,: all incident radiance distributions like $N_+(u)$

Ai: all incident radiance distributions like $N_-(v)$

Next we consider for an arbitrary pair of depths w,x in X(a,b):

B_l: all response radiance distributions like $N_+(w)$

B2: all response radiance distributions like N_-W

Then the interaction principle asserts the existence of four unique interaction operators.:

sl i -- jr"(u,W;v,x)

S 1 , 2 " - & * R (u , x ; v , w)

a (v , w ; u , x)

w ,'(v,x;u,w)

These operators depend generally on all four parameters, u,v, w,x. The choice of arrangement of the parameters in the symbols is guided by the resultant increased facility in establishing and reading the group properties of the operators presented below. For the present the individual parametric groupings simply may serve as long names for the operators. These operators are called the extended reflectance (,q) and extended transmittance.(X.) operators. The interaction principle goes on to state that for N_+ (w) , N- (x)

N_+(w) = N+(u) O'(u,w;v,x) + N-(v),6?(v,W-u,x) (54)

N_ (x) = N_+(u) A(u,x;v,w) + N_ (v) ,7(v,x;u,w) . (55)

This pair of equations can be written in matrix form. We first write:

it ~(v,x;u,w) 11

for

Y (U , W ; V , X) . ~ (u , X ; V , W) O(v,w;u,x) ~ r (v,x ; u,w)

. (56)

Then (54) , (55) become

(N+ (w) ,N_ (x)) = (N_+ (u) ,N_ (v)) `'l (v,x;u,w)

(57)

Equation (57) is the generalized invariant imbedding relation. We denote by 'r$_4$(a,b)" the collection of all operators

(v,x;u,w) with each variable u,v,w,x in the interval [a,b].

Before going on to deduce various consequences from (57) it may be well to add some explanatory comments on the structure of (57) in addition to those motivating its deduction from the interaction principle. The question that seems most likely to arise i s t h i s : if it was assumed that N_(a), N_+(b) were incident radiometric quantities on x(a,b), and if X(a,b) was isolated just prior to the invocation of the interaction principle, why weren't the radiance distributions N-(a),N+(b) explicitly counted among the sets of incident quantities on x(a,b)? The answer to this question is that the choice of the sets of incident and response radiometric quantities on an isolated subset of an optical medium is quite arbitrary and subject only to the choice of the user of the interaction method. A rereading of the interaction principle at this time may help make this answer clear. There is, in short, a precise logical basis for (57) in the statement of the principle, and (57) follows mechanically, so to speak from the principle under the present choice of the sets A andB.

The answer to the preceding question can be put into. a more intuitive, less formal tone by means of the following observation: the light field within X(a,b) it is true, is generated ab initio and sustained by the hypothesized incident radiance distributions N-(a),N+(b)• Once generated and in the steady state, the light field within x(a,b) has a strong internal structural unity in the sense that the radiance distributions over any-two separate planes of X(a,b) are closely and subtly interconnected one with the other; that is to say, the slightest change in the lighting over one plane is generally accompanied by a readjustment of the lighting in the other plane. Equation (57) is the formal expression of this intuitive insight into the internal unity of natural light fields. By our separating radiometric cause and effect in this extreme manner, the extended reflectance and transmittance operators of (57) have placed on them a relatively heavy burden to connect these distant radiometric causes and effects in one part of the light field with another. However, it is enough that the analytic connection--however complex or tenuous in reality--exists; for then a rich analytical harvest of results and techniques are available for use, especially those in the theory of continuous groups and semi-groups, and which we shall state below and in subsequent examples, The first deduction we wish to make from (57) is the invariant imbedding relation (36).

The details ,of this deduction will add substance to the general comments above concerning the internal unity of the light field within X(a,b). It appears that for didactic purposes the deduction of the invariant imbedding relation from (57) is best made in reverse -that is we shall start from the invariant imbedding relation (36) and deduce (57). Then it will be observed that the path traversed from (36) to (57) is reversible. This mode of approach to (57) was the one actually followed in its discovery. Toward this end we use (36) to represent each of the radiance distributions occurring in (57):

N_+ (w) N_+ (^{b}) ~"(b.w. a) + N.. (a) ~(a.w ~b) (58)

- ."'(a,x,b) + N_+ j?(b ix! a)

N $_+$(u) '~ N (b) X(b,u,a) + N.. (a).i~?(a,u,b) (60)

N_- (v) '~ N_- (a) g'(a,v,b) + N_+ (b)a(b,v, a) (61) Equations (54) and (58) are two ways of representing N_+(w).

Let us use (60) and (61) to replace N_+ (u) and N- (v) in (54) a s follows:

N_+(w) n N_- (a) f . T (a , v , b) , g (v , w ; u , x) + 4 R (a , u , b) v r ? " (u , w ; v , x)

+ N_+ (b) Jg (b,u, a) ,~r(u,w;v,x) + .R(b ,v, a).t (v,w;u,x) J (62)

Since N- (a) and N_+(b) are arbitrary, (58) and (62) imply:

.T(a.v.b) A(v,w;u,x) + 0(a,u,b) X(u,w;v.x) wj1(a.w.b) (63)

7(b.u.a) X(u,w;v.x) + a (b . v . a) . (v . w ; u . x) ' . , T (b . w , a) (64)

Next we observe that Equations (55) and (59) are two ways of representing N- (x) .

We use (64) and (61) to replace N+(u) and N- (v) in (55)

N_- (x) ' N.. (a) j AR(a.u.b).0(u.x;v.w) + :r(a,v,b) X(v,x;u.w) I

+ N+(b)j 9'(b,u.a), (u,x;v.w) +.'(b.v,a) %7(v.x;u,w))

Since N- (a) and N_+(b) are arbitrary, this result and (59) imply:

6?(a,u,b)4R(u,x;v,w) + Zr(a,v,b) ,T(v,x;u,w) ' X(a.x,b) (65)

.7'(b.u.a) sa(u,x;v,w) +a(b.v.a) #*Y-(v,x;u,w) ' a(b,x,a) . (66)

The sets of equations (63) , (64) , and (65). (66-) govern the extended reflectance and transmittance operators in terms of the complete reflectance and transmittance operators. We may view these equations in the present discussion as algebraic equations in the unknown extended operators with the known complete operators as "coefficients". This view is heuristic and will lead us correctly to results which can be established
rigorously using advanced operator theory. Thus we are led to write (63) and (64) in matrix form:

7 (b , u , a) . (b .v. a) j j,7(u,w; v,x) ~
F , r (b e w , a)
60 on$
a(a,u,b) g(a.v,b) I k g p , (v , w ; u , x) C
st(a• w, b) b)
. (67)

Let us denote the 2x2 matrix of operators in (67) by " ??(a,b;u,v)". This matrix has an inverse provided
I - fj(a,b;u,v) is norm contracting, a condition which can be shown to generally hold in all natural optical media. Hence:

" [I - (I - ~l(a,b;u,v))]

and by the norm contracting theorem (see, e.g., Sec. 40 of Ref. [251]):

r
00
_:(a,bfu.v) . (I - ?7(a,b;u.v))J (68) J00

Therefore

(69)

Next, we go on to write (b5), and (b6) in matrix form:

~"(a,v,b) ,4R(a.u.b) %07(v,x;u,w) T, (a ¯x:b)

. (b.v.a) (b.u.a) a(u,x;v,w) 'R(b,x,a)

whence:

·7(VVx^{1}'U3'w g(a,x,b)

'M 77 $^{-1}$

a(U,X;V,W) .R(b,x,a)

Equations (69) and (71) express the f-our extended operators in terms of the complete operators in a manner analogous to that in (40)-(43) wherein the complete operators were represented in terms of the standard operators for subsets of X(a,b) of the form X(a,b), X(y,b). Hence we may conclude that: in all natural optical media, the extended operators of the general invariant imbedding relation are ultimately representable in terms of the standard operators of the form M-01) o f Sec. 3.6. An alternate proof of this conclusion, along with specific formulas establishing the asserted representations, is given in Sec. 7.4.

Some observations on the preceding results will now be made. One observation that is immediately forthcoming from (69) and (71) is the somewhat startling fact that Or(u,w;v,x)
and 0(v,w;u,x) are independent of x, and C-(v,x;u,w) and
T_(u,x;v,w) are independent of w. However, some reflection
on the equations (54) and (55) and the choice of notation will show that there is no compelling reason why one response function should depend on another response function. Dependence of response functions on chosen incident functions must certainly be the case, but not necessarily on response functions. Therefore the right-end variables x and w in (69) and (71) are superfluous in the extended ~and 7 operators in the sense just observed, The extended notation
(v,x;u, w)" with all four variables shown is still desirable for reasons which will become clear in the group theoretic discussions of Example 7 below. Hence the individual extended operators inherit an added loose variable which, like a human appendix, has meaning only when the entire domain of evolution of the operator ;~7(v,x;u,w) and the radiometric activity over X(a,b) is considered. This freedom of choice of the right-end variables will be utilized in deriving special semi-group relations subsequently (cf., e.g., Ex. 4, Sec. 8,7).

The second observation is on the manner in which the extended operators reduce to the complete operators upon suitable confluence of the variables u,v,w,x. In this way we fulfill our obligation of showing that the invariant imbedding relation is a special case of (57). This can most readily be seen by returning to (63)-(66). For example, let w = x, a -= v, and b = u in (63) with ' a.. w!sb . The result is
~'(a, a,b)k((a,w;b,w) + a(a,b,,b),?,'(b,w';a,w) = ,6?(a.w.b)

SEC. 3.7 APPLICATIONS TO PLANE-PARALLEL MEDIA

From (44)-(47) we have

~a,w;b,w) _ a(a,w,b)

The three remaining relations are obtained similarly:

$a(b,w_V\text{-}a,w) = \text{-}\sim,?(b,w,a)$

L r (a , w ; b , w) = ' T - (a , w , b)

'7-(b,w;a,w) = ~'(b,w,a)

Hence:

,T(b,w.,a) ZR(b,wfa)

';~`l(afw;b,w) = 7)~(a,w=b) a

~(a,w,b) 7 (a , w , b)

which is an instance of the operator (56). It is clear that by suitable choice of parameter values in (69) and (71) we can once again retrace our steps to the invariant imbedding relation. However, the route just taken is certainly equivalent and somewhat less arduous.

Finally, it should be observed that in general we need not restrict the parameters a,w,b in (76) or in (72)-(75) to have the orders a :sw :sb or b s w !sa . Thus it is possible to explore the properties of r3(a,b) under less restrictive cond i t i o n s o n the three parameters than given at the outset in (36). One such extension will be made in Sec. 7.4 as a matter of course. However, unless specifically noted otherwise, we shall work only with members of r_3(a,b) whose parameters x, y, z are ordered either as x s; y !:-z or z s y :=-x . In the following example we shall introduce a new set (namely r2(a,b)) of operators which will supply a powerful working tool free from any restrictions on the parameters of the operators.

A final word on the choice of notation for 77(v,x;u,w) may be in order. A choice was made in (56) between the displayed order of variables and the alternative "9,~1(u,w;v,x)". The latter would look more natural in (57). However, our current choice works better in remembering the reductions (72)-(76), and was accordingly made with that in mind.

Example 7: Group-Theoretic Structure of
Natural Light Fields

The interaction principle, via the generalized invariant imbedding relation of Example 6, leads to some interesting properties of natural light fields--group theoretic properties--which appear to offer not only some novel analytical means for the numerical determination of light fields in practice, but also some fundamental ways of formulating radiative transfer theory. We illustrate the basis of these new means in this example. The sense in which we use the term "group theoretic" is best explained by going on directly to the derivations of these properties, Some further discussion will follow the derivations.

The group-theoretic structure will first be considered for a relatively easily visualized case--cone which we shall return to in Chapter 7 (in particular Sec. 7.11) as a base for a new computational method of determining radiance distributions. For the present we are concerned only with the bare logical structure of that method. Our setting may be a general plane-parallel medium X(a,b)--that of Example 6 once again--

or it may be a general one-parameter optical medium (ref.: Ex. 2, Sec. 3,.9). We begin by setting w = x and u o v in (57). The result is.

(N+(xl,N.(x)) = (N+(u),.N.(u))7~(u,x;u,x)

(N + (u) , N - (u)) M (u , x , u) 0 (77)

The latter equality follows formally from (76). For brevity, let us write:

'IN (Y) [11] for (N+ (y) p N. (y))

for every level u in X(a,b) and ad hoc.

"?IT (u,x) " for 7/7(u,x, u)

Then (77) becomes:

N (x) " N (u) 9((u, x) (78)

for every level x and u in X(a,b). Before going on, it must be pointed out that equation (78) has been obtained from (57) by a purely formal tactic--that is, the operator ?yr(u,x,u) was obtained by a formal change of parameters in 74(v,x;u,w), and it turns out that '/!P((u,x,u) as we have agreed to use it (i.e., as a member of rs(a,b)) is strictly not defined for x ¢ u. This may be seen by recalling the usual domain of definition of the operator ?~7(x,y, z) of the invariant imbedding relation in Example 4, wherein x,y,z are constrained to have the relations x s y sz. However, our goal at present is to draw certain necessary conclusions from (78), assuming it is possible to extend the domain of the invariant imbedding operator (35), and we shall now use the observations made at the close of Example 6 to justify this extension. The rewards for such a tactic are occasionally great and one interesting precedent for such tactics was in drawing certain necessary conclusions from the equation $x^z + y = 0$, assuming it possible to extend the meaning of .the equation to positive numbers y and values of x other than real numbers. The result, as is well known, was the theory of complex numbers. It is in such a heuristic spirit that we now proceed, using (78) as a premise.

Select any three levels x,y,z in X(a,b) and apply (78) to these levels in the following way.

N (Y) = N (x) 71~(x f Y) N (z) ' N (Y) '?1(y z) N(z) ' N(x) ?~(x,z)

From the first two of these equations.

N(z)

` (N(x) [2]j(x,y)) [7,0]7 (y , z)

= N (x) (~ f (x . Y) ;;% y . z))

Comparing this with the third equation in the preceding group, and using the uniqueness of interaction operators, we have:

'0~(x q z) ' ;V x 1Y) 7~(Y, z)

(79)

This is the closure property of the operators of the type M[u,v].. Furthermore, uniqueness also yields the fact that: for every level w,x,y,z in X(a,b)

P)r(wPx)?VxVY))';Ny9z) _ ?r(w,x) (5Nx)Y) 7I(Ys z)

(80)

which lis-the associativity of the operators of the type ?I7[u,), In addition for every level x in X(a,b)

(81)

Equation (81) expresses the identity property and I is the identity operator, where I± are identity operators which act on upward (+) and downward (-) radiance distributions. Finally, for every level x,y in X(a,b):

7 / 7 (^{x} tY) 7T(Y 0 x) a ? N x P X) W I
(82)

which is the inverse property. If we now denote by 'r_2(a,b)" the set of all operators of the form M(x,y), a 5 x 5b, as $_{ys}$ b, we see that f2 (a,b) forms a partial group in the sense that the closure, associativity, identity, and inverse properties hold. The product of elements -l(w,x) acid 7rl(y, z) of T2 (a,b) is defined whenever x = y or w = z. (Mathematical readers will recognize r2(a,b) as an instance of a local topological group under suitable regularity conditions. See, e.g., t208l. Physicists will note the pertinence of the group-theoretic approach to the foundations of quantum mechanics. See [150] and problem IV of [251] .)

The findings so far may be summarized as follows: by setting w = x and u = v in the generalized invariant imbedding relation we obtain an equation (78) which displays a formal extension '/7t(u,x) of the invariant imbedding mapping '/~t(x,y,z) and which shows that the set T2(a,b) of such extended operators has group structure. Its differential properties will

be explored in Sec. 7.S. These will lead to practical methods for numerical studies of r2(a,b).

The reader will now find it instructive to return to the generalized invariant imbedding relation and use the alternate definition:

"M(u,x)" for ?M(u,x;u,x)

for M(u,x) with no restrictions on u,z in the interval [a,b]. Alternatively,. one may return to the interaction principle, using collections of radiance distributions like N(u) for the incident set A, and collections of radiance distributions like N(x) for the response set B. We choose to call the resultant interaction operator s 1 I so obtained, by the same name, " 2~P((u,x)"; there should be no confusion henceforth with the ad hoe definition of W(u,x).above, which has finished serving its introductory purposes. In either of these ways the reader can place on solid footing the heuristic procedure between (78) and (82). It is also readily seen using the relation (78) and the properties of the invariant imbedding relation that the new operators M (u , x) i n r_2(a,b) for X(a,b) have the following representation in terms of the invariant imbedding operators 7 ~ P (x , y , z)

(u , x) ~ - ; (a , u , b) 7~^{1}(a,x,b) (83)

for every- u,x, as u 5 b, a :sx s b, provided Z7(a,u,b) or I - '7/7(a,u,b) is norm contracting. (See (68) for the general case.) Equation (83) will be established as a matter of course in (40) of Sec. 7.4.

We now can make clear the opening statement of this example, to the effect that natural light fields have group-theoretic properties. From (78) we see that if we fix the parameter u, say let u = a, then the radiance distribution N (x)at every level x in X (a, b) , is associated with a unique interaction operator 7,,1j(a,x), such that (a,x) is a member o f the partial group r 2 (a , b) . As a consequence of this, if N(x) and N(y) are radiance distributions at any two levels x and y in X(a,b), they are connected by the group products:

;~j (x f a) /-*"[' (a v y) (m 7 - 4 r ' (x , y))

or

M (y f a) •`[(a ~x) (" (y s x))

which are clearly group inverses of each other. The first of these acts on N(x) to yield N(y), the second acts on N(y) to yield N (x) .

The possibility of such intimate group-theoretic interconnections between one part and another of a natural light field such as that summarized above stems from the fact that a given natural light field in a well-defined optical medium with non-pathological values of its optical constants or its geometric form, has a strong inner structural bond, so to speak, such that if the light field over a small region is known, the form of the light field in the remaining regions of

the medium can be inferred. This property was commented upon once before in the discussions of Example 6. Because of the importance of this property, we shall pause to discuss it in some more detail by means of two main examples, the first of which is as follows.

As an everyday example of what is meant by "strong inner structure" of some conceptual object, consider a sphere. To make the illustration take on practical overtones consider one of those silvered spheres seen reposing on concrete pedestals in certain gardens. Suppose that one such specimen is dropped and shattered. The gardener takes a remnant shard to a dealer in silvered garden spheres. The dealer is now confronted with the task of inferring from the shard the diameter of the sphere from which it came (so that the new sphere will sit on the old concrete pedestal). Now, such dealers are aware at least on an empirical level, of the "strong inner structure" of spheres in the form of their constant positive curvature over their extents. Hence if the curvature of the shard is estimated (and there are little tripod-like devices which are designed just for such tasks) this estimate is then the numerical reciprocal of the radius of the entire sphere.

Perhaps the preceding illustration will now serve to direct with new insight the-reader's attention to the matter of verifying the intuition of the "strong inner structure" of natural light fields. Mathematical readers will recall that an outstanding example of such a property is possessed by analytic functions on open connected sets of the complex plane.

Further examples of strong inner structures in everyday light fields can be verified without excessive theoretical preliminaries: take the case of a finite sphere of uniform surface radiance in a vacuum. By measuring the normal irradiance at one point at a known distance from the center of the sphere it is possible to infer the normal irradiance produced by the sphere's radiant output at every other point in the space around the sphere (cf. Example 4, Sec. 2.11). Further, if the radiant intensity of the sphere is known, then by measuring the normal irradiance over a known interval of distance, however small, the reconstruction of the whole irradiance field is possible. The light field in this case is represented by an analytic function of a simple kind. The approximate but practically effective exponential law of decay of downward irradiance with depth in natural waters is still another basis for a group-theoretic property of natural light fields. We shall return to these ideas in Chapters 7 and 8. For the present we give one final illustration of a group theoretic property of light fields.

The second of the two main illustrations of the group-theoretic structures of natural light fields to be given in this example will now be considered. Recall that ,r4(a,b)" denotes the set of all operators of the form 7,~('(v,x ; u,w) , with u,v,w,x arbitrary levels in X(a,b). Then, as in the case of r_2 (a,b) , the set r_4 (a,b) is a partial group in the sense that the closure property (79), the associativity property (80), the identity property (81), and the inverse property (82) can be verified to hold for r_4(a,b). In particular, the closure

property is of the form:

(v'z;ufY) = (vix;u'w) 0'(x' z;w=Y)

(84)

using standard matrix multiplication.

Here., at last, the full role of all four parameters defining the members of r4(a,b) can be seen. Notice, e.g., how the transition from v to z on the left is made on the right side of the equation in two steps: v to x and x to z, similarly for the transition from u to y.

From (84), four operator equations are obtainable, namely the fully symmetrized fourth-order versions of (52) and (53), and the proper generalizations of (63)-(06). These are

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This set of equations or (84) will be called the fourth-order semigroup relations.

The preceding two main illustrations of the group theoretic structure of natural light fields constituting the present example will serve to show the novel directions in which the interaction principle can lead us. In particular we are led toward useful modes of representing and computing light fields in radiative transfer problems. The operators introduced in these illustrations will be explored further in Chapters 7 and 8.

We close this discussion on the group-theoretic structures of various sets of interaction operators with some observations on general partial groups of the form r3(a,b); i.e., the collection r s (a,b) of operators M (x,y, z) where x -.r,-y - z. First of all, r3(a|b) does not satisfy the group properties of the type (79)-(82). Under the standard definition of matrix product the closest we can come to closure--the first important group property required of r_9(a,b) is:

~(a,z,b) = 71((a0Yvb) 7I(Y,z.Y) (89)

which clearly requires y = z for the second operator on the right to be defined in the usual invariant imbedding relation. This special relation follows immediately from (84) and (76). However, despite the failure of r3(a,b) to have group structure (a defect adequately remedied by rz(a,b) and r4(a,b) introduced above) using standard matrix multiplication, the

component a and O .operators of /W(x,y, z) have the important and useful semi-group properties (52) and (53). Still further group-theoretic ,possibilities for interaction operators of rs(a,b) will be studied in Secs. 7.4 and 8.7. In particular, certain "non standard" products will be defined in order to obtain various desired group structures.

Finally, we observe two useful connections between the members of r :j(a,b) and r. (a, b) which follow from (76) and (84). The first connection links two members of r3(a,b) via a member of rk (a,b) ˉthus

27(a.Y,b) T(a,x;b,z) /"J(x,Y,z)
(90)
where a 5 x Ey!sz fib. The second connection links two members of r_4(a,b) via a member of r3(a,b) thus:

,7(a, z fY) ?~(a,x,b) ?~(x, z x fY)
(91)
where levels x,y and z are arbitrary within X(a,b).

Group Theory, Radiative Transfer and Quantum Theory

In conclusion, it Can hardly be overemphasized that the group-theoretic formulation of radiative transfer problems in the manner of this example, and as summarized in (79) (82), gives rise to perhaps the most fundamental physical formulation possible at present. This formulation, when suitably generalized (see (102) of Sec. 7.4), begins to indicate a completely unified approach to radiative transfer theory foundations which can be made to rest on quantum mechanics. A study of Landé's formulations of quantum mechanics will illustrate the general manner of approach envisioned. See Chapter VI of [151 and in particular equation (5a), which can be written in matricial form as ID (A, C) = * (A, B) * (B , C) , and which may be compared with (79). It now appears possible that problems II and IV of Sec. 141 in Ref. [251] may not only be successfully solved, but also in an elegant, unified mathematical manner. The program would in outline be as follows: starting with Land6's quantum mechanical formulation, one derives r_2 (a,b) (and, for generality, also its coherent-flux form), and obtains (79)-(82). Then using the techniques to be developed in Sec. 7.4, one can construct the operators in r4(a,b). These operators yield all the operators of radiative transfer theory presently known, including r,/(x;a,b) as defined in (6) of Sec. 3.8 or in Sec. 25 of [251]. Using the steps of Sec. 3.15 below or those of Sec. 126 in [251], one arrives at the equation of transfer. Once the equation of transfer is obtained, then the gateway to the classical theory has been entered.

The approach to the foundations of radiative transfer just outlined is but one of the many possible approaches that

may be developed. The preceding approach is specifically designed to play up the deep group-theoretic similarities of the quantum and phenomenological levels of radiative transfer theory. Unquestionably, the simplest connection between quantum mechanics and radiative transfer theory--the connection that would require a minimum of re-doing of existing constructions, is that which would derive the interaction principle (Sec. 3.2) from the tenets of quantum mechanics with a specific representation of the interaction operator in terms of the quantum properties of matter. Then all the constructions of [251] and the present work would stand ready-made for use without any further effort. In this way one can go on to solve important remaining problems of radiative transfer theory (Sec. 141 of [251]) with a minimum of duplication of effort. For further observations on the similarity of the structures of radiative transfer equations and quantum dynamical equations, see the closing remarks of Sec. H.2.