3.9 Applications to General Spaces

The applications of the interaction principle will now be extended to general optical media. We will begin with some relatively simple but important extensions of the principles of invariance to curvilinear media such as spherical, cylindrical and toroidal media. Then the abstract versions of these media--one-parameter carrier spaces--are considered, and finally the illustrations culminate in the principles of invariance for completely arbitrary media which are not represented explicitly as one-parameter media. Throughout this section, the proceedings may best be viewed once again from the two vantage points defined and discussed in the introduction to Sec. 3.7. In regard to these vantage points, Sections 3.4-3.8 and the present section begin to illustrate the efficacy of the interaction principle, not only as a theoretical tool, but as one which shows promise in fostering novel methods of numerical computations in radiative transfer problems.

FIG. 3.25 Illustrating some applications of the interaction principle to various optical media.

PIG, 3.25$_1$, concluded

Example 1: Principles of Invariance on Spherical Cylindrical, Toroidal Media

our present goal is to use the interaction principle to formulate the principles of invariance on three common types of curvilinear media. Figure 3.25 depicts four instances of a curvilinear inhomogeneous optical medium x and one linear inhomogeneous optical medium. Part (a) depicts a spherical medium in the form of a spherical shell with inner radius a, and outer radius b. Adjacent to the schematic cutaway of the spherical shell is a diagram showing a partition of x into concentric spherical shells of radii x,y,z, with

a s: x =·y =-z : b . Similar descriptions can be made of the hollow cylindrical medium x in part (b) of Fig. 3.25, the hollow toroidal medium in part (c), the rectangular parallelepiped medium in part (d), and the solid vertical cylindrical medium of part (e). In the case of the hollow cylindrical medium, its axial length may be finite or infinite. In the case of the parallelepiped, it may be of infinite extent in one or both lateral dimensions. In all five cases we may have a = 0. However, for the present illustration, we consider for generality a B:o.

We shall use as a prototype for the present formulations, the four principles of invariance derived in Example 3 of Sec. 3.7 for the case of plane-parallel media. As in that earlier example, we shall for brevity use the letters "a", "x",. "y", etc., as names for both the parameter of the associated surface and the surface itself. Each medium in Fig. 3.25 will, be designated by the name "x(a,b)", and subsets of x (a,b) as "x (x, z)" , etc., just as in the plane-parallel case. Each medium is irradiated over surface a and b by incident external radiance distributions; N--(a) for a, N+(b) for b. No other sources are incident on or within x(a,b). The direction conventions are also analogous to the

plane-parallel cases: we agree that at each point x on a parameter surface, the unit normal k (x) is directed toward the direction of decreasing parameter values.
Now, isolating X(a,y) and considering it black convexified, we enumerate the sets of incident radiance distributions:
Ax: all field radiance distributions like N_(a) A2: all field radiance distributions like N+(y)
Enumerating the response radiance distributions, we have: Bi: all surface radiance distributions like N+(a) Bz: all surface radiance distributions like N±(y) The four interaction operators s_{ij} are:

s 1 1 ~ ` - R(a,Y) S 1 2 - - T(a,Y) S 2 1 - - T(y,a)
S 2 2 - - R(Y,a)

These four operators are instances of the standard operator rd (X;a,b) in (6), where X is now X(a,y) and "b" is replaced by "a" where y is now a spherical surface in X(a,b). For the standard reflectance operator R(a,y) we have, explicitly:
f [lsb(X;x',&' ;x,E) dQ(&')dA(x')

where x is in spherical surface a, E is *in $^{-}=_{+}$(x), and X is X(a,y). Similar constructions are made for the remaining three standard R and 1' operators. The R-T notation has been chosen so as to be uniform with the plane-parallel case of Sec. 3.7.
The interaction principle then states that:

N_+ (a) = N- (a) R(a,Y)
N_+ (y) T (y., a)
N+ (y) R(Y,a)

By repeating this process now for X(y,b) we arrive at the analogous pair of statements:

N+ (Y)= N_+(b)T(by) ,+ N_ (y) R(y,b)
(3)
N±(b) = N+(b)R(b,y) + N_ (y)T(Y,b) (4)

The similarity of (1)-(4) with (15)-(18) of Sec. 3.7 is unmistakable: the interaction principle unifies all these instances. When we append the following two auxiliary equations:

N_+ (y) = N_+ (Y) (5)
N+ (y) = N_ (y) (b)

the set (1)-(6) becomes autonomous, as usual. The remaining discussion of Example 2 of sec. 3.7 now holds--virtually unchanged--including the definition (27) of iterated operators. Now, however, we use the standard V-operator. It is not necessary to rewrite the principles of invariance I-IV of Example 3 of Sec. 3.7. They apply, as they stand to the present context. The only salient change is in the basis of the R and T operators: use now use the standard a-operator, as defined in (6) o f Sec. 3. B, as a basis. As in the plane-parallel case, the four principles of invariance are instrumental in allowing one to solve for N_+ (y) for every y, a_-s^{-}y :5b, assuming

the standard R and T operators are known. These, in turn, are obtained from solutions of functional equations of the kind to be studied in Chapter 7.

Example 2: Invariant Imbedding Relation for
One-Parameter Media

The comprehensiveness of the principles of invariance, as extended from their classical plane-parallel settings by means of the interaction principle, begins to emerge as the five specific media in Example 1 are re-examined. In this example we systematically extend the results of Example 1 to their immediate logical limit. To do this, we ask: what is common to all the specific instances of Example 1? The answer is that these media are all constructed by assembling layer upon layer of surfaces of geometrically similar shapes. In part (a) of Fig. 3.25, we can imagine the hollow sphere to be built up from spherical surfaces of radii y, a sx y :sb, much in the way an onion is built up layer by layer. Parts (b) and (e) of Fig. 3.25 show that the cylindrical medium can be built up from cylindrical surfaces or circular plane surfaces. This two-way slice can be done for every instance shown in Fig. 3.25, and many others not shown. In each of the five instances displayed in Fig. 3.25, the medium X(a,b) may be imagined to consist of a set of geometrically similar surfaces X_x with a s x c b, i.e., with x a point in the interval [a, b] of real numbers. Thus we may set:

X(a,b) = Xx: x e a,b 7

i.e., X(a,b) is equal to the set of all geometrically similar surfaces X_x, each being indexed (identified) by a single parameter x drawn. from an interval [a, b] of real numbers.

The examples of Fig. 3.25 only begin to illustrate first of all the great number of three-dimensional subsets of Euclidean space which are one-parameter spaces and available for study, and secondly the multiplicity of ways in which a given solid can be represented as a one-parameter space (viz. (b) and (e) of Fig. 3.25). Indeed, as can readily be verified any solid of Euclidean three-space may be represented as the union of a one-parameter family of two-dimensional surfaces, and in many distinct ways!

Despite this great variety of shapes and sizes for each set X(a,b) and each source-free subset X (x, a) of X (a, b) , we can isolate X (x, z) , consider. X(x,z) black convexified if it is concave, and enumerate the sets of incident radiance distributions on $X(x, z)^-$:

A,: all incident (surface) radiance distributions like N+(z)

Az: all incident (surface) radiance distributions like N,(x)

where we are now following the pattern established in Example 4 of Sec. 3.7 and using. surface radiances throughout (see, e.g., (21)-(24) of Sec. 3.7). The sets of response functions of interest are

B1: all response (surface) radiance distributions like N_+(Y)

Ba: all response (surface) radiance distributions like N _(Y)

In the present enumerations, N+(z) is the outward radiance distribution over the parameter surface X_z, a :!Ez mob. The unit outward normal k(p) at point p on X_Z is in the direction of decreasing parameter values.

The interaction principle then asserts the existence of four interaction operators sib:

sl i -- T(ZPYPX) S12 -` Jg\(zpYVX) $s2_1$ -- 9(x,Y, Z)

S22 r(x,Y; z)

These four operators are not instances of the operators defined in (b) of Sec. 3.8. Rather, they are exactly analogous to the complete reflectance and transmittance operators (40)(43) of the plane-parallel case in Example 4 of Sec. 3.7, The interaction principle now yields the two statements:

I . N+ (Y) " N+ (z) 7(z,Y,x) + N_ (x) .~k(x,Y,z)
II. N_ (Y)a N+(Z).~? (zIYfx) + N_ (x) %7(x.Yyz) which we can write as:
(N+(y)pN_(y)) ' m (N + (z) , N _ (x)) M (x , y , Z) where we have written:
1 1 7,~ (x,Y, z)" for
~r(z f y , x) . 4 ? (Z f y g x) jg(xIY0Z) ~7(xfypz)
The preceding equation is the invariant imbedding relation for one-parameter media. It is exactly analogous to (36) of Sec. 3.7. On the strength of this analogy, we summarize the precoding results as follows
Let X(= {Xx: A c [a,b] j) be a one-parameter optical medium where [a,b] is a closed interval in the extended real-number system. For every y € [a,b], there is a pair (N+(Y),N_ (Y))
of (real or vector valued) response function 8 on X_y. Let "/7" denote the set of all ordered pairs (N+ (z) ,N_ (x)) o f incident functions, [x, z] c [a,b] with subsets 71+ and ~j_ defined as
{N+(z) : z c [a,b]}and {N-(x): x c [a,b]), respectively.
Then for every x, y, z with y c [x, z] c:{a,b] there exists an interaction operator ?f (x,y, z) o f ?I into V~ such that.

(N+(y)j,N_(y)) = (N+(z),N.(x));`(x,,y,z)

where we have written:
J'(z,Y,x) -6W(z,Y,x)
~~ (x,y,z)" for (11)
in which &R(z,Y,x),.(-,*?(x,y,z) are the complete reflectance operators with domains and ranges 7j_, 9j+, respectively; X(z,y,x), O(x,y, z) are the complete transmittance operators with domains 7'+, 2r_ and ranges respectively. In addition, '7(x, z, z) = T(x, z) is the standard transmittance operator for X(x, z) and T (x,x, z)= I., the identity operator; a(x,x,z) = R(x,z) is the standard reflectance operator and ,&~(x, z, z) = 0, the zero operator.
The preceding statement of the invariant imbedding relation is essentially that given in Ref. [233]. It is now a simple matter to deduce from (10) the semi-group properties.
g-(a, z,b) = 7(a,Y,b),7(Y, z,b)
.R(a, z,b) = 9_(a,Y,b),R(Y, z,b)
(12)
for complete transmittances (cf. (52), (53) in Example 5, Sec. 3.7). Furthermore, the principles of invariance for one-parameter media are readily forthcoming from (10)--or the equivalent set (8) , (9) .Indeed, setting x = y in (8)
N+(Y) = N+(z)T(z,Y) + N_ (Y)R(Y,z)
Setting z = y in (9):
I I. N_ (Y)= N_ (x) T(x,Y)-+ N+ (Y) R(Y,x)
Principles III and IV now follow from I, II, as in Example 3 of Sec. 3.7. The present instances of the principles are identical in form to those in Sec. 3.7 and therefore need not be repeated in detail here. Furthermore, the representations of the present complete reflectance and transmittance operators in terms of the standard operators are identical

in form to those given in (40)-(43) of Sec. 3.7 for the plane-parallel setting. Furthermore, the properties (44)-(47) also are easily shown to hold for the present complete reflectance and transmittance operators. The present forms of the standard R and T operators are important enough to repeat here. Thus for an arbitrary one-parameter optical medium x (a , b) we write:

"R(a, b) f ~ for

f f [ ] S_b (fix' ! ' ;x, E) ds1(~') dA(x')

i f x i s in a and ~ i s in $E=_+$ (x) .

"T(a,b)" for [ ]5b(x.x"&' - x 1) dQ(.')dA(x') a .¯_(x')

if x is in b and ~ is in w_ (x) .

"R(b.a)rfor I]5_b(X;x[1],E • ¯x,E) dQ(~')dA(x') f f

b 1 + (x ')

if x is in b and E is in E.. (x) .

" T (b , a) « for I I S_b (X;x' .E' ;x,~) dM(E') dA(x') f f

b : (x')

if x is in a and ~ is in '+(x) .

Example 3: One-Parameter Media with Internal Sources

In this example we show how the interaction principle may be used in the task of formulating the equations governing the radiance distribution N(y) over a parameter surface X_y in

a one-parameter optical medium X(a,b) which has internal sources generally distributed over an internal parameter sur

face X_s .9 a!s s sww¯b.. To see at the outset the essential structure of the resultant equations, we assume that no other sources are incident on X(a,b).

Figure 3.26 depicts the one-parameter optical medium

X a,b) with the incident source (field) radiance distributions N (s) and N^O-(s) over level s in X(a,b). We imagine N_+(s) to irradiate X(a,s) and N_(s) to irradiate X(s,b). Thus, it is

FIG. 3.26 Taking into account internal sources in general one-parameter media.

as if the incident source radiance distribution N°(s)

[_ (N?(s),N°(s))) were placed (like a thin transparent luminous vanilla filling) into X(a,b) after the latter had been momentarily sliced open (like a layer cake) along X_S. It follows that the light field generated by this source may be viewed as being distinct from NO(s). We assume NO(s) to vary from point to point over Xs, and to be of arbitrary directional structure at each point of X_s. Thus in particular, NO(s) could consist of a narrow pencil of radiation at one point only, or it could be of uniform radiance over all directions at each point, etc. As usual X(a,b) is generally inhomogeneous. The only requisite regularity in X(a,b) is its geometric one-parameter structure (and even this can eventually be relaxed); optical properties and radiometric properties are left unconstrained--except for a modicum necessary to define integration and to have the usual additivity and continuity properties on which to build the operator algebra.

The given internal source- over X_s suggests a partition of X(a,b) into two parts X(a,s) and X(s,b). In order to invoke the interaction principle we could employ the usual no tation "N+(y)11 for surface radiance of X_y, and "N^-(y)" for field radiance over X , a s y s b; however, now that some specific examples have sown how to systematically use surface radiance, we shall limit our use mainly to that kind of radiance. When "N" has no superscript, surface radiance is understood. The outward and inward directions over Xy for radiance distributions are as defined in Example 2,

Isolating X(a,b) and enumerating the sets of incident radiance distributions on X(a,b) we have:

A,: all radiance distributions like N0 $_+$(S)

A2: all radiance distributions like N^O(s)

Enumerating the sets of response radiance distributions:

B_l: all radiance distributions like N_+(Y)

B2: all radiance distributions like N_(y)

Then m = 2, n = 2, and the interaction principle yields four interaction operators sib such that:

si i _- T++(s,Y) S12 - - $T_{+.}$ (Spy) SzI - f_+(Spy) S22 'F-_ (s,Y)

The fact that these four operators belong to the medium X(a,b) is implicit in the notation. Occasionally it will be desirable to explicitly denote this fact (see, e.g., Sec. 7.13) and we shall then write I'T++(s,y:a,b)" for f++(s,Y); "T+_(s,y:a,b)" for T+_(s,y), etc. The interaction principle then states that, for every pair of levels y,s in X(a,b)

N_+(Y) = N0(s)'~++(s,Y) + $_{N0}$ (S) I_+(S,Y)(13)

N_ (Y) = No (s) T + No (S) (s vY)
.. (s ~Y) T_ _ (14)

In matrix form, (13) and (14) become:

N(y) = N° (s) T (S,Y) (15)

where we have written:

'I N (Y) I f for

? I N ^{0} (s) " for

(N+ (Y) N _ (Y) (NO(s) ,N°(s))

,.

T_{++}(s,Y) T_+_ (s rY)

11T (s t y) " for

T_+(sly) is pY)

We next show how the four operators T_{++} (s ,Y), ...,1Y_ _ (s,y) can be represented in terms of the standard operators associated with the space X(a,b) and its subsets X(x,z). The derivation of the representation will proceed in two parts. The first part obtains a representation of T(s,s). The second part obtains the representation of T(s,y) with s ~ y.

We turn now to the case of T(s,s). Consider, for example, the subset X(a,s). Isolating this subset and enumerating its incident functions and response functions under the present hypothesized conditions, we have N°(s) and the surface radiance N+(s) of X(s,b) as incident functions which both act on the lower boundary of X(a,s). These are the only incident functions on X(a,s). Hence by principle of invariance II in Example '2 (with z = y = s, x = a) we have

$N_{.}$ (s) _ (N° (s) + N_+ (s)) R(s, a) . (16)

Similarly, for subset X(s,b) and principle I of Example 2 (with x=y=s, z=b)

N_+ (s) = (N^o (s) + N_ (s)) R(s ,b) (17) From (16) and (17):

N_+(s) = [N^O(s) R(s,b)

N_ (s)_ [N^O(s) R(s, a)

N^0 (s)R(s,a)R(s,b)] [I -R(s,a)R(s,b)j -~ (18) N0(s)R(s,b)R(s,a)~ D-R(s,b)R(s,a)]-i (19)

Comparing (118) and (19) with (15) (in which s = y) and recalling that N_+(s) are arbitrary, we deduce for the case* as mob:

T++(s,s) R(s,a)R(s,b) [I-R(s,a)R(s,b)] -i

T_+_ (s q' s) = R(s,a) [I ¯R(S ,b) R(s ,a)] -;

T_{+} (s, s) RCS,b) [I -R(S, a) R(S,b)] -' T_-_(s,s) = R(s,b)R(s,a) [I-R(s,b)R(s,a)] -1

(20) (21) (22) (23)

We now go on to the second part of the representation derivation for T(s,y) with s # y. For definiteness we first assume a s y r·¯s sb, as in Fig. 3.26. Then by the invariant imbedding relation (l0) applied to X(a,s) we have:

(N+()r),,N.(y)) _ (N_(s),N_(a))7.j(a,y,s) +_ (24)

Here, by our present source conditions and choice of notation, we have:

NT (a) = 0 (25)

N+ (s) = N_+ (s) + N°(s) (26)

From (24) - (26) we have

N_+ (Y) = (N ^{0}(s) + N_+(s)) 9(s~Y~a) (27)

N_(y) = (NO(s) + N_+(s) 3J9(s,,Y,,a)(28)

Using (13) and (14) to give a representation of N+(s), (27) and (28) become:

N+(Y) = [N^O(s) + (N° (s) T++ (s f s) + N0 (s)'F-$_+$(s, s))] ACS y, a)

No (s) [~(sfy~a) + I`++ (s, s) L'(S~Yfa)]+
+ No (s)1Y_+(s,,S) 3-(S,,Yfa) (29)
Further,
N_(y) = [N° (s)+ (NO(s)T++(s,s) + N°(s)'Y _+ (s, s)~]. '(SIYfa)
= N (S) [(S ,y., a) + T++(s ,, s).Q(S,,Y,, a)] +
+ N_ (S)T_+(s,5)J4 CS j y f a) (30)

*In the notation of Sec. 7.13., "T++ (s , s) " becomes "T++(s,s:a,b)"; "T+.. (s,s)" becomes "T+_ (s,s:a,b)", and so on. The abbreviated notation is used whenever fa,b] is understood, as in the present discussion.

We return now to (13) and (14) which are the alternative representations of Nt(y) in terms of the T-operator. Since NOW are arbitrary, a comparison of (29) and (30) with the earlier representations implies that:

y

T++(s'Y) M (I + f++(s's)) 7(spy.a) (31)

T+_ (s , W Y) = (I + T++(s,s)) a(s,,Y,a) (32)

T_+(s•Y) '
T_+(s,s) 9-(s,Y,a) (33)

T__ (s,Y) _ Y _ + (s , s) 0 (s M'Y9a)(34)

for the case a s y oc s iscb . Here "I" denotes the identity operator for radiance distributions over Xs. With this set of equations the representation of the operator T(s,Y), s 0 Y, is essentially complete; for by interchanging "a" with "b", and "+" with "-" everywhere in (3l)-(34), the case a :ss -=y -.r-b is obtained (see (39)-(42) of Sec. 7.13). Two observations should be made on the nature of the operator T ($,y) , s ¢ y. First T(s,y) is not continuous at the diagonal points (s,s) of its domain. This can be seen from a study of its transmission component in (31). This discontinuity may be traced back to the equations (27) and'(28), and the meanings ascribed to the radiances N+(s) and Nj(Y).

A careful re-reading of the derivations will show that for every level ~', N+(y) is the surface radiance of the boundary of the subset X(Yfb) of X(a,b). A similar reading holds for N_ (y) . Hence, as we approach the surface X_5 from below we+at first see N+(s), the surface radiance of X_s; then, abruptly, the source radiance NQ(s) is added as we continue to move upward through the surface. In symbols, by (27), and for y _w ; s:
lim (N+(y) - N+(s)) N°(s) (35)
yes

Since we have concentrated the origin of the source radiance N?(s) on a surface X_s, it is small wonder then that the outward (or inward) light field receives a jolt across X_s as we move upward (or downward) through level s. This jolt is duly recorded in T++(s,y) in the manner shown in (31) (for y < S) to wit*
lim (T++(S,Y) - T + + (s , s) 3 ' ~ I Y+s

The second observation is that the use of the concept embodied in $T_{++}(s,y)$ can be extended by postulating a continuous distribution of source radiance over a parameter interval (s-c,s+r.), s> o, or simply defining continuous functions
In the notation of Sec. 7.13, "T_{++} (s ,y)" becomes
"T s :ab" "T s ' s becomes ",y s :ab" and so on.
n r ~ y ~ a w a m t e w ~ x ~ _ _ i - s , ; t , : ~ n y x . 1 . _ v , .
L , : . ~ r 2 , - W . : : a . . . . ~ . . i . . . . i . : a : . i . . .
4 ' . i 3 I t 8 1 1 R . ~ ~ ~

NOW for all s, a -qrs !rmb and adjusting them to represent given physical situations as needed. The latter treatment is more general. The resulting operator equation will then be of the form:

b
N (y) 'f N^o(s)T(s,y) ds (36)
a

The interaction method then yields the integral operator:

f b
I T(s,y) ds (37)
a

as a matter of course and the detailed decomposition of the new 2x2 matrix T(s,y) of operators proceeds analogously to that of the original T(s,y) above. The basis of this new integral operator will not be discussed here. The interested reader will find the general theorems leading to (37) in Sec. 3.16. Furthermore, readers interested in the discrete space version of (36) are referred to Chapters IX and X of Ref. [251]. A representation of (37) will be obtained in the irradiance context in Sec. 8.5.

We shall leave it as a simple exercise for the reader to show that, if there are a finite number of distinct external or internal sources N+(si), i a 1,.,.,n, of radiant flux
of the discontinuous kind N9(s) considered above, then by virtue of the interaction principle, the resultant light field at level y within X(a,b) is given by

n
N (y) = E N^0(si)T(siIY) , 1 1 . (38)

where we have written:

'IN (y)"
IfN0 (si) it for
for
(N+ (y) ,N_ (y)) (NO(si) NO (S_i))

and where each T(si,y), i = 1,...,n is a 2x2 matrix of operators of the same structure as T (s, y) in (15), and where for
every i , 1:!= i ~ n, the components T_{++} (s i , y) _ • . . , T _ _ (s i , y) are as defined in (20)-(23) and (3l)-(34). If n = 1, then (38) reduces to (15).

The net result of this example is the demonstration that the problem of the internal sources of radiant flux in an arbitrary one-parameter medium can be reduced to a straightforward, albeit nontrivial, calculation using only the standard reflectance and transmittance for the medium (cf. Example 4 of Sec. 3.7). These standard reflectance and transmittance operators, in turn, are governed by certain differential equations which, when solved, yield the standard reflectance and transmittance functions for each given medium in terms of the inherent optical properties of the medium. These differential

equations will be developed in Chapter 7.

Example 4: Principles of Invariance for General Media

What is the geometric limit of validity of the principles of invariance? Can the principles of invariance be written down for radiance distributions in highly irregular media such as clouds, lakes, ponds, and wind-blown regions of the sea and other irregularly bounded natural hydrosols? The purpose of this example is to show that the answer to the latter question is in the affirmative. Once a few preliminary geometric conventions have been dispatched, an application of the interaction method yields the requisite principles of invariance.

Figure 3.27 depicts a general connected optical medium One may envision x as a cloud or a part of some natural hydrosol. As usual we assume no internal sources or reflecting boundaries. The boundary surface Y of X may be concave or convex.

In order to invoke the interaction principle we must have some idea of-our goal. Let us re-examine a simple geometric setting in which the principles of invariance were derived. In such a simple case we know what the goal looks like. Fig. 3.20--the setting for Example 2 in Sec. 3.7--is a good starting point. We ask: what are the bare essentials of the

FIG. 3.27 The directional and spatial conventions for applying the principles of invariance to arbitrary optical media,

geometric setting there? First of all examine the medium: it is a plane-parallel medium; its boundary consists of two planes a and b. Thus if we are to emulate this for the case of X we should also divide the boundary of X into two parts a and b. Furthermore, the plane-parallel medium X(a,b) in Fig. 3.20 is partitioned into two parts: X(a,y) and X(y,b) by the internal plane y. The corresponding activity in X of Fig: 3.27 would be a partitioning of X into two parts: A and B by an internal hypothetical surface y. By combining these two requirements we take the first step in the present derivation: the arbitrary optical medium X is partitioned into two parts A and B by an internal surface yi this also partitions the surface o f X into two parts a and b, the exterior boundaries of A and B, respectively. Therefore the total boundary of A is the union of a and y; that of B is the union of b and y. There is one final geometric essential in Fig. 3.20 to be taken into account. That is the matter of direction. Our present goal is to emulate the classical plane-parallel case as far as possible. Hence we establish the directionality conventions depicted in Fig. 3.27. The vector k(x) at each representative point o f a,y, and b is an "outward" (or "upward" or "forward") direction (the appropriate adjective is governed by the context within which one encounters X). As long as one is consistent in choice of directions, the following principles can be recast using any reference system an investigator cares to choose. The choice of directions depicted in Fig. 3.27 is internally consistent and coincides ,with the plane-parallel conventions in the .limit as

X approaches X(a,b), i.e., when a,b, and c are laterally extended and continuously deformed to become parallel planes.

With the foregoing partitioning and orientation of partition elements of the medium X established, we can specify the incident radiance distributions over X. Incident radiance distributions over part a of X will be denoted by t'N:(a)", outward radiance distributions over a will be denoted by "N+(a)". On the internal surface y, "N+(y)" will denote outward surface radiance distributions, "N-(y)" will denote outward field radiance distributions, etc. Over b, incident radiance distributions will be denoted by "N;(b)", etc. As usual, N+(y) is a function which assigns to each point x on surface y and direction E in + (k (x)) the surface radiance N+(x,~) of y directed into part A. At the same point and same direction N+ (y) has the value N+ (x , Q , and by (32) of Sec. 2.5, N+ (x,) = N+ (x,Q .

The geometrical prerequisites are sufficiently established to permit us to apply the interaction method to X. In particular part A is isolated and we enumerate the sets of incident radiometric quantities:

A1: all incident radiance distributions like N_(a) Ax: all incident radiance distributions like N+(y) . The sets of response radiometric quantities are:

B;: all response radiance distributions like N+(a)

Bz: all response radiance distributions like N±(y)

In the present case m = 2, n = 2, and the interaction principle yields the four operators si_b:

5 1 1 -- t/(A;a,a) S 1 2 _ - & (A; a,Y) $21 -" a(A;y,a) S 2 2 ` ' V(A;y,y)

The four operators t/(A;a,a), . . . , x/(A;Y,y) are instances 'of the V-operator (6) of Sec. 3.8 (and hence the appropriate black convexification of A has been achieved). The interaction principle also yields the two interaction equations:

N_ (a) %/(A; a, a) + N_+ (Y)V '(A;Y pa)

N± (Y) " N- (a) .V((A; a,Y)+ N_+ (Y) .k/(A*Y vY)

(39) (40)

In a similar way the interaction method is applied to the subset B of X, with the following interaction equations as a result:

N+ (Y) = N_+ (b) X/(B;b,Y) + N_ (Y) J(B;Y pY)

N+ (b) = N+ (b) k/(B;b,b) + N- (Y) r/(B;Y tb)

(41) (42)

The auxiliary equations for the present formulation are:

N_+ (Y.) = N_+ (Y)(43)

N+ (Y) = N_ (Y) (44)

The set of six equations (39)-(44) is autonomous. We can. use (43)'and (44) to reduce the set of six equations to four in terms of surface radiance only, and so the superscripts are no longer needed, Thus, with incident source radiances N_(a)

and N_+ (b), we have:

=

N_ (a) a(A; a,a) + N_+ (Y) s/(A,Y, a) (45)

N_ (Y) = N_ (a) a(A;a,Y) + N_+ (Y) %J(A;Y pY)(46)

N_+ (Y) =
N_+ (b) A/(B ;boy) + N_ (Y) _j(B;y ~y) (47)

N_ (b) =
N_+ (b) a(B,b,,b) + N_ (Y))j/(B }Y ib) • (48)

Equations (45)-(48) are the requisite principles of invariance for X under the present partition into parts A and B. The middle two equations are autonomous. Their solutions are:
¢-;-.0- 6r,>i9W;.I:,:.:. '-.s: --- .:;:_0:. vr~:m..1! :

N_+ (Y) [N+(b)t/(B;b,y)+N_(a),J(A;a,y)yj(B;y,y)I
N _ (Y)
[N_(a)j/(A;a,y)+N+(b)jf(B-b,y)I/(A-y,y) [I -3I(B;Y,Y))I(A;Y:Y) - (So)
[I -3d(A;Y .Y)Af(B;Y '9Y) -i (49)

A comparison of these solutions with (25),(26) in Example 2 of Smc. 3.7 is instructive. Furthermore, the remaining part of the discussion of that example concerned with computation details pertains also, in essence, to the present situation.

Example S: Invariant Imbedding Relation in General Media

By writing the solution N±(y) of (49),(50) in Example 4 in matrix form, the invariant imbedding relation for the general medium X of that example is obtained. Thus, we write:

11,140(bvyf-a)~for [I - j(B;y,y)t/(A;ySyT1 Then (49), (50) may be put into matricial form:
(N_+ (Y) ,vN_ (y)) _ (N_+ (b) ,N_,(a)) ??r(a,Y,b) (51) where we have written
11;~7 (a,Y.b)11 for
vT
(bvy~a) 4R(bIy,a)
e(a.Y,b) T
, (a fY,b)
(52)

We could have obtained (51) directly by means of the interaction method and deduced the form of f (a,y,b) , J'(a,y,b), etc., as in Example 4 of Sec. 3.7. We took the present route for the purpose of illustrating the manner in which (51) was first obtained (cf. Sec. 23, Ref. [251]). At any rate, the invariant imbedding relation is seen in its most general form in (51) and its basic role in radiative transfer theory is clear: the invariant imbedding relation (51) relates a given incident boundary radiance distribution on X to a requisite internal radiance distribution in X.

The specific evaluation of (52) is contingent on actual knowledge of the V-operators for x. Functional relations for the 1~/-operator on a general medium may be found in I
II' of Sec. 25 of Ref. [2511, The development of a practical general solution procedure in specific media. of the functional relations I'-IV', especially

relation I', constitutes one of the present outstanding (i.e., unsolved) applicational problems of modern radiative transfer theory (see Problem VIII, Sec. 141, Ref. [251]).

Example 6: Reflecting Boundaries and Interfaces

Hitherto we have examined all manners of radiometric interactions of surfaces with surfaces (Sacs. 3.4, 3.5) and solids with solids (Sacs, 3.7 and the present section). In this example we shall illustrate how the mixed radiometric interaction equations for surfaces and solids are derived using the interaction method. We shall consider a medium x with reflecting boundaries and an internal reflecting interface, but with no internal sources. Fig. 3.27 will serve to establish the geometric situation within X, ire consider external sources incident on x over boundary surfaces a and b. In particular N°(a) and NO(b) are the two external sources.

When the radiometric problem at hand is analyzed into its essential components, it is seen that the medium x may be considered to consist of five interacting parts:

boundary

medium A interface y medium B boundary b

The interaction method then indicates the following desideratum: Each of these five subsets of x is required to be isolated and black-convexified, if concave; its sets of incident and response radiometric functions enumerated; the associated interaction equations written down; and the-requisite auxiliary equations stated which will make the resulting system of interaction equations autonomous.

We begin by isolating the boundary part a. We first black-convexify, surface a (see Sec. 3.8). This has the effect of reducing its interaction equations, (14) - (17) of Sec. 3.5.

to a form identical to those of plane surfaces. In particu lar, the interaction equations for part a are:

$$N_+ tai = No\ (a)\ r\_\ (a) + N_+(a)\ t_+(a) \qquad (53)$$

$$N\pm\ (a) = N\_\ (a)\ t_-(a) + N+(a)r+(a) \qquad (54)$$

which follow from (14). (15) of Sec. 3,5 in which the second two terms (the self-interaction terms) are zero by virtue of the black-convexification. The notation "N°(S)" is now replaced by "N? $(_a)$"; "$_N$O $(_S)$ it by "N+ $(_a)$"; etc.

Next, the medium part A is isolated and then black convexified. In isolating A we are to imagine it separated from the boundary parts a and y as if we were carefully separating the meat of a two-slice piece of orange (A + B) from the peel (a + b) and inters l i r e membrane (y) (see Fig. 3.27). Thus let us write, ad hoe, "N_+(A)II for the outward surface radiance of medium part A--i.e., without boundary part a. Similarly, 'IN-(A)" will be the inward field radiance distribution over medium part A, just below the boundary part a, etc. We shall, however, retain the notation " V(A;a,a)" for the interaction operator which describes the responses of the medium part A (i.e.,the boundaryless A). The interaction method then yields the two equations (cf. Fig. 3.27):

N+(A) = .N~ (A) k,/(A;a,a) + N+(A) V(A;y,a) (55)
N+ (A) = N- (A) t/(A; a.Y) + N+ (A) a(A;Y #Y) •(56)
Next, the interface part y is black-convexified if necessary (i.e., if non planar) and generally treated exactly in the manner of boundary part a. The resultant interaction equations are (cf. Fig. 3.27):
N+(Y) N- (Y) r_ (Y) + N+(Y) t+(Y) (57)
N± (Y)" N- (Y) t_ (Y) + N+(y)r+(y)
The operators rt(Y),t±(y) are instances of those defined in
(10),(11) of Sec. 3.3.
The medium part B is isolated and black-convexified exactly in the manner of medium part A. The resultant equations are (cf. Fig. 3.27)
N+ (B) a N_ (B) t/(B ;Y vY) + N+ (B) a(B ; b ,Y) (59)
N ±(B) = N_ ($) x/(B;Y9b) + N +(B) x/(B;b,b) • (64)
Finally the boundary part b is" then isolated and black convexified, if necessary, and generally treated in the manner of boundary part a. The resultant interaction equations are (cf. Fig'. 3.27):

N_ (b) r_ (b) + No (b) t+ (b) (61)

= Nw (b) t- (b) + N° (b) r+ (b) (62)

N+ (b) N+ (b)
The interaction method is brought to its final stage by appending the appropriate auxiliary equations. For the present problem, we have (cf. 'Fig. 3.27) :
N+ (A)
-
N+ (a) (63)

= N_ (A) (64)

N_ (a)
which couple sets (53) , (54) and (55),(56). Further:

N+ (Y) =
N+ (A) (65)

which couple sets (55), (56) and (57),(S8). Further:

$N_+(Y)$ (67)

N± (Y) = N_ (68)

which couple sets (57), (58) and (59),,(60). Finally:

N+ (b) = N+ (B) (69)

N±(B) = N_(b) (70)

which couple sets (59), (60) and (61),(62). The set of 18

equations (53)-(70) is autonomous. Using the eight auxiliary equations (63)-(70) (all instances of (32) of Sec. 2.5), the set reduces to a less complex set of ten equations, which,

written uniformly in response surface radiance form are:

N_+ (a) = No (a)r_ (a) + N_+ (A) t_+ (a) (71)

N_ (a) M N_W(a)t_(a) + N+(A)r+(a) (72)

N_+(A) _ N_ (a) yj(A;a,a) + N_+(Y) ..*,g/(A~y.a) (73)

N_ (A) = N., (a) a (A;a0Y) + N_+ (Y) S A A;Y.Y) (74)

N_+ (Y) = N r +
_ (A)
(Y) N. (B) t. (Y) (7S)

N_ (y) " N_(A)t_(y) + N+(B)r+(Y) (76)

N+ (B) " N_ (Y)u/(B fY pY) + N_+ (b) 3 j (B ib *Y) (77)

N_ (B) =
N_ (Y) %/(B oY~b) + N_+(b) (B ;b,b) (78) 3

N_+ (b) =
N_ (B) r_ (b) + No +(b)t+(b) (79)

$$N_-(b) = N_-(B)t_-(b) + N^0(b)r_+(b) \qquad (80)$$

Of these ten equations, the eight "interior" equations, i.e., (72)-(79) form an autonomous system. Hence the problem of a general optical medium with two reflecting boundaries and one reflecting interface, requires the solution of eight simultaneous integral equations. As in the case of earlier examples in this section, the solution of the present formulation is contingent on knowledge of the various operators $r_\pm(a)$,
r b t a t b j/(A·a a .,. ,4(B;b br,

Example 7: The Unified Atmosphere-Hydrosphere Problem

The atmosphere of the earth and the surface of the earth (over both land and sea- ~_.form a system of radiometrically interacting optical media. In this example the interaction equations for the atmosphere and the hydrosphere are obtained as an autonomous system using the interaction method of formulating radiative transfer problems.

The formulations of Example 6 are readily adapted to the present task. It remains to specify the physical meanings of the five parts a,A,y,B,b of space X in that example. Consider Fig. 3.28 which is a schematic cross section of the earth. Let "A" denote the atmosphere, and "B" the hydrosphere, i.e., natural waters. Let "y" denote the air-water interface. Let "b" denote the bottom of the hydrosphere; at those places where B is solid earth, then b shall coincide with y. Finally, "a" denotes the transparent upper boundary of A.

With these choices, the system of equations (71)-(80) reduces to seven equations: (73)-(79) with $N_-(a) = N^\circ(a)$ and $N^O M = 0$. For the solution procedure of this problem using the techniques of discrete space theory, see Sec. 71 of Ref. (251]. (whenever media of differing indices of refraction are considered, it is implicitly understood that radiance functions are divided by the nz of the medium in which they are defined, see (4) of Sec. 2.b.) Because of the lateral extensiveness of A and B, the problem of solving the set (75)(79) can be considered within the-domain of the plane-parallel ease if attention is restricted to a region such as that enclosed in the dashed radial lines of Fig. 3.28.

i i

FIG. 3.28 A schematic diagram for the unified atmosphere-hydrosphere problem.

Example 8: Several Interacting Separate Media

The quantitative evaluation of the radiometric interaction of separate clouds in the atmosphere, or separate portions of lakes, oceans, or other natural hydrosols forms an interesting and difficult radiative transfer problem. The methods of discrete space theory have been used to develop a systematic means of computing the V-operators of such irregular types of media (see, e.g., Chapter X, Ref. [251]). For actual computations, the latter formulation may use the estimates of the Ap/-operators as supplied by the methods of Chapter X of Ref. (251] .