## 5.1 The n-ary Radiometric Concepts

In this section we shall define those radiometric concepts associated with the scattering order decomposition of a light field which will be needed in the developments of the present chapter. Throughout this section we work with a general source-free optical medium X in the steady state irradiated by a steady incident radiance function $N_o$ defined on the boundary o f X. The medium X 'i s generally inhomogeneous , of arbitrary shape and extent, and with general volume attenuation and scattering functions defined throughout. The incident radiance associated with $N_o$ penetrates into X and generates radiant flux of arbitrarily great scattering orders, which we now proceed to analyze,

### n- ary Radiance

The systematic construction of the radiance functions associated with the families Pn(t) of photons described in the introductory section starts with the incident radiance $N_o$ on the boundary of X. In particular, the radiance No(xo,g) defined for a boundary point $x_o$ and the direction ~ at $x_o$ can be extended to each point x of X by writing:

"$N^o$(x,~)" for $N_o$($x_o$,~)$T_r$($x_o$,00 ~I) where x = x + r~. The meanings of these terms are shown in Fig. 5.10 ?n this way we can construct a radiance distribution NO(x,•) at each point x inside and on the boundary of X.

We call No the initial (residual or unscattered or reduced) radiance function within X. No represents radiance which, relative t o the radiance $N_o$ incident on the boundary o f X, has undergone no scattering operations within X.

When some of the flux which comprises the initial radiance distribution No (x•) at x undergoes a scattering operation there is generated first order (or primary) scattered radiant flux. The amount generated per unit length in the direction & at x is represented by writing:

"N*(x,~)" for No(x,~')o(x;~';g) &2(E') (2) r

This may be written succinctly in operator form using the path function operator R of Sec. 3.17:

N* = $N^o$R (3)

FIG. 5.1 Geometric details for computing n-ary radiance.

In other words, the operator R acts on No to ienerate N* ; alternately, we may say that R maps No into $N_*$. The amount of primary scattered radiance accumulated over a path (Pr (x, 0 in X is then represented by writing:

r
"N a (x, )" for r N* (x' ,E) Tr-r' (x' 0E) dr' (4)
0

This may also be written succinctly using the path radiance operator s of Sec. 3.17:

The general pattern of construction of the radiance functions comprising the scattering order decomposition of the light field should now be clear. Thus, for every integer na 4,1,2, ... , we agree to write:

and

„N n+ 1„ for $N^nR$ (6)

'IN n+I" for $N^{*+I}s$ (7)

The function NJ is called the n-ary path function and Nn is the n-ary radiance function , in X relative to N u, By means of (6) and (7) we can construct the (n+I)-ary radiance function on X once we know the n-ary radiance function on X, for n thus:

~'

$N^{n+I} = N^{*+j}T = (NnR)s$

for every scattering-order n > 0, The composition RT of the two operators R and Z occurs F f ten in our studies of radiative transfer theory. We shall then write, for brevity r1slot

The reader should verify that

Sl = I F o(xf t~' t Idn( t1 1T Cx' )dr'

r„r

r

Now, using the notation for $S^1$, (8) may be written:

and if n is an arbitrary integer greater than 4, then it follows that we can apply the statement (8), or statement (11). once again to obtain

Nn+l = (Nn-lsl)sl. (12)

If n-1 T 1, then $n+_l$can apply (11) again, with the eventual conclusion that N is represented as the result of operating on No with S I at total of n+1 times in succession. That is, if we write

1rsn+ III

for $S^is^n$ (13)

for-every integer n, n > U, then it is an easy application of the principle of complete induction to show that

(14)

for every scattering order (nonnegative integer) n. The sense in which (13) and (14) are to be understood is the obvious one:

Operate on No and S ' to obtain N'; then once
N' is obtained, operate on N' with S $^1$ to obtain N'; and so on until Nn _.is obtained. The total combined integration operation of obtaining n-ary radiance -Nn from the initial radiance No is summarized by the operator Sn defined recursively in (13).

n-ary Scalar Irradiance

Now that the n-ary radiance functions have been defined it is a relatively easy matter to define the n-ary counterparts to all the radiometric concepts. For example, by recalling the integral representation of scalar irradiance h(x) at a point x in the optical medium X (cf. Sec. 2.7), i. e. , the definition in which we have written

"h (x)" for I N (x, E) do (E)
M

we are then led to write analogously

11$_h$n$_{(x)}$ ~ for $N^n$(x,E) do() (Is) M
w

for every nonnegative integer n. We call hn (x) the n-ary scalar irradiance in X relative to No.

n-ary Radiant Energy

The connection between scalar irradiance h (x) and radiant density u(x) at each point x of X was seen in Sec. 2.7 to be •

h(x) = v(x)u(x)

where v(x) is the speed of light at x in X. Furthermore the definition of the radiant energy content U(x) of. X was defined by writing:

for j u (x) dV(x)
X

This leads us to write analogously:

"$u^n$(x)" for I $u^n$(x)dV(x) (16)

for every nonnegative integer n where, in turn, we have written:

"$u^n$ (x) " for $h^n$ (x) Iv (x)
(17)

for ever rK nonnegative integer n. Combining the definitions of h , u and U 1. we have the following representation of U .

Un (X)

W
$N^n$(x, dV(x) (18)

for every nonnegative ¯integer n, and where the n-ary radiance Nn is re resented in terms of the initial radiance N° throughout X ty means of (14).

General n-ary Radiometric Functions

The n -ary radiance and radiant energy functions 'constructed above will not be the only n-ary radiometric concepts used in the present work. For example the two-flow equations of Sec. 8.4 are studied by means of n-ary irradiance concepts. I t is a sample matter to extend the type of definition exhibited for hn and Un to an arbitrary function C obtained from the radiance function by an appropriate linear operator -t associated with C; that is

For example, the operator at in the case where C i s scalar irradiance was

Then in general we write analogously
fi $_C$ne,
for $N^n$ L , (20)
for every non ne gat ive integer n. We call en the n -ary radiometric function of C, in y, and relative to N°. It follows from (14) and (2) that
= No (Sn
(21)
$C^n$

is the representation of the n-ary radiometric function Cn associated with the general radiometric concept C. In particular, we write:
"C*" for N* 44 (22) where N* is the path radiance (the scattered) component of N, as it occurs in (5) of Sec. 3.13. C* is the diffuse radiometric function o f C in X and relative to No. Together, C* and Cn are the decomposed radiometric functions. Radiometric functions which have not been decomposed are call undecomposed.