5.3 Canonical Equations for n-ary Radiance

We pause in the present development of the natural solution .of the equation of transfer to present the canonical form of the transfer equation for n- ary radiance. We shall be particularly interested in the case of n a .l. that is,,---in: the case o f the canonical equation for primary radiance.,........ From this case we can derive an expression which has often..:..: formed an integral part of expressions which attempt to approximately represent radiance distribution with a Modicum of analytic complications. The derivations below aro patterned on those in Sec. 4.5. Hence we can proceed with a minimum of motivation and explanation for the present discussion. Let us write:

n

,,Rn,, for _ VN 1 $_N$n

Then (1) of Sec. 5.2 becomes

_ R^nN^n _ _ aN^n + N whence, for every integer n with n > 1:

N n N A_ , . (2) $_K$n

and consequently:

n

$_N$n $_N$n$_T$ _ al + N* 1-T [_ (a- ~ •K^n) 1 (3)

r 0 r $_{a}$ _ $_e$in r

which are respectively, the desired canonical form o f the equation of transfer for n-ary radiance and its canonical representation for a path r (x, 0 .

If the medium X is assumed to be a plane-parallel stratified optical medium, then following the pattern established in equations (16) - (19) of Sec. 4.5, (2) and (3) reduce to

N^n

$_N$n ! _ A

a $_{+K}$ n cos 8

and the associated canonical representation of Nn, over

(Jr[1](x,&), analogous to N_r of Sec. 4.5 is

a path

$_N$n r

n

r-$_a$l + '-- N* - -, 1-T [- ($_a$ + Kn cos 6)

1 (53

o r a + K^n cos 0

Equations (2) and (3), and their special cases (4) and

(5), are the alternate (canonical) ways of representing Nn; the usual way being summarized in (14) of Sec. 5.1 by:

$_N$n = N0Sn

To see how (2), (3), and (6) throw light on one another, let us consider the case of a homogeneous source-free plane parallel medium X irradiated by narrow beams of radiance N_o

incident at each point of its upper boundary through a small solid angle B of magnitude Qo, as shown in Fig. 5.2. The radiant flux rom N_o initiates a multiple scattering process within X and eventually all scattering orders o f radiant flux are present within X. We direct attention now to N ^{l} and first compute its value at depth z in the direction ~ using (6). Thus, from (6) with n = 1

N^{1} = $(N^{O}R)$ s

For the present case NOR is readily evaluated:

N°(zsC')a(z~F,' ;&)dQ(~')

Since for each V in V_0 ,

No(zt~') = N_0(OOC')T_r(0 P ')

$_e$az sec 8_0

▪ N

0

where

cos 0_0 = ~ o•k

FIG. 5.2 Computing the primary scattered radiance in a hydrosol, induced by a collimated source N_o.

we have

N*(z~ = N eaz sec eo Q(z ~~' ~)d~C '~ 0

"0

eaz sec eo a $_{(\&o}$ rl (7) 0 0

Here E^{7}_{0} is the set of directions, of solid angle n_o, over which the incident beam has uniform radiance N_o. Note also that we have used the homogeneity of X in freeing o of depth dependence. Next, we apply the path radiance operator? over the path depicted in Fig. 5.2:

N*(z=,&)Tr_r, (z' t)drl

s

z

- sece N 0 a [& 0 ;] a 0 $_e$az'$sece_o$ $_e$a(z-z') sece dz'

U

z

sece N_o a ~ $_o$;) ~o $_e$azsece $_e$az' ($sece_o$- sere) dz' To

Therefore:

eta a~ ; n $_e$az seceo LI -e- az ($sece_o$- sece)

]

.. s

1 tz o 0 00

' a sec $_o$-sec

This canonical representation of N I (z , E), in which cos 6 = •K, holds for all paths such that e $0_0$0' For the case .6 = io, we return to the penultimate equally and evaluate the integral anew, or use L'Hospital's rule in

(8). Clearly, the new integral value is simply z for the case e_o = e. Comparing (8) with-the canonical form, with the latter now evaluated for the case n = l:

a+K case

we see that the following equality must hold:

1 _ -a(secoo-sere) a K case

seta[1-e-az sec $_o$-sec8

(1D)

From this we can, if required, solve for K^1 (which generally is a function of z,e, and, also in the present case, the parameter eo) .Observe that for $sece_o$ I sece, i . e.. ,for

e t e_o.

1 imz-+Wx.'(z,&). = - asece-o

and for 9 > eo,

1im $_+$~IC^1 tZ ,0 = - asece z

This shows directly that the K-function for primary radiance eventually, i.e., for sufficiently great depths, becomes

 independent of ~ over large sets of directions when 0 E 00). This phenomenon of the eventual partial independence of K^1 with respect to direction, presages an analogous behavior of the complete .K-function for observable radiance; we will study this depth behavior of K in more detail in Chapter 14.

We now summarize the main results of our illustrative example: By evaluation of (6) for the case of n = 1 and comparing the resultant representation of N^1 with that given by the canonical form J4), we deduce the necessary form of the K-function K for N . The usual classical method of looking at Ni is by means of formulas of the structure of (8). Our studies of the canonical equation of transfer in Chapter 4, extended to the present setting, now show that (8) is but a special form of the canonical equation for primary radiance N , as given in (9). Hence (8) may be given the $_i$ compact and intuitively useful canonical form (9) provided K is as given implicitly by (10) .

Concluding Observations

In conclusion we note that the integrations leading to (8) may be redone now over a path $g_r(z_o,\&)$ with initial point at depth $z_o > 0$. The result will be a path radiance N_r expression, the special case of Nn for n = 1, leading to an instance of (6). Observe that Nnfz,~) in (4) and $N_r(z,\&)$ in (5) are equal for every z and & , being but two ways of expressing the same radiance: Whereas (4) expresses the radiance Nn (z, ~) as a value of the radiance distribution Nn (z , •) at depth z for the direction ~, equation (5) , on the other hand, expresses the same radiance now by conceptually partitioning it into two parts associated with an arbitrary path

. .($?_r$(zo,~) in the medium. In other words, we can carry over without change .from the discussions of Chapter 4 to the present setting of n-ary concepts, all interpretations of path radiance N_r, transmitted residual radiance N , and apparent radiance N $_r$, arrived at in those earlier discussions. It is of interest to emphasize in particular a powerful but simple model for 'radiance distributions that arises when we represent N* rather than N by means of the general equation (2) of Sec. 4.4. For such a model "N_*" in (1) of Sec. 4.4 is replaced by "N*". The correct basis for this model is (?) of Sec. 5.2.