## 5.4 The Natural Solution for Radiance

We return now to the main thread of the argument, begun in 5.2, leading to the development of the natural solution of the equation of transfer. Our most basic intuitions about light fields in the sea and the air and generally for any optical medium, lead us to think of the radiance perceived by our eyes and our instruments as consisting of multiply-scattered light, i.e., light which has undergone one, two, three, and generally very large numbers of scattering operations after its entrance into the medium and before its incidence on the retina or photocell located somewhere in the medium. It is natural then (hence the name of the present



mode of solution) to attempt to construct a solution of the equation of transfer for radiance by constructing all the n-ary radiance functions Nn within a given optical medium X and to sum them to obtain the requisite radiance field throughout the medium. Thus we are led to write:

0 0 a
1 1 N ( Z , ~ ) ' f for I $N^3$ (x,O j=o

and hope that the function N so defined satisfies the equation of transfer. We call N defined by (1) the natural solution of the equation o f transfer. We now show that the word "solution" in the name for N is indeed justified.

We begin by using (14) of Sec. 5,1 to write N(x,E) in (1) as:

N (x ,&) = I $N^o$sj (x , &) =0

or more compactly in functional form as

In this way we come to define the basic operator S for the natural solution, i, e. , we can now write: for I 53 js0

where "So" denotes the identity operator I, with the-property fI = f for every radiance function. With this definition the natural solution representation takes the form

By means of this representation, the formal that N in (3) is a solution of the equation readily forthcoming via the following eight
verification of transfer is main steps

N = $N^o$S = $N^o$ ( I
$N^o$ (I
( I S] )s1 ) j=0



w
$N^o$+N5[1] $N^¢$ + (N R) T
N
° +N⋆T
$N^o$+N* We have therefore shown that:
N=$N^o$+N* 0 (4)

which is the integral form of the equation of transfer (re (1) of Sec. 3.1S). An alternative approach to the above demonstration is to show that N as defined by (1) is a solution of the integrodifferential equation of transfer. The basis for such a demonstration is given by (?) of Sec. 5.2. It remains only to add (2) of Sec. 5.2 to each side of (T) and reduce the results.

To summarize our findings: We have shown that the natural mode of constructing the radiance function N from the n-ary radiance functions Nn, n>0, leads to a solution

the natural solution--of the equation of transfer. It also may be seen that N so constructed is a unique solution in the sense that whenever N' is also a solution of (4), then N'= N. The mathematical basis for the existence and uniqueness of the natural solution will be described in Sec. 5.12.

We conclude by observing that the natural solution of the equation of transfer is not only fundamental from an intuitive physical point of view, but that it in essence exemplifies a mode of function construction which has been of increasing importance in the logical foundations of mathematics in recent years. This mode of construction--the enumerably [innumerably] recursive mode of construction--is very closely related to the natural mode of construction defined above and is coming under intensive study principally because of the current strides in developing ultra-fast mechanical aids to numerical and logical computations. These developments will eventually make feasible the computation of relatively high scattering orders -n for ¯Nn, so that finite sums of the form

$No + N^1 + N^2 + \dots + Nn$

will constitute appropriately adequate approximations to the ideal natural solution N. Thus we will eventually be able to go far beyond the first order solutions

$_N o$

$_N l$

$N^o$

N*

a + K1 cos e

 (cf. (8) , (9) of Sec. 5.3) to which many classical studies

in atmospheric and hydrologic optics were hitherto limited because of the relatively heavy demand on manipulative skill (and time t ) needed to evaluate $N^2$ , N' and higher order n-ary radiance functions 4,