

5.5 Truncated Natural Solutions for Radiance

We now investigate the effect of truncating the natural solution of the equation of transfer after a finite number of terms. While the natural solution is an ideal conceptual tool in the study of radiative transfer theory, as has been demonstrated at length in Chapter III of Ref. [251], the solution can almost never be evaluated completely either numerically or theoretically, because of the infinite number of terms comprising the solution. We are then in practice obliged to stop the accumulation of the terms after a finite number of them have been evaluated. The question then arises as to the closeness of the resultant truncated solution to the natural solution. We shall now consider this question in detail.

Throughout the remainder of this section we shall choose as our setting a source-free homogeneous plane parallel optical medium X of arbitrary depth with a steady internal light field induced by arbitrary incident radiance distributions N_0 at each point of the upper boundary of the medium. The volume scattering function s and the volume attenuation function a are otherwise arbitrary.

Now, starting with the natural solution N of the equation of transfer as defined in (1) of Sec. 5.4, we write:

$$N = \sum_{j=0}^{\infty} N_j \quad (1) \quad j=0, 1, 2, \dots$$

The central question of the present discussion may now be phrased as follows. Writing $N(k)$

for $\sum_{j=0}^k N_j$,

we ask: by how much does the finite sum $N(k)$ differ from the infinite sum N ; or in other words, what is the general order of magnitude of

$N - N(k)$. To answer this question we obtain an upper bound on the values of the difference $N - N(k)$. This upper bound shall serve as a measure of the difference between the functions N and $N(k)$.

We begin by letting N_0 denote the upper bound of the initial radiance function N^0 within X (re: (1) of Sec. 5.1). This upper bound is easily evaluated in general, and in particular in all natural hydrosols this upper bound is actually

46 NATURAL SOLUTIONS VOL. III

attained by N_0 at the air-water boundary of the medium. Indeed, for sunny days, N_0 is almost invariably the apparent radiance of the sun as seen just below the surface of the medium.

The upper bound of the primary radiance function W is obtained by first 'bounding' N^* . Thus, starting with (5) of Sec. 5.1 in which $n = 0$, we have for every x in X and direction \sim in W :

$$N^0(x, \sim) = \int_V \int_{\Omega} s(x; \sim, \sim') d\Omega' dV$$

$$N^0(x; \sim) \leq \int_V \int_{\Omega} s(x; \sim, \sim') d\Omega' dV$$

Here- " s " denotes the value of the volume total scattering function defined in (3) of Sec. 4.2. The reader will discern that it is sufficient at this stage to assume that

for every ω and E at each point x of X , in order that we have

$$dW = \int a(x-E) dQW$$

This is not an unusual requirement on a (it is called a reciprocal condition) and is readily met by all a from natural hydrosols. (For related conditions on a , see Sec. 7.12.

Next, use is made of (7) of Sec. 5.1 and the equality (2) just deduced to obtain:

$$N_l(x, \omega) = \int_{D'} N_j(x', C) T_{r-r'}(x', E) dr' D$$

$$N_0 = \int_{D'} T_{r-r'}(x', C) dr$$

$$N_0^s$$

$$, r_0$$

$$e^{-a(r-r')ar}$$

SEC. 5.5 TRUNCATED SOLUTIONS 47

$$N^p \tag{3}$$

for every point x in X and direction E in; and where we have written:

$$"p" \text{ for } s/a \tag{4}$$

The ratio p is called the albedo for single scattering or more accurately the scattering-attenuation ratio. By our agreement in Sec. 4.2, namely that about the nonnegativity of the volume absorption function a , it follows that p satisfies the inequality $0 < p < 1$. For the present discussion we assume in particular that $0 < p < 1$.

When we repeat the results (2) and (3), but now applied to $N^p(x, \omega)$ we obtain:

$$N_z(x, \omega) < N_0 p^2$$

for every x in X and E in E . From this we can see a pattern emerging and we readily prove that

for every scattering order n , every point x in X and direction ω in E .

The inequality (5) is the main result needed for determination of the upper bound for the difference $N - N^p$. Indeed, by direct computation, we have

$$N(x, E) - N^p(x, E) =$$

$$j=k+1$$

$$N_0 p^{k+1} \int P_j j-D$$

$$N_0 p^{k+1} \int -P$$

which holds for every x in X and \sim in E . Summarizing, we may say that:

48 NATURAL SOLUTIONS VOL. III

holds for every nonnegative integer k , every point x in X and direction C in E .

As an example of the use of (6), suppose a given lake has a scattering-attenuation ratio of $p = 0.4$ for wavelength 550μ , and that N_0 for that wavelength is 10^6 watts/($m^2 \times$ steradian). We require for a particular computation that

$N(x, \sim)$ be not more than 10^4 watts/($m^2 \times$ steradian) for every x and \sim . What is the least scattering order k at which the natural solution must be truncated so that this condition is met? By (6) we require k such that:

$$10^6 (0.4)^{k+1} < 10^4$$

$$10^2 < (0.4)^{k+1}$$

$$\log_{10} 10^2 < \log_{10} (0.4)^{k+1}$$

or that

$$2 < (k+1) \log_{10} 0.4$$

$$\log_{10} 10^2 < (k+1) \log_{10} 0.4$$

This implies that to the nearest integer, $k+1=6$, so that $k=5$. Hence the truncation solution is required to be carried out to five scattering orders, at least.

A useful alternative formula to (6) is obtained by first noting that for media in which $p > 0$, we certainly have the maximum value R of $N(x, E)$ greater than the maximum value N_0 of $N_0(x, \sim)$. Then (6) implies

$$N(x, \sim) - N_0(x, \sim) < p N_0(x, \sim)^{k+1}$$

for every x in X and E in w . The comparative merit of (7) over (6) consists in equation (7)'s ability to express the error of truncation in terms of a relative error, that

is the error relative to the prevailing magnitude N of the light field.

Hence for the medium at hand, carrying out the natural solution to five terms results in a relative error of less than 1 percent,

Before closing we shall examine the inequalities (5) and (6) for some insight they may yield about the relative importance of the various components of the decomposition of

the natural light field. For example, (5) shows that n -ary radiances are on the whole less by a factor of p than $(n-1)$ -ary radiances. Thus if $p = 1/2$, say, then $N_1(x, \sim)$ is on the whole, about half the magnitude of $N_0(x, \sim)$, and the magnitude of $N^2(x, \sim)$ in turn, is about half that of $N_1(x, \sim)$ and so on. Thus the overall magnitude of n -ary radiances decrease exponentially with scattering order n .

Inequality

(6) also shows that for small p (near 0), a given n -ary radiance varies directly as the n th power of p , whereas for large p (near 1), the n -ary radiances vary essentially hyperbolically

SEC. 5.5 TRUNCATED SOLUTIONS 49 with $1-p$, i.e., as $1/(1-p)$. Similar observations can be made using (6) or (7). We shall return to the matter of truncated natural solutions in the following section and reconsider them for transient light fields. The reader wishing radiance bounds in a slightly more general steady state case than that considered in this section, may consult Sec. 22 of Ref. [251].