5.7 Optical Ringing Problem. Three-Dimensional Case

We examine next how the natural mode of solution of the equation of transfer can be applied to the problem of determining the time-dependent radiance field in a natural optical medium. The program to be followed-here is that which systematically generalizes the developments of Sec. 5.1 to the time-dependent case; in particular the generalizations of the R and T operators will be the key steps in the present discussion. We begin by introducing an important geometrical concept connected with the time-dependent problem.

The Characteristic Ellipsoid

Let x and y :be two points in an extensive natural optical medium x. Suppose that at time t = 0, a spherical pulse of light is emitted from x. This pulse expands about x as center and at time r/v passes point y, where r is the distance from x to y. Here v is the speed of light in y, assumed independent of location and time throughout this discussion. Just after the wave front of the pulse passes y, a

multiply-scattered radiant flux field is generally incident on y from all directions about y. We now ask: What is the region of points in x which can send radiant flux to y at an arbitrary time t > r/v? It is easy to see that at exactly t a r/v, this region is the straight line segment between x and y. Any points x of x off this line segment could not send scattered flux to y because the detour, however, slight, would delay the scattered flux's arrival time at y. For times t of arrival at y such that t > r/v, such detours are possible to some extent. The region in which the scattering detours are possible and which allow arrival at y at time t is generally an ellipsoid of revolution with x and y as foci. This may be seen by studying Fig. 5.8, and recalling that definition of an ellipsoid which characterizes it as the locus of points z such that the sum of distances d(x, z) + d(z,y) is a constant,

CHARACTERISTIC ELLIPSOID AT TIME t

FIG. 5.8 The characteristic ellipsoid relative to the source at x and receiver at y at time t.

For the case at hand these distances are all initially considered in terms of times of travel t (x, z) and t (z ,y) across the respective distances and we are interested in all those points z in x such that

d (x , z) + d (z ,Y) = vt (1)

This defines at each instant t ? r/v an ellipsoid of revolution in x, with foci x and y. From (1) we see that the major axis of the ellipsoid is of length vt. We call the ellipsoid so defined, the characteristic ellipsoid e(x,y ; t) associated with x and y at time t > r/v. A useful polar representation of E (x,y ; t) with y as pale, is given by the equation

D - d2

r(Yt ~t) '~ - cos

where e is the angle between the unit vectors & and n, as in Fig. 5.8, and where we have written:

'r d"

for vt

for d(x,y)

The eccentricity e of the characteristic ellipsoid o?(x,y ;t) turns out to be d/D. At time t such that t = d(x,y) /v = r/v, we have e = 1. As time increases indefinitely, e decreases to zero, so that-if if the space is infinite in all directions about y--the characteristic ellipsoid approaches a sphere which takes on very nearly the polar form:

D vt

r (Y ~t) t 7. " -T

The exact spherical form of e(x,y;t) occurs at finite times if x = y, i.e., whenever d = 0. In such a case ,,F(x,x;t) becomes the characteristic spheroid S(x;t) with radius vt/2.

Time-Dependent R and T Operators and the Natural Solution

With the necessary geometrical preliminaries out of the way we can now adapt the R and T operators of Sec, 5.1 to the time-dependent case. We shall limit the present discussion to a homogeneous steady medium x with point source at a fixed point 0 and such that the characteristic ellipsoid 4f(O,x;t) is contained in x for all t under discussion. We shall then write

"R"

for

t)QCxf~' ~~) d~(~')

for I [ ]Tr-r' (x' ~C) dr' (4)

0

Comparing this pair of operators with their namesakes in Sec. 5.1, we see that the essential difference between the two pairs rests in the limit of integration for T. Now we can limit the integration to the characteristic ellipsoid 1(0,s;t) , whereas before (see Fig. 5.1) the limit of integration for T was generally the distance from x to the boundary of y in the direction -

If we go on to write:

It_s ill

for RT

and then: (5)

"N n+ 1„

for Nn 51

for every n > 0, it follows that we can construct the time dependent natural solution for the time-dependent equation of transfer (4) of Sec. 3.15, just as in 5.4. In particular the solution verification may be repeated line for line and culminating as in (4) of Sec. 5.4, with the form:

N(xvEft) '~ N^0(x,E,t) + N*(x,&,t)

(5a)

but now each term has a-time-dependent interpretation.

Truncated Natural Solution

Just as in the steady case in Sec. 5.5 we may now truncate the time-dependent natural solution and obtain an estimate of the accuracy of the truncated solution. I t turns out that the truncation estimates of the time-dependent solution can be much sharper than their steady state counter p arts, owing to the use of the characteristic ellipsoid in the timedependent computations, In this discussion suppose the source starts at t = 0

and emits in an arbitrary manner thereafter. The light field sweeps out from 0 as center in the form' of a spherical field, building up radiant flux of all scattering orders within the sphere as time goes on.
Let N° be the maximum (or supremum, if need be) of the initial radiance function No over the sphere of radius vt, center 0. See Fig. 5.9, Then observe that:
$N^{o}s^{1}$(xt ~ ft) < g^{o} p (l-e-ar (max) (6)
for every E in 0 at x and time t, where p = s/a and where we have written

FIG. 5.9 The spherical wave front of the pulse has radius vt. The characteristic ellipsoid relative to 0 and x at time t defines those points of the medium which can send flux to x from 0 at time t.
"r(max)" for max r(x,E,t) e ..
Hence:
r (max) = (D + d) /2
vt
By letting x vary over the spherical region of radius vt, center 0, (b) leads to:
N1 (x,&,t) = $N^{o}s^{l}$(x,&,t) < N° P($1^{-}e^{-avt}$) (7)

for every x in y and E in I. This may be compared with (3) of Sec. 5.5. Using (7) we can estimate the upper bound of primary scalar irradiance and radiant energy over y in terms of that of residual scalar irradiance or radiant energy. Using the basic_ idea contained in (7), we can construct a chain of inequalities for n-ary radiances. For (7) yields an upper bound o f primary radiance over the sphere of radius vt, center 0, and this upper bound now can be turned around to play the role 2f No in the estimate of the next scattering order, namely, N (x, ~ , t) . Thus in general, since
$_{N}n$ $_{N}n\text{-}1_{5}1$
it readily follows that
N^{n}(x,~ 3,t) < N^{o} [p(1-e-avt)]n

for every x in X, in H., and integer n > o. This inequality reduces to (5) of Sec. 5.5 in the steady-state, i.e. , when t' ~, The inequality (8) shows that for x sufficiently close to 0 and for small times t,
Nn (x'E't) _- (svt) nRo
where s is the total volume scattering function.
Now, just as in the. steady state case of Sec. can estimate the error of truncation of the natural series. Thus using (8), we have
5.5, we solution

$N^{(k)}$(XIE,t)

00

N^{o} [p(l-e-avt)] j = +1
Hence

1 - [p(1-'e`avt)]
(l0)
for every x in y, and & in HE at time t. For large times, (l0) reduces to (6) of Sec. 5.5. The space and source conditions giving rise to this estimate are stated at the outset of this 'discussion.

It should, now be a relatively simple matter to reduce the preceding analysis to pulse-like sources at 0, such as that considered in Sec. 5.6. The general method of analysis and its results developed between (6) and (10), of course remain the same for such sources, but sharper time-dependent estimates of N° are now possible. These truncation estimates are evidently capable of a large variety of treatments and

with the general mode of analysis now clear, each special case is best left to individual treatment by the interested investigator*